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Stochastic evolution of inviscid Burgers fluid

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Dedicated to Henry McKean with admiration

ABSTRACT. We study a stochastic Burgers equation using the geometric point of view initiated by Arnold for the incompressible Euler flow evolution. The geometry is developed as a Cartan-type geometry, using a frame bundle approach (stochastic, in this case) with respect to the infinite-dimensional Lie group where the evolution takes place. The existence of the stochastic Burgers flow is a consequence of the control in the mean of the energy transfer from low modes to high modes during the evolution, together with the use of a Girsanov transformation.

Introduction

Many distinguished authors have made notable contributions to the stochastic Burgers equation, of which a small sample appears in our very short bibliography. It is not our purpose to review those contributions; it is perhaps appropriate that we underline here that which seems to us the novelty of our approach.

We start from the viewpoint of geometrization of inertial evolution initiated in [Arnold 1966] and systematically developed in [Ebin and Marsden 1970; Brenier 2003; Constantin and Kolev 2002], based on infinite-dimensional Riemannian geometry; the classical approach of [Ebin and Marsden 1970] is to use Banach-modeled manifold theory; inherent difficulties appear in the construction of exponential charts and in the introduction of appropriate function spaces. We circumvent these difficulties by using the viewpoint [Malliavin 2007] of *Itô charts, Itô atlas*; in short Itô calculus makes it possible to compute any derivative of a smooth function f on the path p of a diffusion from the unique knowledge

Cruzeiro developed her work in the framework of the research project POCI/MAT/55977/2004.

of its restriction $f_{|P}$. Then no more function spaces are a priori introduced: the path of the diffusion constructs dynamically its canonical tangent space, built from the evolution of the system.

How do we make explicit computations without local coordinates? We take the viewpoint of [Arnold 1966; Cruzeiro and Malliavin 1996; Airault and Malliavin 2006; Cruzeiro et al. 2007], using the parallelism defined by the infinite-dimensional Lie group structure.

In fluid dynamics the escape of the energy from low modes to higher modes induces a lack of compactness which ruins the advantage of energy conservation for inertial evolution. The key point of our approach is the control of this ultraviolet divergence. We control the ultraviolet divergence in the case of the stochastic Burgers equation with vanishing initial value. Then symmetries appear which, as in [Airault and Malliavin 2006; Cruzeiro et al. 2007], make it possible to compute exactly the expectation of the energy transfer by the exponentiation of a numerical symmetric matrix.

Then we have solved our stochastic Burgers equation for vanishing initial data: We reduce, as in [Cruzeiro et al. 2007], the nonvanishing initial data case to this trivial case by a symmetry breaking expressed at the level of probability space by a Girsanov functional.

We emphasize that the noise that we use is neither an external force nor a damping. This important point is made explicit in the next section.

1. Random regularization of nonlinear evolution

In order to clarify our objectives, we shall proceed in this section at a conceptual level, which has the disadvantage that we cannot produce at this level of generality a single mathematical statement: the considered objects will not be exactly defined; the reader will have to wait until Section 2 before getting into mathematics.

Numerical integration of an evolution equation through a time discretization scheme introduces at each step a numerical error; if the scheme is "well chosen", it will be unbiased: therefore the cumulative effect of numerical errors will converge *locally* to a Brownian motion.

Let us axiomatize the previous empirical situation. Denote by $\mathcal{S}$ the infinitesimal generator of an evolution equation, which is not assumed to be linear; the operator $\mathcal{S}$ is operating on Cauchy data; then consider the Stratonovitch SDE

$$d_t u_t^\varepsilon = \mathcal{S}\big(u_t^\varepsilon\, dt + \varepsilon dx(t)\big),\ u_0^\varepsilon \text{ deterministic and independent of } \varepsilon, \qquad (1.1)_a$$

where x is a suitable Brownian motion modeling *the instantaneous discretization error* and where $\varepsilon > 0$. We call the solution of $(1.1)_a$ the *random regular-*

ization of the evolution equation

$$d_t u_t = \mathcal{S}(u_t\ dt), \quad u_0 \text{ given.} \tag{1.1$_b$}$$

The disadvantage of $(1.1)_a$ versus $(1.1)_b$ is to replace an ODE by an SDE; this disadvantage is balanced by the advantage that the introduction of a small noise can smooth out resonances leading to the system explosion.

The terminology used, *random regularization*, is parallel to the classical terminology *elliptic regularization*. This choice of terminology can be justified by the fact that dealing with the Brownian motion x is equivalent to dealing with some infinite-dimensional elliptic operator defined on the path space of $x(*)$.

2. The Burgers equation as a geodesic flow

Consider the group G of C^∞ diffeomorphisms of the circle S^1, denote by $\mathcal{G}$ its Lie algebra of right invariant first order differential operators on G; we identify $\mathcal{G}$ to vector fields on S^1; define on $\mathcal{G}$ the pre-Hilbertian metric

$$\|u\|^2 = \frac{1}{\pi}\int_0^{2\pi} |u|^2(\theta)\ d\theta; \tag{2.1}$$

then G becomes an "infinite-dimensional Riemannian manifold".

THEOREM [Arnold 1966; Constantin and Kolev 2003]. *Let $v_t(\theta)$ a be smooth vector field defined on S^1, depending smoothly on time t, which is assumed to satisfy the Burgers equation*

$$\frac{\partial v}{\partial t} = v \times \frac{\partial v}{\partial \theta}. \tag{2.2}$$

Let g_t be the time dependent diffeomorphism of S^1 defined by the family of ODEs

$$\frac{d}{dt} g_t(\theta) = v_t(g_t(\theta)); \quad g_0(\theta) = \theta. \tag{2.3}$$

Then

$$t \mapsto g_t \textit{ is a geodesic of the Riemannian manifold } G. \tag{2.4}$$

3. Structure constants of $\mathcal{G}$

The vector fields

$$A_k = \cos k\theta, \quad B_k = \sin k\theta, \quad k > 0, \quad A_0 = \frac{1}{\sqrt{2}} \tag{3.1}$$

constitute an orthonormal basis of $\mathcal{G}$. In this basis, the Lie brackets are as follows:

$$\begin{aligned}
&[A_0, A_k] = -(k/\sqrt{2})\, B_k,\\
&[A_0, B_k] = (k/\sqrt{2})\, A_k,\ k > 0,\\
&[A_s, A_k] = \tfrac{1}{2}((s-k) B_{k+s} + (s+k) B_{s-k}),\\
&[B_s, B_k] = \tfrac{1}{2}((k-s) B_{k+s} + (s+k) B_{s-k}),\\
&[A_s, B_k] = \tfrac{1}{2}((k-s) A_{k+s} + (s+k) A_{k-s}), \quad s \neq k,\\
&[B_k, A_s] = \tfrac{1}{2}((s-k) A_{k+s} - (s+k) A_{s-k}), \quad s \neq k,\\
&[A_k, B_k] = \sqrt{2} k A_0.
\end{aligned}$$

PROOF.

$$\begin{aligned}
[A_s, A_k] &= -k A_s \times B_k + s A_k \times B_s = \tfrac{1}{2}(-k(B_{k+s} + B_{k-s}) + s(B_{k+s} + B_{s-k}))\\
&= \tfrac{1}{2}((s-k) B_{k+s} + (s+k) B_{s-k}),\\
[B_s, B_k] &= k B_s \times A_k - s B_k \times A_s = \tfrac{1}{2}(k(B_{k+s} + B_{s-k}) - s(B_{k+s} + B_{k-s}))\\
&= \tfrac{1}{2}((k-s) B_{k+s} + (s+k) B_{s-k}),\\
[A_s, B_k] &= k A_s \times A_k + s B_k \times B_s = \tfrac{1}{2}(k(A_{k+s} + A_{k-s}) + s(-A_{k+s} + A_{s-k}))\\
&= \tfrac{1}{2}((k-s) A_{k+s} + (s+k) A_{k-s}),
\end{aligned}$$

Analogously,

$$[B_k, A_s] = \tfrac{1}{2}((s-k) A_{k+s} - (s+k) A_{s-k}). \qquad \square$$

4. The Christoffel tensor

We have on G two connections:

(i) the algebraic connection defined by the right invariant parallelism on G;
(ii) the Riemannian connection defined by the Levi-Civita parallel transport.

The difference of two connections defines a tensor field $\Gamma^{*}_{*,*}$.

We have the key general lemma:

LEMMA [Arnold 1966; Cruzeiro and Malliavin 1996; Airault and Malliavin 2006]. *Let G be a group with a right-invariant Hilbertian metric, and let $\{e_k\}$ be an orthonormal basis of its Lie algebra $\mathcal{G}$. Then*

$$\Gamma^{l}_{s,k} = \tfrac{1}{2}(c^{l}_{s,k} - c^{s}_{k,l} + c^{k}_{l,s}), \quad \textit{where } [e_s, e_k] = \sum_{l} c^{l}_{s,k} e_l. \tag{4.1}$$

We deduce immediately from the structural constants the identities

$$\begin{aligned}
2\Gamma^{A_l}_{A_s A_k} &= ([A_s, A_k]|A_l) - ([A_k, A_l]|A_s) + ([A_l, A_s]|A_k) = 0,\\
2\Gamma^{B_l}_{A_s B_k} &= ([A_s, B_k]|B_l) - ([B_k, B_l]|A_s) + ([B_l, A_s]|B_k) = 0,\\
2\Gamma^{A_l}_{B_s B_k} &= ([B_s, B_k]|A_l) - ([B_k, A_l]|B_s) + ([A_l, B_s]|B_k) = 0,\\
2\Gamma^{B_l}_{B_s A_k} &= -2\Gamma^{A_k}_{B_s B_l} = 0.
\end{aligned}$$

It remains to compute

$$\Gamma^{B_l}_{A_s A_k}, \quad \Gamma^{A_l}_{B_s A_k}, \quad \Gamma^{B_l}_{B_s B_k}, \quad \Gamma^{A_l}_{A_s B_k}.$$

THEOREM.

- *Assume* $0 < s < k$. *Then*

$$\begin{aligned}
\Gamma_{A_s A_k} &= -\left(k - \tfrac{1}{2}s\right)B_{k-s} - \left(k + \tfrac{1}{2}s\right)B_{k+s},\\
\Gamma_{A_s B_k} &= \quad\left(k - \tfrac{1}{2}s\right)A_{k-s} + \left(k + \tfrac{1}{2}s\right)A_{k+s},\\
\Gamma_{B_s A_k} &= -\left(k - \tfrac{1}{2}s\right)A_{k-s} + \left(k + \tfrac{1}{2}s\right)A_{k+s},\\
\Gamma_{B_s B_k} &= -\left(k - \tfrac{1}{2}s\right)B_{k-s} + \left(k + \tfrac{1}{2}s\right)B_{k+s}.
\end{aligned}$$

- *Assume* $0 < k < s$. *Then*

$$\begin{aligned}
\Gamma_{A_s A_k} &= \quad\left(k - \tfrac{1}{2}s\right)B_{s-k} - \left(k + \tfrac{1}{2}s\right)B_{k+s},\\
\Gamma_{A_s B_k} &= \quad\left(k - \tfrac{1}{2}s\right)A_{s-k} + \left(k + \tfrac{1}{2}s\right)A_{k+s},\\
\Gamma_{B_s A_k} &= -\left(k - \tfrac{1}{2}s\right)A_{s-k} + \left(k + \tfrac{1}{2}s\right)A_{k+s},\\
\Gamma_{B_s B_k} &= \quad\left(k - \tfrac{1}{2}s\right)B_{s-k} + \left(k + \tfrac{1}{2}s\right)B_{k+s},
\end{aligned}$$

In each case the two first lines define an antisymmetric operator $\boldsymbol{\Gamma}(A_s)$ *and the two last lines define an operator* $\Gamma(B_s)$.

- *For* $k > 0$,

$$\Gamma_{A_k A_k} = -\Gamma_{B_k B_k} = -\tfrac{3}{2}k B_{2k},$$

$$\begin{aligned}
\Gamma_{A_k B_k} &= \tfrac{3}{2}k A_{2k} + \tfrac{\sqrt{2}}{2}k A_0, &\quad \Gamma_{B_k A_k} &= \tfrac{3}{2}k A_{2k} - \tfrac{\sqrt{2}}{2}k A_0,\\
\Gamma_{A_0 A_k} &= -\sqrt{2}k B_k, & \Gamma_{A_k A_0} &= -\tfrac{\sqrt{2}}{2}k B_k,\\
\Gamma_{A_0 B_k} &= \tfrac{\sqrt{2}}{2}k A_k + \tfrac{\sqrt{2}}{2}k A_0, & \Gamma_{B_k A_0} &= \tfrac{\sqrt{2}}{2}k A_0,
\end{aligned}$$

- *Finally*, $\Gamma_{A_0 A_0} = 0$.

PROOF. Consider the case $0 < s < k$. We have $4\Gamma^{B_l}_{A_s A_k} = I - II - III$, with

$$I = 2([A_s, A_k]|B_l), \quad II = 2([A_k, B_l]|A_s), \quad III = -2([B_l, A_s]|A_k).$$

The term I is equal to $s-k$ when $l=k+s$ and to $-(s+k)$ when $l=k-s$.

Other contributions to the component B_{k+s} are $s+2k$ from II in the case $k<l$ and $-(2s+k)$ from III corresponding to the case $s<l$. Concerning the component B_{k-s} we have to consider the contribution $2k-s$ from II when $l<k$ and the contribution from III in the case $s<l$, which is equal to $2s-k$. Summing up all the terms gives the result.

In more detail, introduce for $s>0$ the new Kronecker symbol

$$\varepsilon_p^s=\delta_p^s,\ p>0,\quad \varepsilon_p^s=-\delta_{-p}^s,\ p<0,\quad \varepsilon_0^s=0.$$

Take $s,k,l>0$; then $4\Gamma_{A_sA_k}^{B_l}$ equals

$$(s-k)\delta_{k+s}^l+(s+k)\varepsilon_{s-k}^l+(k-l)\delta_{k+l}^s-(l+k)\delta_{|l-k|}^s+(s-l)\delta_{l+s}^k-(s+l)\delta_{|s-l|}^k.$$

Consider first the case $0<s<k$; then $4\Gamma_{A_sA_k}^{B_l}$ equals

$$(s-k)\delta_{k+s}^l-(s+k)\delta_{k-s}^l+(k-l)\delta_{k+l}^s-(l+k)\delta_{|l-k|}^s+(s-l)\delta_{l+s}^k-(s+l)\delta_{|l-s|}^k.$$

(1) Take the subcase $0<s<k<l$. Then $4\Gamma_{A_sA_k}^{B_l}$ equals

$$(s-k)\delta_{k+s}^l-(s+k)\delta_{k-s}^l+(k-l)\delta_{k+l}^s-(l+k)\delta_{l-k}^s+(s-l)\delta_{l+s}^k-(s+l)\delta_{l-s}^k;$$

expressing the δ functions relatively to l, this expression becomes

$$(s-k)\delta_{k+s}^l-(s+k)\delta_{k-s}^l+(k-l)\delta_{s-k}^l-(l+k)\delta_{k+s}^l+(s-l)\delta_{k-s}^l-(s+l)\delta_{k+s}^l,$$

so

$$4\Gamma_{A_sA_k}^{B_l}=\big((s-k)-(l+k)-(s+l)\big)\delta_{k+s}^l=-2(k+l)\delta_{k+s}^l.$$

(2) In the subcase $0<s<l<k$, we obtain for $4\Gamma_{A_sA_k}^{B_l}$ successively

$$\begin{aligned}&(s-k)\delta_{k+s}^l-(s+k)\delta_{k-s}^l+(k-l)\delta_{k+l}^s-(l+k)\delta_{k-l}^s+(s-l)\delta_{l+s}^k-(s+l)\delta_{l-s}^k=\\&(s-k)\delta_{k+s}^l-(s+k)\delta_{k-s}^l+(k-l)\delta_{s-k}^l-(l+k)\delta_{k-s}^l+(s-l)\delta_{k-s}^l-(s+l)\delta_{k+s}^l=\\&\quad\big(-(s+k)-(l+k)+(s-l)\big)\delta_{k-s}^l=-2\big(k+l\big)\delta_{k-s}^l=-2\big(2k-s\big)\delta_{k-s}^l.\end{aligned}$$

(3) In the subcase $0<l<s<k$, we obtain for $4\Gamma_{A_sA_k}^{B_l}$

$$\begin{aligned}&(s-k)\delta_{k+s}^l-(s+k)\delta_{k-s}^l+(k-l)\delta_{k+l}^s-(l+k)\delta_{k-l}^s+(s-l)\delta_{l+s}^k-(s+l)\delta_{s-l}^k=\\&(s-k)\delta_{k+s}^l-(s+k)\delta_{k-s}^l+(k-l)\delta_{s-k}^l-(l+k)\delta_{k-s}^l+(s-l)\delta_{k-s}^l-(s+l)\delta_{s-k}^l=\\&\qquad\big(-(s+k)-(l+k)+(s-l)\big)\delta_{k-s}^l=-2(k+l)\delta_{k-s}^l.\end{aligned}$$

Finally, still for $0<s<k$ we have

$$\Gamma_{A_sA_k}=-\big(k-\tfrac{1}{2}s\big)B_{k-s}-\big(k+\tfrac{1}{2}s\big)B_{k+s}.$$

We now consider a rotation of angle φ. Define

$$A_k^\varphi = A_k \cos k\varphi - B_k \sin k\varphi, \quad B_q^\varphi = B_q \cos q\varphi + A_q \sin q\varphi.$$

The metric on $\mathcal{G}$ is invariant under translation by φ. Therefore the Christoffel symbols commute with this translation:

$$\begin{aligned}-\big(k-\tfrac{1}{2}s\big)B_{k-s}^\varphi-\big(k+\tfrac{1}{2}s\big)B_{k+s}^\varphi &= \Gamma_{A_s^\varphi A_k^\varphi} \\ &= \Gamma_{A_s A_k}\cos s\varphi\cos k\varphi+\Gamma_{B_s B_k}\sin s\varphi\sin k\varphi \\ &\quad -\Gamma_{A_s B_k}\cos s\varphi\sin k\varphi-\Gamma_{B_s A_k}\sin s\varphi\cos k\varphi.\end{aligned}$$

On the other hand,

$$\begin{aligned}&-\big(k-\tfrac{1}{2}s\big)B_{k-s}^\varphi-\big(k+\tfrac{1}{2}s\big)B_{k+s}^\varphi \\ &\quad=-\big(k-\tfrac{1}{2}s\big)\big(B_{k-s}\cos(k-s)\varphi+A_{k-s}\sin(k-s)\varphi\big) \\ &\qquad\qquad -\big(k+\tfrac{1}{2}s\big)\big(B_{k+s}\cos(k+s)\varphi+A_{k+s}\sin(k+s)\varphi\big) \\ &\quad=-\big(k-\tfrac{1}{2}s\big)\big(B_{k-s}(\cos k\varphi\cos s\varphi+\sin k\varphi\sin s\varphi) \\ &\qquad\qquad +A_{k-s}(\sin k\varphi\cos s\varphi-\cos k\varphi\sin s\varphi)\big) \\ &\qquad -\big(k+\tfrac{1}{2}s\big)\big(B_{k+s}(\cos k\varphi\cos s\varphi-\sin k\varphi\sin s\varphi) \\ &\qquad\qquad +A_{k+s}(\sin k\varphi\cos s\varphi+\cos k\varphi\sin s\varphi)\big).\end{aligned}$$

Identifying the coefficients of $\cos k\varphi\cos s\varphi$, $\sin k\varphi\sin s\varphi$, $\sin k\varphi\cos s\varphi$, and $\cos k\varphi\sin s\varphi$, we get the formulae for the Christoffel symbols in the case $0 < k < s$.

For $0 < k = s$, we have, for example,

$$\Gamma_{A_k A_k}^{B_l} = -([A_k, B_l]|A_k) = -\tfrac{1}{2}(k+l)\delta_{2k}^l = -\tfrac{3}{2}k\delta_{2k}^l.$$

The other expressions are proved in a similar way. □

5. Stochastic parallel transport; symmetries of the noise

Consider for each $k \geq 0$ a $\mathbb{R}^2$-valued Brownian motion $\zeta_k(t) = (x_k(t), y_k(t))$; all these Brownian motions are taken to be independent. Choose a weight $\rho(k) \geq 0$ and consider the $\mathcal{G}$ valued process

$$p_t = \sum_{k>0} \rho(k)\big(x_k(t) \times A_k + y_k(t) \times B_k\big). \tag{5.1}$$

Consider the Stratonovitch SDE

$$d\psi_t = -\boldsymbol{\Gamma}(dp_t) \circ \psi_t, \quad \psi_0 = \text{Identity}. \tag{5.2}$$

As the $\boldsymbol{\Gamma}$ are antisymmetric operators this equation takes formally its values in the unitary group of $\mathcal{G}$.

The geometric meaning of (5.2) is to describe in terms of the algebraic parallelism inherited from the group structure of G the Levi-Civita parallelism inherited from the Riemannian structure of G; for this reason we call (5.2) the *equation of stochastic parallel transport.*

Symmetries of the noise. The translation $\tau_\varphi : \theta \mapsto \theta + \varphi$ is a diffeomorphism

$$[(\tau_\varphi)_*(z)](\theta) = z(\theta - \varphi)$$

The collection $(\tau_\varphi)_*,\ \varphi \in S^1$, constitutes a unitary representation of S^1 on $\mathcal{G}$ which decomposes into irreducible components along the direct sum of two-dimensional subspaces

$$\bigoplus_{k>0} \mathcal{E}_k, \quad \mathcal{E}_k := (A_k, B_k), \quad \mathcal{E}_0 := A_0,$$

the action of $(\tau_\varphi)_*$ on $\mathcal{E}_k$ being the rotation

$$\mathcal{D}_k(\varphi) := \begin{pmatrix} \cos k\varphi & -\sin k\varphi \\ \sin k\varphi & \cos k\varphi \end{pmatrix}, \quad \mathcal{D}_0(\varphi) := \text{Identity}.$$

Furthermore τ_φ preserves the Lie algebra structure. The Christoffel symbols are derived from the Hilbertian structure and from the bracket structure of $\mathcal{G}$. Therefore they commute with τ_φ in the sense that

$$(\tau_\varphi)_*[\Gamma(\xi)(\eta)] = \Gamma((\tau_\varphi)_*\xi)[(\tau_\varphi)_*\eta)], \quad \xi,\ \eta \in \mathcal{G};$$

or, denoting $\boldsymbol{\Gamma}(z)$ the antihermitian endomorphism of $\mathcal{G}$ defined by the Christoffel symbols, we have

$$\boldsymbol{\Gamma}((\tau_\varphi)_*(z)) = (\tau_\varphi)_* \circ \boldsymbol{\Gamma}(z) \circ (\tau_{-\varphi})_*.$$

Denote by $\mathrm{su}(\mathcal{G})$ the vector space of antisymmetric operators on the Hilbert space $\mathcal{G}$.

PROPOSITION. *Let p_t the $\mathcal{G}$-valued process defined in* (5.1) *and set* $(\tau_\varphi)_* p =: p_*^\varphi$; *then p_*^φ and p have the same law.*

PROOF. The rotation $\mathcal{D}_k(\phi)$ preserves in law the Brownian motion on $\mathcal{E}_k$. □

COROLLARY. *The processes $(\tau_\varphi) \circ \psi_t \circ (\tau_{-\varphi})$ and ψ_t have the same law.*

PROOF. Denote by ψ_t^p the solution of (3.3) associated to the noise p_t. Then

$$(\tau_\varphi) \ \circ \ \psi_t \ \circ \ (\tau_{-\varphi}) = \psi_t^{p^\varphi}$$ □

The Stratonovich SDE (5.2) corresponds to the Itô SDE

$$d\psi_t^p = \big(\boldsymbol{\Gamma}(dp) + \mathcal{B}\, dt\big)\, \psi_t,$$
$$\mathcal{B} = \sum_{k\geq 0} \frac{[\rho(k)]^2}{2} (\boldsymbol{\Gamma}(A_k) * \boldsymbol{\Gamma}(A_k) + \boldsymbol{\Gamma}(B_k) * \boldsymbol{\Gamma}(B_k)).$$

We get $\mathcal{B} = (\tau_\varphi)_* \circ \mathcal{B} \circ (\tau_{-\varphi})_*$, which implies that $\mathcal{B}$ diagonalizes in the basis $\bigoplus \mathcal{E}_k$. More precisely:

THEOREM. *The operator*

$$[\boldsymbol{\Gamma}(A_s)]^2 + [\boldsymbol{\Gamma}(B_s)]^2$$

is diagonal and on the mode k it has eigenvalue

$$\lambda_s(k) = -(4k^2 + s^2), \quad k > 2s.$$

PROOF. We have

$$\begin{aligned}
[\boldsymbol{\Gamma}(A_s)]^2(A_k) &= -\big(k - \tfrac{1}{2}s\big)\boldsymbol{\Gamma}(A_s)(B_{k-s}) - \big(k + \tfrac{1}{2}s\big)\boldsymbol{\Gamma}(A_s)(B_{k+s}) \\
&= -\big(k - \tfrac{1}{2}s\big)\big((k - \tfrac{3}{2}s)A_{k-2s} + (k - \tfrac{1}{2}s)A_k\big) \\
&\qquad\qquad -\big(k + \tfrac{1}{2}s\big)\big((k + \tfrac{1}{2}s)A_k + (k + \tfrac{3}{2}s)A_{k+2s}\big),
\end{aligned}$$

$$\begin{aligned}
[\boldsymbol{\Gamma}(B_s)]^2(A_k) &= -\big(k - \tfrac{1}{2}s\big)\boldsymbol{\Gamma}(B_s)(A_{k-s}) + \big(k + \tfrac{1}{2}s\big)\boldsymbol{\Gamma}(B_s)(A_{k+s}) \\
&= -\big(k - \tfrac{1}{2}s\big)\big(-(k - \tfrac{3}{2}s)A_{k-2s} + (k - \tfrac{1}{2}sA_k)\big) \\
&\qquad\qquad -\big(k + \tfrac{1}{2}s\big)\big((k + \tfrac{1}{2}s)A_k - (k + \tfrac{3}{2}s)A_{k+2s}\big).
\end{aligned}$$

Hence

$$[\boldsymbol{\Gamma}(A_s)]^2(A_k) + [\boldsymbol{\Gamma}(B_s)]^2(A_k) = -2(k - \tfrac{1}{2}s)^2 - 2(k + \tfrac{1}{2}s)^2. \qquad \square$$

We want to take, as in [Cruzeiro et al. 2007], a finite-mode driven Brownian motion, which means that $\rho(k) = 0$ except for a finite number of values of k.

6. Control of ultraviolet divergence by the transfer energy matrix

THEOREM. *Let e be a trigonometric polynomial, and define*

$$\xi_k(t) = E\big([(\psi_t(e) \mid A_k)]^2 + [(\psi_t(e) \mid B_k)]^2\big).$$

Then $\xi(t)$ satisfies the ordinary differential equation

$$\frac{d\xi(t)}{dt} = \mathcal{A}(\xi(t)), \tag{6.1}$$

where the matrix $\mathcal{A}$ has diagonal entries

$$\mathcal{A}_l^l = -4\sum_k \rho(k)^2\big(2l^2 + \tfrac{1}{2}k^2\big) - \tfrac{9}{8}l^2\rho^2(\tfrac{1}{2}l)$$

and nondiagonal entries

$$\mathcal{A}_s^l = 2\sum_k \rho(k)^2\big((l-\tfrac{1}{2}k)^2\delta_s^{|k-l|} + 2(l+\tfrac{1}{2}k)^2\delta_s^{k+l}\big) + \tfrac{9}{8}l^2\rho^2(\tfrac{1}{2}l),$$

with $s, l > 0$. *The sum of the coefficients in each column vanishes.*

PROOF. We have, explicitly,

$$d\psi_t^{A_l} = -\sum_m (\Gamma^{A_l}_{A_k B_m}\psi_t^{B_m} odx_k(t) + \Gamma^{A_l}_{B_k A_m}\psi_t^{A_m} ody_k(t)).$$

By Itô calculus,

$$d(\psi_t^{A_l})^2 = 2\psi_t^{A_l} d\psi_t^{A_l} + d\psi_t^{A_l}.\, d\psi_t^{A_l},$$
$$d(\psi_t^{B_l})^2 = 2\psi_t^{B_l} d\psi_t^{A_l} + d\psi_t^{B_l}.\, d\psi_t^{B_l}.$$

Since we are interested in taking expectations we compute only the bounded variation part of this semimartingale. Considering the terms $0 < m \le k$,

$$\begin{aligned} d\psi_t^{A_l} &= -\Gamma^{A_l}_{A_k B_{l-k}}\psi_t^{B_{l-k}} odx_k(t) - \Gamma^{A_l}_{B_k A_{l-k}}\psi_t^{A_{l-k}} ody_k(t) \\ &\qquad -\Gamma^{A_l}_{A_k B_{l+k}}\psi_t^{B_{l+k}} odx_k(t) - \Gamma^{A_l}_{B_k A_{l+k}}\psi_t^{A_{l+k}} ody_k(t) \\ &= -(l-\tfrac{1}{2}k)\psi_t^{B_{l-k}} odx_k(t) - (l-\tfrac{1}{2}k)\psi_t^{A_{l-k}} ody_k(t) \\ &\quad -(l+\tfrac{1}{2}k)\psi_t^{B_{l+k}} odx_k(t) + (l+\tfrac{1}{2}k)\psi_t^{A_{l+k}} ody_k(t) \\ &\qquad -\tfrac{3}{2}\sum_k \rho(k)k\psi_t^{B_k} odx_k(t) - \tfrac{3}{2}\sum_k \rho(k)k\psi_t^{A_k} ody_k(t). \end{aligned}$$

Computing the Itô contractions, we obtain, for example, in the case of the first term,

$$\begin{aligned} -(l-\tfrac{1}{2}k)\psi_t^{B_{l-k}} odx_k(t) &= -(l-\tfrac{1}{2}k)\psi_t^{B_{l-k}} dx_k(t) \\ &\quad -\tfrac{1}{2}\big((l-\tfrac{3}{2}k)\psi_t^{A_{l-2k}} - (l-\tfrac{1}{2}k)\psi_t^{A_l} + (\tfrac{3}{2}k-l)\psi_t^{A_{2k-l}}\big)\, dt. \end{aligned}$$

We can check by explicit computation that all the nondiagonal contributions coming from these Itô contractions cancel in their contribution to the expectation of $\psi_t^{A_l} d\psi_t^{A_l} + \psi_t^{B_l} d\psi_t^{B_l}$. The diagonal ones, for the case $0 < k < m$, sum up to give

$$-2\sum_k (2l^2 + \tfrac{1}{2}k^2)\psi_t^{A_l}\, dt.$$

The terms in $0 < k < m$ give the same expression. The contribution from $k = m$ gives

$$-\tfrac{3}{2}\rho\big(\tfrac{1}{2}l\big)\,\tfrac{1}{2}l\psi_t^{A_{1/2 l}}\, dt.$$

Concerning the B_l component of ψ_t, namely

$$d\psi_t^{B_l} = -\sum_m \big(\Gamma^{B_l}_{A_k A_m}\psi_t^{A_m} o dx_k(t) + \Gamma^{B_l}_{B_k B_m}\psi_t^{B_m} o dy_k(t)\big),$$

analogous computations give rise to the expressions

$$-2\sum_k (2l^2 + \tfrac{1}{2}k^2)\psi_t^{B_l}\,dt$$

for $l < m$ and $m < l$, and

$$-\tfrac{3}{2}\rho\big(\tfrac{1}{2}l\big)\,\tfrac{1}{2}l\psi_t^{B_{1/2}l}\,dt$$

when $k = m$.

The nondiagonal terms of the transfer energy matrix come from computing the contractions $d\psi_t^{A_l}.\,d\psi_t^{A_l}$ and $d\psi_t^{B_l}.\,d\psi_t^{B_l}$. We have, when $0 < k \le l$,

$$\begin{aligned}
d\psi_t^{A_l}.\,d\psi_t^{A_l} &= \sum_k \rho(k)^2\big(\Gamma^{A_l}_{A_k B_{l-k}}\psi_t^{B_{l-k}}\big)^2 dt + \sum_k \rho(k)^2\big(\Gamma^{A_l}_{B_k A_{l-k}}\psi_t^{A_{l-k}}\big)^2 dt\\
&\quad + \sum_k \rho(k)^2\big(\Gamma^{A_l}_{A_k B_{l+k}}\psi_t^{B_{l+k}}\big)^2 dt + \sum_k \rho(k)^2\big(\Gamma^{A_l}_{B_k A_{l+k}}\psi_t^{A_{l+k}}\big)^2 dt\\
&\quad + \rho\big(\tfrac{1}{2}l\big)^2\Gamma^{A_l}_{A_{1/2}l B_{1/2}l}\big(\psi_t^{B_l/2}\big)^2 + \rho\big(\tfrac{1}{2}l\big)^2\Gamma^{A_l}_{B_{1/2}l A_{1/2}l}\big(\psi_t^{A_l/2}\big)^2\\
&= \sum_k \rho(k)^2\big(l - \tfrac{1}{2}k\big)^2\big(\psi_t^{B_{l-k}}\big)^2 + \sum_k \rho(k)^2\big(l - \tfrac{1}{2}k\big)^2\big(\psi_t^{A_{l-k}}\big)^2\\
&\quad + \sum_k \rho(k)^2\big(l + \tfrac{1}{2}k\big)^2\big(\psi_t^{B_{l+k}}\big)^2 + \sum_k \rho(k)^2\big(l + \tfrac{1}{2}k\big)^2\big(\psi_t^{A_{l+k}}\big)^2\\
&\quad + \tfrac{9}{4}\rho\big(\tfrac{1}{2}l\big)^2\big(\tfrac{1}{2}l\big)^2\big(\psi_t^{B_l/2}\big)^2 + \tfrac{9}{4}\rho\big(\tfrac{1}{2}l\big)^2\big(\tfrac{1}{2}l\big)^2\big(\psi_t^{A_l/2}\big)^2.
\end{aligned}$$

Computing the corresponding terms for the indices $0 < l < k$ as well as the contractions $d\psi_t^{B_l}.\,d\psi_t^{B_l}$ gives the desired result. □

7. Ultraviolet divergence and dissipativity of the associated jump process

The ordinary differential equation (6.1) can be integrated quite explicitly by the exponential $\exp(t\mathcal{A})$; nevertheless the effective computation of this exponential is not easy.

It was observed in [Airault and Malliavin 2006, Theorem (3.10)] that $\mathcal{A}$ can be also considered as the infinitesimal generator of a Dirichlet form; therefore its exponentiation is equivalent to construct the jump process associated to this Dirichlet form. Recall how this jump process was constructed in that theorem.

In order to shorten our discussion we shall sketch our proof in the special case where

$$\rho(1) = 1, \quad \rho(k) = 0, \quad k \neq 1.$$

Then the random walk $X(n)$ is a nearest neighbor random walk defined on $\mathbb{N}$, the set of positive integers, as follows:

If $X(n) = k,\ k > 2$ we have

$$\mathrm{Prob}\{X(n+1) = k+1\} = p_k := \frac{1}{2}\left(1 + \frac{k}{4k^2+1}\right),$$
$$\mathrm{Prob}\{X(n+1) = k-1\} = 1 - p_k.$$

The random walk is nonsymmetric, it has a drift $\simeq \frac{1}{k}$ pushing it to escape at infinity. This drift has a negligible effect in our discussion and we shall proceed as if the random walk was symmetric.

The jump process is defined as

$$\eta(t) := X(\varphi(t))$$

where the change of clock $\varphi(t)$ is the integer-valued function defined by

$$\sum_{n \le \varphi(t)} \frac{1}{4[X(n)]^2+1} \times \Lambda_n \le t < \sum_{n \le \varphi(t)+1} \frac{1}{4[X(n)]^2+1} \times \Lambda_n,$$

where $\{\Lambda_k\}$ is a sequence of independent exponential times.

THEOREM. *The jump process is conservative. That is*, $\varphi(t) < \infty$ *almost surely*; *more precisely*,

$$E([X(\varphi(t))]^q) < \infty \quad \textit{for all } q > 0. \tag{7.2}$$

PROOF. What follows is an improved methodology of proof compared to the one used in [Cruzeiro et al. 2007]. The proof of (7.2) will occupy us till the end of Section 7.

Let Ω_1 be the probability space of the random walk; then Ω_1 is a space generated by an infinite sequence of independent Bernoulli variables; let Ω_2 be the probability space generated by an infinite sequence of independent exponential variables. Then the probability space of the jump process is $\Omega_1 \times \Omega_2$. We denote by E^{ω_i} the expectation relatively to Ω_i, the other coordinate being fixed, and we write $\mathrm{Prob}_i(A) := E^{\omega_i}(1_A)$.

We introduce a strictly increasing sequence of stopping times $T_1 < T_2 < \cdots < T_k < \cdots$ on the random walk by the following recursion: T_1 is the first time where the value starting from 1 it reaches 2; T_{k+1} is the first time after T_k where $X(T_{k+1})$ leaves the interval $\left(\frac{1}{2}X(T_k), 2X(T_k)\right)$; we have

$$X(T_k) = 2^{\xi_k}, \quad \xi_k \in \mathbb{N}.$$

Then ξ_k is an unsymmetric random walk on the set of positive integers. We construct on Ω_1 a new random walk $X^*(n)$ by taking

$$X^*(T_{k+1}) = 2X^*(T_k), \text{ then } X(n) \le X^*(n); \quad \inf_{m>n} X^*(m) \ge \tfrac{1}{2}X^*(n).$$

Denote by $\varphi^*(t)$ the time change in the jump process associated to the random walk $X^*(*)$; we obtain a new jump process $\eta^*(t)$, defined on the same probability space as η, and we have

$$\eta(t) \leq 2\eta^*(t);$$

therefore it is sufficient to prove (7.2) for η^*. Introduce the functionals

$$\Phi(p) := \sum_{n\leq p} \frac{1}{4|X^*(n)|^2+1}, \quad \Psi(p) := \sum_{n\leq p} \frac{\Lambda_n}{4[X^*(n)]^2+1}\Lambda_n;$$

then $E^{\omega_2}(\Psi(p)) = \Phi(p)$.

We have

$$\Phi(T_{k+1}) - \Phi(T_k) \geq \frac{T_{k+1}-T_k}{2^{2(\xi_k+2)}+1} \tag{7.3}$$

THEOREM. $\mathrm{Prob}\{\Phi(T_k)-\Phi(T_s)<t\} \leq \exp\left(\dfrac{-3(k-s)^{3/2}}{12\sqrt{12t}}\right)$, $k-s>20(t+1)$.

PROOF. Denote by $\mathcal{S}$ the exit time of the random walk from the interval $I_k := (2^{\xi_k-1}, 2^{\xi_k+1})$ and for $0<\lambda<1$ being fixed, define on I_k the function

$$v(p) = E_p(\lambda^{\mathcal{S}});$$

then v takes the value 1 at the boundary of I_k; by the Bellman programming equation it satisfies

$$v(p) = \tfrac{1}{2}\lambda\big(v(p-1)+v(p+1)\big).$$

Define $\Delta f(n) := \frac{1}{2}\big(f(n+1)+f(n-1)\big) - f(n)$; then

$$\Delta v = (\lambda^{-1}-1)v.$$

Define $f_a(n) := a^n$; then $\frac{1}{2}\big(f_a(n+1)+f_a(n-1)\big) - f_a(n) = cf_a(n)$, $c = \frac{1}{2}(a+a^{-1})-1$. We satisfy these two equations by imposing the condition

$$a^2 - 2\lambda^{-1}a + 1 = 0 \qquad (7.4),$$

which has for roots η, η^{-1}, $\eta<1$. We deduce that

$$v(n) = \alpha\eta^n + \beta\eta^{-n},$$

where α, β are chosen such that the boundary conditions for v are satisfied; we deduce that

$$E(\lambda^{T_{k+1}-T_k}) = v(2^{\xi_k}) < \frac{1}{\cosh(2^{\xi_k-1}\,\log\eta)}$$

Writing this equality with $\lambda = 1-r^{-1}2^{-2\xi_k}$ we get

$$\mathrm{Prob}\{T_{k+1}-T_k \leq r2^{2\xi_k}\} \times (1-r^{-1}2^{-2\xi_k})^{r2^{2\xi_k}} \leq \frac{1}{\cosh(2^{\xi_k-1}\,\log\eta)}$$

where η is obtained from (7.4) and where $\lambda = 1 - r^{-1}2^{-2\xi_k}$, a relation which leads to the asymptotic formula

$$\eta \simeq 1 - \sqrt{2} \times r^{-1/2} 2^{-\xi_k}.$$

Further,

$$\mathrm{Prob}\{T_{k+1} - T_k \leq r 2^{2\xi_k}\} \leq 2e \, \exp\Big(-\frac{1}{\sqrt{2r}}\Big).$$

Finally we have, using (7.3),

$$\mathrm{Prob}(\Phi(T_{k+1}) - \Phi(T_k)) \leq r) \leq 2e \exp\left(- \frac{1}{3\sqrt{r}}\right).$$

Denote by ν the law of $(\Phi(T_{k+1}) - \Phi(T_k))$. Then

$$E \exp(-c(\Phi(T_{k+1}) - \Phi(T_k)) = \int_0^\infty \exp(-\lambda y) \nu(dy);$$

integration by parts yields for this expression the bound

$$\begin{aligned}\int_0^\infty \lambda \exp(-\lambda c) \, \nu([0,c]) \, dc &\leq 2e\lambda \int_0^\infty \exp\left(-\lambda c - \frac{1}{3\sqrt{c}}\right) dc \\ &\leq 2e \exp\big(-\tfrac{1}{3}[\lambda]^{1/3}\big).\end{aligned}$$

Since the $\Phi(T_{k+1}) - \Phi(T_k)$ are independent, we have

$$E(\exp(-\lambda(\Phi(T_k) - \Phi(T_s))) \leq \exp\big(-\tfrac{1}{4}(k-s)[\lambda]^{1/3}\big), \quad \lambda > 16,$$

and

$$\begin{aligned}\mathrm{Prob}\{\Phi(T_k) - \Phi(T_s) < t\} &\leq \inf\nolimits_\lambda \exp\big(\lambda t - 14(k-s)[\lambda]^{1/3}\big) \\ &\leq \exp\left(-\frac{3(k-s)^{3/2}}{12\sqrt{12t}}\right), \quad k - s > 20(t+1). \quad \square\end{aligned}$$

LEMMA.

$$\mathrm{Prob}_2\left\{\frac{\Psi(T_{k+1}) - \Psi(T_k)}{\Phi(T_{k+1}) - \Phi(T_k)} \leq \frac{1}{2}\right\} \leq \exp\left(-\frac{T_{k+1} - T_k}{64}\right) \leq \exp\left(-\frac{2^k}{128}\right). \quad (7.5)$$

PROOF. Let $\xi > 0$ and let $S := \Psi(T_{k+1}) - \Psi(T_k)$. Then

$$\mathrm{Prob}\{\, S \leq a \,\} \times \exp(-a\xi) \leq E(\exp(-\xi S))$$

or

$$\mathrm{Prob}\{\, S \leq a \,\} \leq \inf_{\xi > 0} \exp(a\xi) \times E(\exp(-\xi S)).$$

We have

$$S = \sum_{T_k < n \leq T_{k+1}} \frac{1}{4[X^*(n)]^2 + 1} \times \Lambda_n$$

By the independence of the Λ_n we have

$$E^{\omega_2}(\exp(-\xi S)) = \exp\Big(-\sum_{T_k<n\le T_{k+1}} \log\Big(1+\frac{\xi}{4[X^*(n)]^2+1}\Big)\Big).$$

Now we use the inequality

$$\log(1+u) \ge \tfrac{3}{4}u, \quad u \in \big[0, \tfrac{1}{4}\big],$$

obtaining

$$E^{\omega_2}(\exp(-\xi S)) \le \exp(-\xi \tfrac{3}{4}(\Phi(T_{k+1})-\Phi(T_k))) \quad \xi \in [0,\xi_0], \quad \xi_0 := 2^{2(k-1)}.$$

Taking

$$a = \tfrac{1}{2}(\Phi(T_{k+1})-\Phi(T_k)), \quad \xi = \xi_0,$$

we get

$$\mathrm{Prob}\{\, S \le a \,\} \le \exp\big(-14\xi_0(\Phi(T_{k+1})-\Phi(T_k))\big),$$

that is to say,

$$\tfrac{1}{4}\xi_0\big(\Phi(T_{k+1})-\Phi(T_k)\big) > 2^{2(k-2)}2^{-2(k+1)}(T_{k+1}-T_k),$$

which concludes the proof of the lemma. □

Now, starting from (7.5), Borel–Cantelli proves (7.2). □

8. Towards stochastic fluid motion on the configuration space

The configuration space in Arnold's point of view is G, the diffeomorphism group of the circle. The last section has given rise to a solution of a stochastic Burgers equation on the moment space $\mathcal{G}$; in this section we shall start to integrate this solution from the moment space to the configuration space.

Covariance functionals. Baxendale and Harris [1986] have characterized classical stochastic flows in terms of their covariance. The construction we propose will depend upon the integration of a *delayed* SDE, in contrast to Baxendale and Harris, who develop their study in the framework of classical infinite-dimensional SDE. Nevertheless covariance estimates will be needed.

THEOREM. *Assume that the noise energy ρ has a finite support. Let $\psi_x(t)$ be the stochastic parallel transport defined in* (5.2).

(a) *The covariance is*

$$\mathcal{C}_{x,t}(\theta,\theta') = \sum_k \big([\psi_x^*(t)(A_k)](\theta)\,[\psi_x^*(t)(A_k)](\theta') + [\psi_x^*(t)(B_k)](\theta)\,[\psi_x^*(t)(B_k)](\theta')\big)\rho(k).$$

(b) *Almost surely the map* $t \mapsto \mathcal{C}_{x,t}(*,*)$ *is a* $H^q(S^1 \times S^1)$ *continuous map.*

(c) $E(\mathcal{C}_{x,t}(\theta,\theta')) = \bar{\mathcal{C}}_t(\theta-\theta')$.

(d) $E\left(\sup_{\theta,\theta',\, t<T} \dfrac{\mathcal{C}_{x,t}(\theta,\theta)+\mathcal{C}_{x,t}(\theta',\theta')-2\mathcal{C}_{x,t}(\theta,\theta')}{(\theta-\theta')^2}\right) < \infty.$

PROOF. Part (c) results from the corollary on page 138, and part (b) follows from (7.2) and the continuity property of Brownian martingales. Let

$$p(\theta,\theta') := \mathcal{C}_{x,t}(\theta,\theta)+\mathcal{C}_{x,t}(\theta',\theta')-2\mathcal{C}_{x,t}(\theta,\theta'),$$

then $p(\theta,\theta)=0$. Since $\big[(\partial p/\partial\theta)(\theta,\theta')\big]_{\theta=\theta'}=0$, Taylor's formula gives

$$p(\theta,\theta') = (\theta-\theta')^2 \int_0^1 \frac{\partial^2 p}{\partial\theta^2}(\theta'+t(\theta-\theta'),\theta')(1-t)\,dt. \qquad \square$$

The system of Itô flow equations is not closed. Denote by $\mathcal{G}^s$ the space of vector fields with values in the Sobolev space of vector fields in H^s. Then $t \mapsto y_t$ is an $\mathcal{G}^s$-valued semimartingale. We have to solve a Stratonovitch SDE

$$d_t g_{x,t}(\theta) = (od\, y_t)(g_{x,t}(\theta))$$

(see [Cruzeiro et al. 2007]); there appears then the Itô contraction

$$Y_t(g_{x,t}(\theta) \times \mathcal{C}_{x,t}(\theta,\theta)\,dt,$$

where

$$Y_t = \frac{\partial g_{x,t}}{\partial\theta}.$$

In order to write the Itô SDE driving the flow we must know the derivative of the flow itself, an so on: we have an unclosed system of Itô SDE.

A usual procedure of existence for SDE relies on the Itô formalism. We could try the following alternative approach: solutions of Stratonovitch SDE are limits of solutions of corresponding ordinary differential equations. Then it may be possible to implement this limiting procedure in the geometric context of the stochastic development.

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