Cálculo Diferencial e Integral I

LEA, LEM, LEAN, MEAer, MEMec

2º Semestre de 2006/2007

11ª Aula Prática

Soluções e algumas resoluções abreviadas

1.

a) $P(xe^x) = xe^x - P(e^x) = (x-1)e^x$,

b) $$P(x \operatorname{arctg} x) = \frac{x^2}{2}\operatorname{arctg} x - P\left(\frac{x^2}{2}\,\frac{1}{1+x^2}\right)$$
$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}P\left(1 - \frac{1}{1+x^2}\right) = \frac{1}{2}\left(-x + (x^2+1)\operatorname{arctg} x\right),$$

c) $P(\arcsin x) = x \arcsin x - P\left(x\,\frac{1}{\sqrt{1-x^2}}\right) = x\arcsin x + \sqrt{1-x^2}$,

d) $P(x \sin x) = -x\cos x + P(\cos x) = -x\cos x + \sin x$,

e) $P(x^3 e^{x^2}) = P(x^2 \cdot x e^{x^2}) = x^2 \frac{e^{x^2}}{2} - P\left(2x\frac{e^{x^2}}{2}\right) = (x^2-1)\frac{e^{x^2}}{2}$,

f) $P(\log^3 x) = x\log^3 x - P(3\log^2 x) = x(\log^3 x - 3\log^2 x) + P(6\log x) =$
$x(\log^3 x - 3\log^2 x + 6\log x) - P(6) = x(\log^3 x - 3\log^2 x + 6\log x - 6)$,

g) $P(x^n \log x) = \frac{1}{n+1}x^{n+1}\log x - P\left(\frac{1}{n+1}x^{n+1}\,\frac{1}{x}\right) = \frac{1}{n+1}x^{n+1}\log x - \frac{1}{(n+1)^2}x^{n+1}$,

h) $P\left(\frac{x^7}{(1-x^4)^2}\right) = P\left(x^4\,\frac{x^3}{(1-x^4)^2}\right) = x^4\,\frac{1}{4(1-x^4)} - P\left(4x^3\,\frac{1}{4(1-x^4)}\right) =$
$\frac{x^4}{4(1-x^4)} + \frac{1}{4}\log(1-x^4)$.

2.

a) $e^x(e^x + x - 1) - e^{2x}/2$,

b) $e^x(\operatorname{sen} x - \cos x)/2$,

c) $-e^{-x^2}(x^2+1)/2$,

d) $x\arctan x - \frac{1}{2}\log(1+x^2)$,

e) $\frac{2}{3}x^{\frac{3}{2}}\left(\log x - \frac{2}{3}\right)$

f) $\frac{1}{4}(1+x^2)^2\arctan x - x/4 - x^3/12$,

g) $\frac{2}{3}x^3\sqrt{1+x^3} - \frac{4}{9}(1+x^3)^{2/3}$,

h) $x\log(1/x+1) + \log|x+1|$,

i) $\frac{x^3}{3}\log^2 x - \frac{2}{9}x^2\log x + \frac{2}{27}x^3$,

j) $x\log^2 x - 2x\log x + 2x$,

k) $-\frac{1}{x}\operatorname{sen}\frac{1}{x} - \cos\frac{1}{x}$,

l) $\frac{1}{2}\operatorname{sen}(2x)\log(\operatorname{tg} x) - x$,

m) $-(1-x^2)^{3/2}\arcsin x + x - x^3/3$,

n) $-\frac{\log x}{1+x} + \log\left|\frac{x}{1+x}\right|$,

o) $\frac{1}{2}(\operatorname{sh} x\cos x+\operatorname{ch} x\sin x)$,   p) $\frac{1}{1+\log^2 3}3^x(\sin x+\log 3\cos x)$,

q) $\frac{x}{2}(\cos(\log x)+\sin(\log x))$,   r) $-\frac{1}{2}\frac{x}{1+x^2}+\frac{1}{2}\operatorname{arctg} x$.

3. c) $P\left(\frac{1}{(1+x^2)^2}\right)=\frac{x}{2(1+x^2)}+\frac{1}{2}\operatorname{arctg} x$.

$P\left(\frac{1}{(1+x^2)^3}\right)=\frac{x}{4(1+x^2)^2}+\frac{3x}{8(1+x^2)}+\frac{3}{8}\operatorname{arctg} x$.

4.

a) $\frac{1}{2}e^{2x}-\frac{1}{2}\log\left(e^{2x}+1\right)$,   b) $\frac{3}{2}\arctan\sqrt[3]{x^2}$,   c) $2\sqrt{x-1}-2\arctan\sqrt{x-1}$,

d) $\frac{6}{7}x\sqrt[6]{x}-\frac{6}{5}\sqrt[6]{x^5}-\frac{3}{2}\sqrt[3]{x^2}+2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}-3\log|1+\sqrt[3]{x}|+6\arctan\sqrt[6]{x}$,

e) $\frac{1}{4}\log\left|\frac{e^x-1}{e^x+1}\right|-\frac{1}{2(1+e^x)}$,   f) $-2\arctan\sqrt{1-x}$,

g) $\log|\cos x|+\log|\operatorname{tg} x+1|$,   h) $\log|\log x-1|-\frac{1}{\log x-1}$,

i) $3\log(\sqrt[3]{x}+1)$,

5. a) Fazendo a substituição $\sqrt{x}=t\Leftrightarrow x=t^2$, com $x>0$, $x\neq 16$, e $t>0$, $t\neq 4$, temos

$$P\left(\frac{1+\sqrt{x}}{x(4-\sqrt{x})}\right)=P\left(\frac{1+t}{t^2(4-t)}\,2t\right)=2P\left(\frac{1+t}{t(4-t)}\right).$$

Usando a decomposição em fracções simples:

$$\frac{2+2t}{t(4-t)}=\frac{A}{t}+\frac{B}{4-t}$$

temos $A=\frac{1}{2}$, $B=\frac{5}{2}$, logo

$$2P\left(\frac{1+t}{t(4-t)}\right)=\frac{1}{2}P\left(\frac{1}{t}+\frac{5}{4-t}\right)=\frac{1}{2}\log\left|\frac{t}{(4-t)^5}\right|$$

e assim,

$$P\left(\frac{1+\sqrt{x}}{x(4-\sqrt{x})}\right)=\frac{1}{2}\log\left|\frac{\sqrt{x}}{(4-\sqrt{x})^5}\right|.$$

b) Fazendo a substituição $\sqrt[4]{1+x}=t\Leftrightarrow x=t^4-1$, com $x>-1$ e $t>0$, temos

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right)=P\left(\frac{1}{(t^4-1)t}\,4t^3\right)=P\left(\frac{4t^2}{t^4-1}\right).$$

Usando a decomposição em fracções simples:

$$\frac{4t^2}{t^4-1} = \frac{4t^2}{(t-1)(t+1)(t^2+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1},$$

temos $A = 1$, $B = -1$, $C = 0$, $D = 2$. Logo,

$$P\left(\frac{4t^2}{t^4-1}\right) = P\left(\frac{1}{t-1} - \frac{1}{t+1} + \frac{2}{t^2+1}\right) = \log\left|\frac{t-1}{t+1}\right| + 2\operatorname{arctg} t$$

e assim,

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = \log\left|\frac{\sqrt[4]{1+x}-1}{\sqrt[4]{1+x}+1}\right| + 2\operatorname{arctg}\sqrt[4]{1+x}.$$

c) Fazendo a substituição $e^{2x} = t \Leftrightarrow x = \frac{1}{2}\log t$, com $x \in \mathbb{R}$ e $t > 0$, temos

$$P\left(\frac{1}{1+e^{2x}}\right) = P\left(\frac{1}{1+t}\cdot\frac{1}{2t}\right).$$

Usando a decomposição em fracções simples:

$$\frac{1}{(1+t)2t} = \frac{A}{1+t} + \frac{B}{t}$$

temos $A = -\frac{1}{2}$, $B = \frac{1}{2}$, logo

$$P\left(\frac{1}{1+t}\cdot\frac{1}{2t}\right) = P\left(-\frac{1}{2(1+t)} + \frac{1}{2t}\right) = \frac{1}{2}\log\left|\frac{t}{1+t}\right|$$

e assim,

$$P\left(\frac{1}{1+e^{2x}}\right) = \frac{1}{2}\log\left|\frac{e^{2x}}{1+e^{2x}}\right|.$$

d) Fazendo a substituição $e^x = t \Leftrightarrow x = \log t$, com $x \in \mathbb{R}\setminus\{0\}$ e $t > 0$, $t \neq 1$, temos

$$P\left(\frac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right) = P\left(\frac{t^3}{(1+t^2)(t-1)^2}\frac{1}{t}\right) = P\left(\frac{t^2}{(1+t^2)(t-1)^2}\right).$$

Usando a decomposição em fracções simples:

$$\frac{t^2}{(1+t^2)(t-1)^2} = \frac{At+B}{1+t^2} + \frac{C}{t-1} + \frac{D}{(t-1)^2}$$

temos $A = -\frac{1}{2}$, $B = 0$, $C = D = \frac{1}{2}$, logo

$$\begin{aligned} P\left(\frac{t^2}{(1+t^2)(t-1)^2}\right) &= \frac{1}{2}P\left(-\frac{t}{1+t^2} + \frac{1}{t-1} + \frac{1}{(t-1)^2}\right) \\ &= -\frac{1}{4}\log(1+t^2) + \frac{1}{2}\log|t-1| - \frac{1}{2}\frac{1}{t-1} \end{aligned}$$

e assim

$$P\left(\frac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right) = -\frac{1}{4}\log(1+e^{2x}) + \frac{1}{2}\log|e^x-1| - \frac{1}{2}\frac{1}{e^x-1}.$$

e) Fazendo a substituição $\log x = t \Leftrightarrow x = e^t$, com $x \in \mathbb{R}^+ \setminus \{1, e\}$ e $t \in \mathbb{R} \setminus \{0, 1\}$, temos

$$P\left(\frac{2\log x - 1}{x\log x(\log x - 1)^2}\right) = P\left(\frac{2t-1}{e^t t(t-1)^2}\, e^t\right) = P\left(\frac{2t-1}{t(t-1)^2}\right).$$

Usando a decomposição em fracções simples:

$$\frac{2t-1}{t(t-1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2}$$

temos $A = -1$, $B = C = 1$, logo

$$P\left(\frac{2t-1}{t(t-1)^2}\right) = P\left(-\frac{1}{t} + \frac{1}{t-1} + \frac{1}{(t-1)^2}\right) = \log\left|\frac{t-1}{t}\right| - \frac{1}{t-1}$$

e assim

$$P\left(\frac{2\log x - 1}{x\log x(\log x - 1)^2}\right) = \log\left|\frac{\log x - 1}{\log x}\right| - \frac{1}{\log x - 1}.$$

f) Fazendo a substituição $\operatorname{sen} x = t \Leftrightarrow x = \operatorname{arcsen} t$, obtem-se (verifique)

$$P\left(\frac{1}{\operatorname{sen}^2 x \cos x}\right) = -\frac{1}{\operatorname{sen} x} + \frac{1}{2}\log\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right|.$$

6.

a) $\frac{1}{2}\log\left|\frac{1+\operatorname{sen}x}{1-\operatorname{sen}x}\right|$, b) $\sqrt{1-\frac{1}{x^2}}$, c) $\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\arcsin x$,

d) $\log\left|1+\operatorname{tg}\frac{x}{2}\right|$, e) $-\frac{1}{3}\left(\frac{1}{x^2}-1\right)^{3/2}$, f) $-2\arcsin\sqrt{1-e^x}$,

g) $-x+\operatorname{tg}x+\sec x$, h) $2\arcsin\sqrt{x}$, i) $\log|\frac{1+2\sin x}{1-\sin x}|$,

j) $\frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right|+\frac{1}{4(1-\sin x)}-\frac{1}{4(1+\sin x)}=\frac{1}{2}\log\left|\frac{1+\sin x}{\cos x}\right|+\frac{\sin x}{2\cos^2 x}$
$=\frac{1}{2}\log|\sec x+\operatorname{tg}x|+\frac{1}{2}\sec x\operatorname{tg}x$, k) $\log|x+\sqrt{x^2+1}|$,

l) $\log\left|\frac{\operatorname{sen}x}{1+\operatorname{sen}x}\right|$, m) $\log\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|$, n) $\log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|$,

o) $\frac{1}{2}\log\left|\sqrt{\left(1+\frac{x}{2}\right)^2}+\frac{x}{2}\right|+\frac{x}{4}\sqrt{\left(1+\frac{x}{2}\right)^2}$,

p) $\frac{\sqrt{x^2-1}}{2}(x-2)+\frac{1}{2}\log\left|x+\sqrt{x^2+1}\right|$.

7. a) $f(x)=\frac{1}{2}\operatorname{arctg}^2 x+c$, com $c\in\mathbb{R}$; $\lim_{x\to+\infty}f(x)=\frac{\pi^2}{8}+c$, logo $c=-\frac{\pi^2}{8}$.

   b) $g(x)=\frac{1}{2}\log|\frac{\sqrt{x}}{(4-\sqrt{x})^5}|+c$, para $x>16$ (Ex. 4.a)); $\lim_{x\to+\infty}g(x)=+\infty$, logo não existe g nas condições do enunciado.

8. (ver Ex. 4.c).)

9. a) $\frac{1}{2}x|x|$,

   b) $\frac{x^2}{2}\arcsin\frac{1}{x}+\frac{1}{2}x\sqrt{1-\frac{1}{x^2}}$, (por partes, por ex.)

   c) $\frac{x}{2}\operatorname{sen}(\log x+1)-\frac{x}{2}\cos(\log x+1)$, (por partes, por ex.)

   d) $\frac{x}{8}-\frac{1}{32}\operatorname{sen}4x$,

   e) $\frac{2}{3}x^{3/2}\arctan\sqrt{x}-\frac{1}{3}x+\frac{1}{3}\log(1+x)$, (por partes, por ex.)

   f) $-\log x+2\log|1+\log x|+\frac{\log^2 x}{2}$, (substituição $t=\log x$, por ex.)

   g) $\frac{x}{2}-\frac{1}{2}e^{-x}-\frac{1}{4}\log(e^{2x}-2e^x+2)$, (substituição $t=e^x$, por ex.)

   h) $\frac{2}{3}\sqrt{x^3}-x+4\sqrt{x}-4\log(\sqrt{x}+1)$, (substituição $t=\sqrt{x}$, por ex.)

   i) $\operatorname{sen}x-\frac{1}{3}\operatorname{sen}^3 x$,

j) $\frac{3}{8}x + \frac{1}{4}\operatorname{sen} 2x + \frac{1}{8}\operatorname{sen} 4x$,

k) $\frac{1}{2}(x^2-1)\log\left|\frac{1-x}{1+x}\right| - x$,

l) $\frac{1}{2}\log\left|\frac{(x-1)(x+3)}{(x+2)^2}\right|$,

m) $\frac{1}{2}\log^2(\log x)$,

n) $x\log(x+\sqrt{x}) - x + \sqrt{x} - \log(1+\sqrt{x})$, (substituição $t=\sqrt{x}$ e por partes, por ex.)

o) $-\left(\frac{1}{x}+1\right)e^{\frac{1}{x}}$, (por partes, por ex.)

p) $\sin x\log(1+\sin^2 x) - 2\sin x + 2\operatorname{arctg}(\sin x)$,

q) $\log x\log(\log x) - \log x$,

r) $\frac{x^2+1}{2}\operatorname{arctg}^2 x - x\operatorname{arctg} x + \frac{1}{2}\log(1+x^2)$,

s) $2\sqrt{1+x}(\log(1+x)-2)$,

t) $\log\left|\frac{\sin x}{\cos x+1}\right|$,

u) $-\frac{x}{\operatorname{sen} x} + \log\left|\frac{\sin x}{\cos x+1}\right|$,

v) $-\frac{\sqrt{3}}{3}\arctan\left(\sqrt{3}\cos x\right)$,

w) $-\frac{1}{2}\log^2(\cos x)$,

x) $\log\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right|$ (substituição $t=\sqrt{x+2}$, por ex.),

y) $x(\operatorname{arcsen} x)^2 + 2\sqrt{1-x^2}\operatorname{arcsen} x - 2x$ (por partes, por ex.),

z) $\frac{1}{4}\log\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right| + \frac{1}{2(1-\operatorname{sen} x)}$ (substituição $t=\operatorname{sen} x$, por ex.).

10. $\log\left(1+e^{-x}\right) + \frac{\pi}{2}$.