Cálculo Diferencial e Integral I

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10ª Aula Prática

Soluções e algumas resoluções abreviadas

1.

a) $\dfrac{2}{3}x^3 + \dfrac{3}{4}x^4$, b) $2\sqrt{x} - \log x - \dfrac{1}{x}$, $x > 0$,

c) $P\left(\dfrac{x^2 - x + 1}{\sqrt{x}}\right) = P\left(x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) = \dfrac{2}{5}x^{\frac{5}{2}} - \dfrac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} =$
$\dfrac{2}{5}\sqrt{x^5} - \dfrac{2}{3}\sqrt{x^3} + 2\sqrt{x}$, d) $-\dfrac{3}{4}\sqrt[3]{(1-x)^4}$,

e) $P\left(\dfrac{\sqrt[3]{x^2} + \sqrt{x^3}}{x}\right) = P\left(x^{-\frac{1}{3}} + x^{\frac{1}{2}}\right) = \dfrac{3}{2}\sqrt[3]{x^2} + \dfrac{2}{3}\sqrt{x^3}$,

f) $\dfrac{5}{6}\sqrt[5]{(x^2-1)^6}$, g) $\dfrac{1}{3}\log(3 + x^4)$, h) $\dfrac{1}{2}\log(1 + 2e^x)$, i) $\log(1 + \operatorname{sen} x)$,

j) $-\dfrac{1}{2}\cos(2x)$, k) $P\left(\dfrac{\sin(2x)}{1 + \operatorname{sen}^2 x}\right) = P\left(\dfrac{2\sin x\cos x}{1 + \operatorname{sen}^2 x}\right) = \log(1 + \operatorname{sen}^2 x)$,

l) $P\left(\cos^2 x\right) = P\left(\dfrac{\cos(2x) + 1}{2}\right) = \dfrac{1}{4}\operatorname{sen}(2x) + \dfrac{x}{2}$,

m) $\operatorname{tg} x$, n) $e^{\operatorname{tg} x}$, o) $\dfrac{1}{2}\sin(x^2 + 2)$, p) $-\cos(e^x)$,

q) $\dfrac{3}{4}\sqrt[3]{(1 + x^3)^4}$, r) $-\dfrac{1}{1 + e^x}$, s) $-\operatorname{arctg}(\cos x)$,

t) $P\left(\dfrac{1}{\sqrt{1 - 4x^2}}\right) = P\left(\dfrac{1}{\sqrt{1 - (2x)^2}}\right) = \dfrac{1}{2}\arcsin(2x)$,

u) $P\left(\dfrac{x + 1}{\sqrt{1 - x^2}}\right) = P\left(\dfrac{x}{\sqrt{1 - x^2}}\right) + P\left(\dfrac{1}{\sqrt{1 - x^2}}\right) = -\sqrt{1 - x^2} + \arcsin x$,

v) $P\left(\dfrac{x^3}{(1 + x^4)^2}\right) = -\dfrac{1}{4(1 + x^4)}$, w) $P\left(\cos^3 x\sqrt{\operatorname{sen} x}\right) =$

$= P\left(\cos x(1 - \operatorname{sen}^2 x)\sqrt{\operatorname{sen} x}\right) = P\left(\cos x(\sqrt{\operatorname{sen} x} - \operatorname{sen}^{\frac{5}{2}} x)\right) = \dfrac{2}{3}\operatorname{sen}^{\frac{3}{2}} - \dfrac{2}{7}\operatorname{sen}^{\frac{7}{2}}$,

x) $P(\operatorname{tg}^2 x) = P(\sec^2 - 1) = \operatorname{tg} x - x$.

2.

a) $\dfrac{1}{7}x^7 + \dfrac{3}{5}x^5 + x^3 + x;$ b) $e^{x+3};$ c) $\dfrac{1}{\log 2}2^{x-1};$

d) $P\left(\dfrac{1}{\sqrt[5]{1-2x}}\right) = -\dfrac{1}{2}P\left(-2(1-2x)^{-\frac{1}{5}}\right) = -\dfrac{5}{8}(1-2x)^{\frac{4}{5}};$

e) $P\left(\dfrac{x}{1+x^2}\right) = \dfrac{1}{2}P\left(\dfrac{2x}{1+x^2}\right) = \dfrac{1}{2}\log(1+x^2) = \log\sqrt{1+x^2};$

f) $P\left(\dfrac{x^3}{x^8+1}\right) = \dfrac{1}{4}P\left(\dfrac{4x^3}{(x^4)^2+1}\right) = \dfrac{1}{4}\operatorname{arctg}(x^4);$

g) $P(\operatorname{cotg} x) = P\left(\dfrac{\cos x}{\sin x}\right) = \log|\sin x|;$

h) $P\left(3^{\sin^2 x}\sin 2x\right) = P\left(3^{\sin^2 x}2\sin x\cos x\right) = P\left(3^{\sin^2 x}(\sin^2 x)'\right) = \dfrac{1}{\log 3}3^{\sin^2 x};$

i) $P\left(\dfrac{\operatorname{tg}\sqrt{x}}{\sqrt{x}}\right) = 2P\left(\dfrac{1}{2\sqrt{x}}\operatorname{tg}\sqrt{x}\right) = 2P\left((\sqrt{x})'\operatorname{tg}\sqrt{x}\right) = -2\log|\cos x|;$

j) $\arcsin e^x;$ k) $\dfrac{1}{2(1-\alpha}\dfrac{1}{(1+x^2)^{\alpha-1}}$, se $\alpha \neq 1, \log\sqrt{1+x^2}$, se $\alpha = 1;$

l) $P(\cos x\cos 2x) = P\left(\cos x(1-2\operatorname{sen}^2 x)\right) = P\left(\cos x - 2\cos x\operatorname{sen}^2 x\right) =$
$= \sin x - \dfrac{2}{3}\sin^3 x;$

m) $P\left(\operatorname{sen}^3 x\cos^4 x\right) = P\left(\operatorname{sen} x(1-\cos^2 x)\cos^4 x\right) = P\left(\operatorname{sen} x(\cos^4 x - \cos^6 x)\right) =$
$= -\dfrac{1}{5}\cos^5 x + \dfrac{1}{7}\cos^7 x;$

n) $P(\operatorname{tg}^3 x + \operatorname{tg}^4 x) = P((\sec^2 x - 1)\operatorname{tg} x) + P((\sec^2 x - 1)\operatorname{tg}^2 x) =$
$P(\sec^2 x\operatorname{tg} x) - P(\operatorname{tg} x) + P(\sec^2 x\operatorname{tg}^2) - P(\operatorname{tg}^2 x) =$
$\dfrac{1}{2}\operatorname{tg}^2 x + \log|\cos x| + \dfrac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x.$

3.

a) $\sqrt{2x^3}$, b) $-3\cos x + \dfrac{2}{3}x^3$, c) $\dfrac{1}{3}\log\left|1+x^3\right|$,

d) $-\dfrac{1}{2}e^{-x^2}$, e) $\dfrac{3}{1+\cos x}$, f) $\dfrac{1}{3}\left(1+x^2\right)^{3/2}$,

g) $\dfrac{1}{2}e^{2\operatorname{sen} x}$, h) $-\log(1+e^{-x})$, i) $-\log|\cos x|$,

j) $\dfrac{1}{\sqrt{2}}\arctan\dfrac{x}{\sqrt{2}}$, k) $\dfrac{1}{3}\sec^3 x$, l) $\dfrac{1}{4}\operatorname{sen}^4 x - \dfrac{1}{6}\operatorname{sen}^6 x$,

m) $\log|\arctan x|$, n) $\dfrac{1}{2}\arctan\left(x^2\right)$, o) $2\arctan(\sqrt{x})$,

p) $\dfrac{\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right)$, q) $\dfrac{1}{2}\arctan\left(\dfrac{1}{2}e^x\right)$, r) $\dfrac{2}{3}\sqrt{(\arcsin x)^3}$,

s) $\dfrac{1}{2\sqrt{2}}\arcsin\left(\sqrt{2}x^2\right)$, t) $\log\sqrt[3]{\left|\dfrac{x-2}{x+1}\right|}$, u) $-\dfrac{1}{x+1}$,

v) $\sin(\log x)$, w) $\log(\log x)$, x) $\operatorname{tg} x + \dfrac{1}{3}\operatorname{tg}^3 x$.

4. a) Calculamos primeiro uma primitiva de $\frac{1}{4+9x^2}$:

$$P\left(\frac{1}{4+9x^2}\right) = \frac{1}{4}P\left(\frac{1}{1+\left(\frac{3}{2}x\right)^2}\right) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x.$$

Temos então, para $x \in \mathbb{R}$, $f(x) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x + c$, com $c \in \mathbb{R}$. Para determinar c temos $f(0) = c = 1$, logo $f(x) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x + 1$.

b) $P(\left(\frac{1}{x-1}\right) = \log|x-1|$, para $x \neq 1$. Temos então

$$g(x) = \begin{cases} \log(x-1) + c_1, & \text{se } x > 1 \\ \log(1-x) + c_2, & \text{se } x < 1. \end{cases}$$

com $c-1, c_2 \in \mathbb{R}$. Para determinar as constantes, temos $g(0) = \log 1 + c_2 = 0$, logo $c_2 = 0$, e $g(2) = \log 1 + c_1 = 3$, logo $c_2 = 3$.

c) O domínio da secante é $\mathbb{R}\setminus\{\frac{\pi}{2}+k\pi \ : \ k \in \mathbb{Z}\}$. Neste conjunto temos $P(\sec^2 x) = \operatorname{tg} x$, e portanto para $x \in \left]\frac{\pi}{2}+(k-1)\pi, \frac{\pi}{2}+k\pi\right[$, para cada $k \in \mathbb{Z}$, temos $h(x) = \operatorname{tg} x + c_k$. Como $k\pi \in \left]\frac{\pi}{2}+(k-1)\pi, \frac{\pi}{2}+k\pi\right[$, temos que $0 + c_k = k$, ou seja, $c_k = k$.

5.
- $P(x\sin(x^2)) = \frac{1}{2}\cos(x^2)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \frac{1}{2}\cos(x^2) + C$, com $C \in \mathbb{R}$.
  a) $F(0) = 0 \Leftrightarrow \frac{1}{2} + C = 0$, logo $C = -\frac{1}{2}$.
  b) $\lim_{x\to+\infty} F(x)$ não existe, para qualquer $C \in \mathbb{R}$, logo não existe uma primitiva nas condições dadas.
- $P(\frac{e^x}{2+e^x}) = \log(2+e^x)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \log(2+e^x) + C$, com $C \in \mathbb{R}$.

a) $F(0)=0 \Leftrightarrow \log 3 + C = 0$, logo $C = -\log 3$.

b) $\lim_{x\to+\infty} F(x) = +\infty$, para qualquer $C \in \mathbb{R}$, logo não existe uma primitiva nas condições dadas. .

- $P(\frac{1}{(1+x^2)(1+\operatorname{arctg}^2 x)}) = \operatorname{arctg}(\operatorname{arctg} x)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \operatorname{arctg}(\operatorname{arctg} x) + C$, com $C \in \mathbb{R}$.

  a) $F(0) = 0 \Leftrightarrow C = 0$.

  b) $\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \operatorname{arctg}(\operatorname{arctg} x) + C = \operatorname{arctg} \frac{\pi}{2} + C$, logo $C = -\operatorname{arctg} \frac{\pi}{2}$.

6.

a) $P\left(\dfrac{1}{1-x}\right) = -\log|1-x|$, b) $P\left(\dfrac{1}{(x-3)^3}\right) = -\dfrac{1}{2(x-3)^2}$,

c) $P\left(\dfrac{x+1}{x^2+1}\right) = \dfrac{1}{2}\log(x^2+1) + \operatorname{arctg} x$,

d) $P\left(\dfrac{x}{1+(x-1)^2}\right) = \dfrac{1}{2}\log(1+(x-1)^2) + \operatorname{arctg}(x-1)$,

e) $P\left(\dfrac{2x+1}{x^2+4}\right) = \log(x^2+4) + \dfrac{1}{2}\operatorname{arctg}\left(\dfrac{x}{2}\right)$, f) $P\left(\dfrac{1}{x^2+2x+2}\right) = \operatorname{arctg}(x+1)$.

7. a) $P\left(\frac{1}{x^2+x}\right) = P\left(\frac{1}{x(x+1)}\right)$. Usando a decomposição em fracções simples $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$ temos

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax + A + Bx}{x(x+1)} = \frac{(A+B)x + A}{x(x+1)}$$

logo $A + B = 0$ e $B = 1$, ou seja, $A = 1$ e $B = -1$. Temos então

$$P\left(\frac{1}{x^2+x}\right) = P\left(-\frac{1}{x} + \frac{1}{x+1}\right) = -\log|x| + \log|x+1| = \log\left|\frac{x+1}{x}\right|.$$

b) Usando a decomposição em fracções simples $\frac{x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} +$

$\frac{C}{(x-1)^2}$, temos

$$\begin{aligned}\frac{x+1}{x(x-1)^2} &= \frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\\ &= \frac{A(x-1)^2+Bx(x-1)+Cx}{x(x-1)^2}\\ &= \frac{Ax^2-2Ax+A+Bx^2-Bx+Cx}{x(x-1)^2}\\ &= \frac{(A+B)x^2+(-2A-B+C)x+A}{x(x-1)^2}\end{aligned}$$

logo $A+B=0$, $-2A-B+C=1$, $A=1$, ou seja, $A=1$, $B=-1$, $C=2$. Temos então

$$P\left(\frac{x+1}{x(x-1)^2}\right)=P\left(\frac{1}{x}-\frac{1}{x-1}+\frac{4}{(x-1)^2}\right)=\log|x|-\log|x-1|-\frac{4}{x-1}.$$

c) Usando a decomposição em fracções simples $\frac{x^2+x-4}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$, temos

$$\begin{aligned}\frac{x^2+x-4}{x(x^2+4)} &= \frac{A}{x}+\frac{Bx+C}{x^2+4}\\ &= \frac{Ax^2+4A+Bx^2+Cx}{x(x^2+4)}\\ &= \frac{(A+B)x^2+4A+Cx}{x(x^2+4)}\end{aligned}$$

logo $A+B=1$, $C=1$ e $4A=-4$, ou seja, $A=-1$, $B=2$, $C=1$. Temos então

$$P\left(\frac{x^2+x-4}{x(x^2+4)}\right)=P\left(-\frac{1}{x}+\frac{2x+1}{x^2+4}\right)=-\log|x|+\log(x^2+4)+\frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right).$$

d) $2\log|x-1|-\log|x|+\frac{1}{x}$, e) $\frac{x^4}{4}+\frac{x^2}{2}+\frac{1}{2}\log|x^2-1|$,

f) $\log\left|\frac{x+2}{x+1}\right|-\frac{2}{x+2}$, g) $\frac{x^2}{2}+\log|x+1|+\frac{1}{x+1}$,

h) $x+\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\operatorname{arctg}x$, i) $\frac{1}{2}\log(x^2+4)+\arctan\left(\frac{x}{2}\right)+\frac{1}{2}\log\left|\frac{x-2}{x+2}\right|$.

8. a) O domínio de $\frac{1}{x^2+x}$ é $\mathbb{R}\setminus\{-1,0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} -\log x + \log(x+1) + C_1, & \text{se } x > 0, \\ -\log(-x) + \log(x+1) + C_2, & \text{se } -1 < x < 0, \\ -\log(-x) + \log(-x-1) + C_3, & \text{se } x < -1, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

b) O domínio de $\frac{x+1}{x(x-1)^2}$ é $\mathbb{R}\setminus\{0,1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \log x - \log(x-1) - \frac{4}{x-1} + C_1, & \text{se } x > 1, \\ \log x - \log(-x+1) - \frac{4}{x-1} + C_2, & \text{se } 0 < x < 1, \\ \log(-x) - \log(-x+1) - \frac{4}{x-1} + C_3, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

c) O domínio de $\frac{x^2+x-4}{x(x^2+4)}$ é $\mathbb{R}\setminus\{0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} -\log x + \log(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_1, & \text{se } x > 0, \\ -\log(-x) + \log(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_2, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias.

d) O domínio de $\frac{x^2+1}{x^2(x-1)}$ é $\mathbb{R}\setminus\{0,1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} 2\log(x-1) - \log x + 1/x + C_1, & \text{se } x > 1, \\ 2\log(1-x) - \log x + 1/x + C_2, & \text{se } 0 < x < 1, \\ 2\log(1-x) - \log(-x) + 1/x + C_3, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

9. a) $\dfrac{1}{2}e^{x^2+2x} + C$, com $C \in \mathbb{R}$.

b) $P\left(\dfrac{x+3}{x^4-x^2}\right) = P\left(\dfrac{x+3}{x^2(x-1)(x+1)}\right)$, para $x \in \mathbb{R}\setminus\{-1,0,1\}$. Escrevendo

$$\frac{x+3}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}$$

tem-se $A=-1$, $B=-3$, $C=2$, $D=-1$ (verifique). Logo,

$$P\left(\frac{x+3}{x^4-x^2}\right) = -\log|x| + \frac{3}{x} + 2\log|x-1| - \log|x+1| = \frac{3}{x} + \log\frac{(x-1)^2}{|x(x+1)|}.$$

A forma geral da primitiva em $]1,+\infty[$ é $G(x) = \frac{3}{x} + \log\frac{(x-1)^2}{x(x+1)} + K$, com $K \in \mathbb{R}$. Tem-se

$$\lim_{x\to+\infty} G(x) = \lim_{x\to+\infty} \frac{3}{x} + \log\frac{(x-1)^2}{x(x+1)} + K = \log(1) + K = K,$$

logo $\lim_{x\to+\infty} G(x) = 3 \Leftrightarrow K = 3$.

10. $P\left(\frac{1}{(x-1)^2}\right) = -\frac{1}{x-1}$, para $x \in \mathbb{R}\setminus\{1\}$. A forma geral das primitivas é:

$$\begin{cases} -\frac{1}{x-1} + C_1, & \text{se } x > 1, \\ -\frac{1}{x-1} + C_2, & \text{se } x < 1, \end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias. Como $F(2)=0$, temos $-1+C_1 = 0 \Leftrightarrow C_1 = 1$. Como $\lim_{x\to+\infty} -\frac{1}{x-1} = 0$, de $\lim_{x\to+\infty} F(x) = 10$ tem-se $C_2 = 10$.

11. Sendo $P\left(\frac{1}{1+x}\right) = \log(x+1)$, para todo o $x \in]-1,+\infty[$, temos

$$\psi'(x) = \log(x+1) + C_1.$$

A condição $\psi'(0) = 1$, resulta em $C_1 = 1$. Usando primitivação por partes (verifique!) temos

$$P\left(\log(x+1)+1\right) = (x+1)\log(x+1),$$

ou seja $\psi(x) = (x+1)\log(x+1) + C_2$. Dado que $\psi(0)=1$, obtém-se o resultado

$$\psi(x) = (x+1)\log(x+1) + 1.$$