Análise Numérica II - Licenciatura em Matemática Aplicada

Formulário para o 1^o exame

I. Aproximação de funções

Interpolação polinomial

$$p_n(x) = \sum_{i=0}^{n} y_i l_i(x), \qquad l_i(x) = \prod_{j=0, j\neq i}^{n} \left(\frac{x - x_j}{x_i - x_j}\right)$$

$$p_n(x) = D_0^0 + \sum_{i=1}^{n} D_0^i (x - x_i), \qquad \begin{cases} D_j^0 = y_j, \quad j = 0, ..., n \\ D_j^k = \dfrac{D_{j+1}^{k-1} - D_j^{k-1}}{x_{j+k} - x_j}, \quad j = 0, ..., n-k, \quad k = 1, ..., n \end{cases}$$

$$p_{2n+1}(x) = \sum_{i=0}^{n} y_i H_i^0(x) + \sum_{i=0}^{n} y_i' H_i^1(x), \qquad H_i^0(x) = [1 - 2l_i'(x)(x - x_i)]l_i^2(x), \quad H_i^1(x) = (x - x_i)l_i^2(x)$$

$$f(x) - p_m(x) = \frac{f^{(m+1)}(\xi(x))}{(m+1)!} \prod_{j=0}^{n} (x - x_j)^{\alpha_i + 1}, \quad m = \sum_{i=0}^{n} (\alpha_i + 1) - 1$$

$$f[x_0, x_1, ..., x_n, x] = \frac{f^{(n)}(\eta(x))}{n!}$$

Interpolação trigonométrica

$$q_n(x) = \sum_{i=0}^{2n} y_i l_i(x), \qquad l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{\sin \frac{x - x_j}{2}}{\sin \frac{x_i - x_j}{2}}$$

Interpolação por splines

<u>Splines cúbicos</u> $\left(K_j := s'(x_j), \quad \tau_i = \dfrac{1}{x_{i+1} - x_i}\right)$

$$\tau_{j-1} K_{j-1} + 2(\tau_{j-1} + \tau_j) K_j + \tau_j K_{j+1} = 3\left[\tau_{j-1}^2 (y_j - y_{j-1}) + \tau_j^2 (y_{j+1} - y_j)\right], \quad j = 1, ..., n-1$$

- Condição da derivada nos extremos: $K_0 := s'(x_0) = a, \quad K_n := s'(x_n) = b$
- Condição livre nos extremos:

$$s''(x_0) = 0, s''(x_n) = 0 \quad \Leftrightarrow \quad K_0 + \frac{1}{2}K_1 - \frac{3}{2}\tau_0 (y_1 - y_0) = 0, K_n + \frac{1}{2}K_{n-1} - \frac{3}{2}\tau_{n-1} (y_n - y_{n-1}) = 0$$

$$\max_{x\in[a,b]} |f(x) - s(x)| \leq \frac{h^2 \max_{t\in[a,b]} |f^{(2)}|}{8}, \quad s \in \mathcal{S}_1([a,b], \Delta_n)$$

$$\max_{x\in[a,b]} |f(x) - s(x)| \leq \frac{5h^4 \max_{t\in[a,b]} |f^{(4)}|}{384}, \quad s \in \mathcal{S}_3([a,b], \Delta_n)$$

Melhor aproximação no sentido dos mínimos quadrados

$$f^* = \sum_{i=0}^{m} c_i \varphi_i, \qquad \sum_{i=0}^{m} c_i < \varphi_i, \varphi_j >=< f, \varphi_j >, \quad j = 0, ..., m,$$

$$< \varphi, \psi >= \int_a^b w(x)\varphi(x)\psi(x)dx, \qquad < \varphi, \psi >= \sum_{i=0}^{n} w_i \varphi(x_i)\psi(x_i)$$

Polinómios ortogonais (mónicos) em relação ao produto interno $<\varphi,\psi>=\int_a^b w(x)\varphi(x)\psi(x)dx$

$$\begin{cases} p_0(x)=1; \quad p_1(x)=x-<1,x>/<1,1>; \\ p_n(x)=(x-\lambda_n)p_{n-1}(x)-\mu_n p_{n-2}(x), \quad \lambda_n=<xp_{n-1},p_{n-1}>/\|p_{n-1}\|^2, \quad \mu_n=\|p_{n-1}\|^2/\|p_{n-2}\|^2 \end{cases}$$

- Polinómios de Chebychev: $[a,b]=[-1,1]; \quad w(x)=1/(1-x^2)$

$$<T_0,T_0>=\pi, \quad <T_n,T_n>=\pi/2, n\geq 1;$$
$$T_n(x)=\cos(n\arccos x), n=0,1,...; T_{n+1}(x_i)=0 \Leftrightarrow x_i=\cos\frac{(2i+1)\pi}{2(n+1)}, i=0,...,n;$$
$$\begin{cases} T_0(x)=1, \quad T_1(x)=1, \\ T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), n=1,2,... \end{cases}$$

- Polinómios de Legendre: $[a,b]=[-1,1]; \quad w(x)=1$

$$<T_n,T_n>=2/(n+1), n\geq 0;$$
$$\begin{cases} T_0(x)=1, \quad T_1(x)=1, \\ T_{n+1}(x)=\dfrac{1}{n+1}[(2n+1)xT_n(x)-nT_{n-1}(x)], n=1,2,... \end{cases}$$

II. Integração numérica

Fórmulas de Newton Côtes

$$\int_a^b f(x)dx-\sum_{j=0}^n A_{j,n}f(x_{j,n})=\begin{cases} h^{n+3}\dfrac{f^{(n+2)}(\xi)}{(n+2)!}\displaystyle\int_0^n s\prod_{i=0}^n(s-i)ds, & n \text{ par} \\ h^{n+2}\dfrac{f^{(n+1)}(\eta)}{(n+1)!}\displaystyle\int_0^n \prod_{i=0}^n(s-i)ds, & n \text{ ímpar} \end{cases}$$

Fórmulas de Gauss

$$\int_a^b w(x)f(x)dx-\sum_{j=0}^n A_{j,n}f(x_{j,n})=\frac{f^{(2n+2)}(\theta)}{(2n+2)!}\int_a^b w(x)\prod_{i=0}^n(x-x_i)dx$$

Regra dos Trapézios

$$T_n(f)=h\left[\frac{f(a)+f(b)}{2}+\sum_{i=1}^{n-1}f(x_i)\right], \qquad \int_a^b f(x)dx-T_n(f)=-\frac{h^2(b-a)}{12}f''(\xi)$$

Regra de Simpson

$$S_n(f)=\frac{h}{3}\left[f(a)+f(b)+4\sum_{i=1}^{\frac{n}{2}}f(x_{2i-1})+2\sum_{i=1}^{\frac{n}{2}-1}f(x_{2i})\right], \qquad \int_a^b f(x)dx-S_n(f)=-\frac{h^4(b-a)}{180}f^{(4)}(\eta)$$

III. Resolução numérica de EDO's: problemas com valor inicial

$$\begin{cases} u'(x) = f(x, u(x)), & x \in]a, b[, \\ u(a) = u_0 \end{cases}$$

- Métodos de Euler: explícito $u_{j+1} = u_j + hf(x_j, u_j)$; implícito $u_{j+1} = u_j + hf(x_{j+1}, u_{j+1})$
- Métodos de Taylor de ordem k: $u_{j+1} = u_j + hf(x_j, u_j) + ... + \frac{h^k}{k!} f^{(k-1)}(x_j, u_j)$
- Métodos de Runge-Kutta de ordem 2: $u_{j+1} = u_j + \left(1 - \frac{1}{2\alpha}\right) hf(x_j, u_j) + \frac{1}{2\alpha} hf(x_j + \alpha h, u_j + \alpha hf(x_j, u_j))$

  $\alpha = \frac{1}{2}$ - Método de Euler modificado; $\alpha = 1$ - Método de Heun
- Método do trapézio: $u_{j+1} = u_j + \frac{h}{2}[f(x_j, u_j) + f(x_{j+1}, u_{j+1})]$

Método com $p+1$ passos: $u_{j+1} = \sum_{k=0}^{p} a_k u_{j-k} + h\Phi(x_{j+1}, x_j, ..., x_{j-p}; u_{j+1}, u_j, ..., u_{j-p}; h)$

$$\tau(x, h) = \frac{1}{h}[u(x+h) - \sum_{k=0}^{p} a_k u(x - kh)] - \Phi(x+h, x, ..., x - ph; u(x+h), u(x), ..., u(x - ph); h)$$

$$r(\lambda) = \lambda^{p+1} - \sum_{j=0}^{p} a_j \lambda^{p-j}$$

$$|u(x_j) - u_j| \leq K_1 \rho(h) + K_2 \tau(h), \quad \rho(h) = \max_{0 \leq k \leq p} |u(x_k) - u_k|$$

Para $u_{j+1} = u_j + h\Phi(x_j, u_j; h)$, a última desigualdade fica $|u(x_j) - u_j| \leq \frac{e^{L(x_j - x_0)} - 1}{L} \tau(h)$

Métodos multipasso lineares: $u_{j+1} = \sum_{k=0}^{p} a_k u_{j-k} + h \sum_{k=-1}^{p} b_k f(x_{j-k}, u_{j-k})$

$$c_0 = 1 - \sum_{j=0}^{p} a_j; \quad c_m = 1 + (-1)^{m-1} \sum_{j=1}^{p} j^m a_j + (-1)^m \sum_{j=-1}^{p} j^{m-1} b_j$$

IV. Resolução numérica de EDO's: problemas com valores na fronteira

$$\begin{cases} u''(x) = p(x)u(x) + q(x)u'(x) + r(x), & x \in]a, b[\\ u(a) = \alpha, \quad u(b) = \beta \end{cases}$$

- método das diferenças finitas:

$$\begin{cases} \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = p(x_j)u_j + q(x_j)\frac{u_{j+1} - u_{j-1}}{2h} + r(x_j), & j = 1, ..., N-1 \\ u_0 = \alpha, \quad u_N = \beta \end{cases}$$

- método dos elementos finitos ($\alpha = 0, \beta = 0$): $\sum_{i=1}^{N-1} u_i \mathbf{a}(\varphi_i, \varphi_j) = \mathbf{l}(\varphi_j), \quad j = 1, ..., N-1,$

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \leq x \leq x_i, \\ \frac{x - x_{i+1}}{x_i - x_{i+1}}, & x_i \leq x \leq x_{i+1}, \\ 0, \quad x \leq x_{i-1}, & x \geq x_{i+1} \end{cases}$$