Maximal subsheaves of torsionfree sheaves

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VBAC conference, Porto, 3 July 2003

Let Y be a reduced irreducible projective curve of arithmetic genus g with at most ordinary nodes as singularities. For a torsionfree sheaf F on Y, let $r(F), d(F), \mu(F)$ denote respectively the rank, degree and slope of F. Consider a torsionfree sheaf E of rank r, degree d, slope μ. For a subsheaf F of E let $s(E, F) = r(F)d - rd(F)$. For a fixed $k, 1 \leq k \leq r - 1$, define $s_k(E) = \min\ s(E, F)$, where the minimum is taken over all subsheaves F of E of rank k. Note that $s_k(E) \equiv kd \bmod r$. The stability of E implies that $s_k(E) > 0$. There is an upper bound $s_k(E) \leq k(r - k)g$. This is due to Mukai and Sakai in the nonsingular case [MS] and Ballico [Ba] in the nodal case. It was generalized to principal G-bundles on nonsingular curves by Holla and Narasimhan [HN]. Recently Biswas and myself have generalized it to parabolic principal G-bundles on nonsingular curves (using a different approach).

The function $E \mapsto s_k(E)$ is an integral valued function on the moduli space $U(r, d)$ of stable torsionfree sheaves of rank r, degree d. It gives a stratification of $U(r, d)$ by locally closed subsets

$$U_{k,s}(r, d) = \{E \in U(r, d) \mid s_k(E) = s\}.$$

In case Y is nonsingular, this stratification has been studied by many authors (Lange, Hirschowitz, Maruyama, Narasimhan, Oxbury, Russo, Teixidor and others). We extend some of these results to nodal curves.

Note that vector bundles form an open dense subset in $U(r, d)$. Its compliment is a subset of codimension 1. General stable bundles, i.e. the general elements of $U(r, d)$, are crucial in the study of the strata $U_{k,s}(r, d)$. We have the following results for general stable bundles.

Theorem 1 *The tensor product of two general stable vector bundles on an irreducible nodal curve Y is nonspecial. More precisely, if F_1, F_2 are two general stable vector bundles with slopes μ_1, μ_2, then*

$$\begin{aligned} H^0(F_1 \otimes F_2) &= 0 \text{ if } \mu_1 + \mu_2 \leq g-1 \\ H^1(F_1 \otimes F_2) &= 0 \text{ if } \mu_1 + \mu_2 \geq g-1 \end{aligned}$$

This was first proved by Hirschowitz in the nonsingular case [H]. There is another proof by Russo and Teixidor [RT] (in the nonsingular case), which fails in the singular case. Our proof is a generalization of the original proof by Hirschowitz (with many modifications).

Theorem 2. (1) *If a general stable bundle E of rank r and degree d on Y is an extension of a vector bundle E'' of rank r'' and slope μ'' by a vector bundle of rank r' and slope μ', then $\mu'' - \mu' \geq g-1$.*
(2) If E', E'' are general stable bundles of slopes μ', μ'', ranks r', r'' with $r' + r'' = r$ and $\mu'' - \mu' \geq g-1$, then a general extension E of E'' by E' is a (general) stable bundle.
(In the nonsingular case, Part (1) was proved by Lange [L], Part (2) by Hirschowitz [H]).

By Theorem 2 , the maximum value of $s_{r'}$ (as E varies over $U(r,d)$) is given by $s_{max} = r'(r-r')(g-1)+\epsilon$, where ϵ is the unique integer $0 \leq \epsilon \leq r-1$ with $s_{max} \equiv r'd \bmod r$.

Theorem 3 *Fix positive integers r', s with $0 < s \leq s_{max}, s \equiv r'd \bmod r$ and $r'' = r - r'$.*

(1) $U_{r',s}(r,d)$ is nonempty.

(2) dim $U_{r',s} = r^2(g-1) + 1 + s - r'r''(g-1)$ for $s \leq r'r''(g-1)$.

Definition A subsheaf $F \subset E$ is called a *maximum subsheaf of rank k* if $s(E,F) = s_k(E)$.

Proposition 1 *(1) A general element in $U_{r',s}(r,d)$ is a vector bundle.*
(2) For $s \leq r'(r-r')(g-1)$, this vector bundle has only finitely many maximum subsheaves E' of rank r'. Such a maximum subsheaf E' and the corresponding quotient E'' are general stable vector bundles (resp. in $U(r',d')$

and $U(r'', d''), d'' = d - d'$*). Further if* $s < r'r''(g-1)$*, then this maximum subsheaf of rank* r' *is unique.*
(3) For a fixed E*, let*
$A_{r',s}(E) = \{E' \mid E' \subset E$ *saturated subsheaf of rank* r'*, degree* d'*,* $s = r'd - rd'\}$.
For $s \geq r'(r-r')(g-1)$*, we show that dim* $A_{r',s}(E) = s - r'(r-r')(g-1)$ *for a general* E*.*

The proofs of these results need some results on torsionfree sheaves on nodal curves which are of independent interest too. For torsionfree sheaves E, F, let $Hom(E, F)$ denote the (torsionfree) sheaf of homomorphisms and let Hom $(E, F) = H^0(Y, Hom(E, F))$. $\mathrm{Ext}^i(E, F)$ will denote Ext groups. The stalk E_{y_j} of a torsionfree sheaf E at a node y_j is isomorphic to $a_j \mathcal{O}_{y_j} \oplus b_j m_{y_j}, a_j + b_j = r(E)$, where $\mathcal{O}_{y_j}$ and m_{y_j} are respectively the local ring and the maximal ideal at y_j. We will say that E is of *local type* $b, b = (b_1, \cdots, b_m), m$ being the number of nodes.

Lemma 2 *Let* B, C *be torsionfree sheaves of local types* b, c*. Then one has the following.*
(1) $d(Hom(B, C)) = r(B)d(C) - r(C)d(B) + \sum_j b_j c_j$
(2) dim $\mathrm{Ext}^1(B, C) =$ dim $H^1(Y, Hom(B, C)) + 2\sum_j b_j c_j$.

Lemma 3 *For* $a = (a_1, \cdots, a_m), 0 \leq a_i \leq r$*, let* $W_a(r, d) \subset U_Y(r, d)$ *be the closed subscheme* $W_a(r, d) = \{F \in U_Y(r, d) \mid$ *For all* $i, F_{y_i} \approx (r-j)\mathcal{O}_{y_i} \oplus j m_{y_i}, j \geq a_i\}$ *(i.e. local type of* F *at* y_i *is bigger than* a_i*). Then*
$codim_{U_Y} W_a(r, d) \geq \Sigma_i (a_i)^2$.
The proof of this Lemma uses generalized parabolic vector bundles.

Sketch of the proof of Theorem 2 .
For fixed E', E'', such extensions are parametrized by the groups $\mathrm{Ext}^1(E'', E')$. As E', E'' vary (over corresponding moduli spaces) dimension of all such extensions $\delta = p(E') + p(E'')+$ dim $\mathbb{P}(\mathrm{Ext}^1(E'', E'))$. If $\delta <$dim $U(r, d)$, then a general E does not come from such extensions. In the nonsingular case, $\mathrm{Ext}^1(E'', E') = H^1(Hom(E'', E'))$, this can be computed by Riemman-Roch theorem (H^0 is zero as E stable). Using $p(E') = r'^2(g-1) + 1$ etc., get $\delta \leq$ dim $U(r, d) - 1$. In the nodal case, E', E'' could be nonlocally free, then Ext^1 shoots up by Lemma 2(2). However, E', E'' vary over sets of smaller dimension (by Lemma 3) compensating this rise (giving $\delta \leq$ dim $U(r, d) - 1 - \Sigma_j (b_j^2 + c_j^2 - b_j c_j)$) and we are through.

This gives an idea how proofs differ from the nonsingular case. The proof of Theorem 3 is similar to that of [RT]. We start with defining certain subsets of $U(r,d)$. Let $W_{r',s}(r,d)$ denote the irreducible subset of $U(r,d)$ consisting of stable torsionfree sheaves E which are extensions of a stable vector bundle $E^{''}$ by a stable vector bundle E' of rank r', degree d'.
Let $V_{r',s}(r,d)$ be the subset of $U(r,d)$ consisting of stable torsionfree sheaves which are extensions of vector bundles $E^{''}$ by vector bundles E' of rank r', degree $d'(E',E^{''}$ not necessarily stable). One has $W_{r',s}(r',d) \subset V_{r',s}(r,d)$ and both these sets consist of vector bundles.
Let $Z_{r',s}(r,d) \subset U_{r',s}(r,d)$ be consisting of vector bundles which have a maximum subsheaf E' of rank r' locally free. In fact $Z_{r',s} = V_{r',s} \cap U_{r',s}$.
For a fixed E, let $A_{r',d'}(E) = \{E' \mid E' \subset E$ saturated subsheaf of rank r', degree $d'\}$.

Proposition 4 *For $0 < s_0 \leq r, Z_{r',s_0}$ is nonempty.*
Proof sketch : Let $E', E^{''}$ be general stable vector bundles of ranks $r', r^{''}$ and degrees $d', d^{''}$. One shows that a vector bundle E which is a general extension of $E^{''}$ by E' is stable. The proof is similar to that of [RT]

Lemma 5 *Assume that $W_{r',s}$ is nonempty and let E be a general element in it.*
1) If $s \leq r'r^{''}(g-1)$, dim $A_{r',d'}(E) = 0$.
dim $W_{r',s} = r^2(g-1) + 1 + s - r'r^{''}(g-1)$.
2) If $s \geq r'r^{''}(g-1)$ then dim $A_{r',d'}(E) = s - r'r^{''}(g-1)$,
dim $W_{r',s} = r^2(g-1) + 1$.

Proof By the construction of $W_{r',s}(r,d)$, there is a morphism p from an irreducible variety P onto $W_{r',s}$ with dim $P =$ dim $U(r',d')+$ dim $U(r^{''},d^{''})+ h^1(E^{''*} \otimes E') - 1$. The fibre of p at E is $A_{r',d'}(E)$ which can be identified to the quot scheme of E corresponding to torsionfree quotients $E^{''}$ of rank $r^{''}$, degree $d^{''}$ of E. The tangent space to the quot scheme at $E^{''}$ is $\mathrm{Hom}(E', E^{''})$. Its dimension can be estimated by Theorem 1 for $E', E^{''}$ general bundles.

Lemma 6 *If $V_{r',s}$ is nonempty, then it is contained in the closure of $W_{r',s}$. Thus a general element of $V_{r',s}$ is in $W_{r',s}$, dim $V_{r',s} =$ dim $W_{r',s}$ and $V_{r',s}$ is irreducible.*

Proof: One shows that any $E_0 \in V_{r',s}$ can be deformed to an element of $W_{r',s}$. Suppose that E_0 is an extension of E'' by E'. There exist irreducible families T', T'' of vector bundles containing E', E'' respectively with generic members stable vector bundles. Let T be the family of extensions of bundles in T'' by bundles in T' and T^s its subset giving stable bundles, $E_0 \in T^s$. The general member of T^s is in $W_{r',s}$.

Lemma 7 *dim* $(U_{r',s} - Z_{r',s}) <$ *dim* $V_{r',s}$.

Proof Let $E \in U_{r',s} - Z_{r',s}$ and let E' be a maximal subsheaf rank r' with $s(E, E') = s, E''$ the corresponding quotient. Let E', E'' be respectively of local type b, c; one has at least one of b_j, c_j nonzero. Since E is stable, $\mathrm{Hom}(E'', E') = 0$, so that dim $\mathrm{Ext}^1(E'', E') = d''r' - r''(g-1) + \sum_j b_j c_j$, by Lemma 2. By Lemma 3, E' and E'' vary over schemes of dimensions $r'^2(g-1) + 1 - \sum b_j^2, r''^2(g-1) + 1 - \sum c_j^2$ respectively. Hence if p is the number of parameters determining $E \in U_{r',s} - Z_{r',s}$ then $p \leq r^2(g-1) + 1 + s - r'r''(g-1) - \sum_j b_j^2 - \sum c_j^2 + \sum_j b_j c_j = \dim V_{r',s} - \sum_j [(b_j - c_j)^2 + b_j c_j] <$ dim $V_{r',s}$.

Proof of Theorem 3 We shall show that $Z_{r',s} \neq \phi$. By Proposition 4, Z_{r',s_0} is nonempty for $0 < s_0 \leq r$. Starting from a vector bundle in Z_{r',s_0} and using elementary transformations repeatedly, we get $V_{r',s} \neq \phi$ for all s. Clearly, $V_{r',s} \subset \coprod_{s' \leq s} U_{r',s'}$ and hence

$$(*) \qquad V_{r',s} = \coprod_{s' \leq s} (U_{r',s'} \cap V_{r',s})$$

Now, dim $U_{r',s'} - Z_{r',s'} <$ dim $V_{r',s'} \forall s'$ (Lemma 7). Since dim $Z_{r',s'} \leq$ dim $V_{r',s'}$, have dim $U_{r',s'} \leq$ dim $V_{r',s'}, \forall s'$.
now, dim $U_{r',s'} \cap V_{r',s} \leq$ dim $U_{r',s'} \leq$ dim $V_{r',s'}$. By Lemmas 5 and 6, dim $V_{r',s'} <$ dim $V_{r',s}$ for $s' < s$. It follows that dim $\coprod_{s' < s} (U_{r',s'} \cap V_{r',s}) <$ dim $V_{r',s}$.
Then $Z_{r',s} = U_{r',s} \cap V_{r',s}$ is nonempty and a general element of $V_{r',s}$ belongs to $Z_{r',s}$. Hence dim $Z_{r',s} =$ dim $V_{r',s}$. Since $Z_{r',s} \subset U_{r',s}$, dim $U_{r',s} \geq$ dim $V_{r',s}$. Thus, dim $U_{r',s} =$ dim $V_{r',s}$. Part (2) now follows from Lemmas 5 and 6. This completes the proof of the theorem.

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