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Paired operators in asymmetric space setting

Contents

1 Introduction

Paired operators in symmetric setting are well-known from the theory of singular operators since the 1960s, see BoeSil06,GF71,GK73,Pro74. In the standard setting a ***paired operator*** has the form

$$T_1 \;=\; P\,A + Q\,B \quad \text{or} \quad T_2 \;=\; A\,P + B\,Q \tag{1.1}$$

where $A, B \in \mathcal{L}(X)$ are bounded linear operators in a Banach space and P, Q are complementary projectors, i.e., $P, Q \in \mathcal{L}(X)$, $P^2 = I$, $Q = I - P$.

The operators T_j are said to be of ***normal type***, if A and B are isomorphisms, in brief $A, B \in \mathcal{GL}(X)$. In this case they are obviously **equivalent** to $PAB^{-1} + Q$ and to $P + QBA^{-1}$ in the first case and to $B^{-1}AP + Q$ and $P + A^{-1}BQ$ in the second, but not to each other in general.

As usual two operators S, T (in topological vector spaces) are said to be **equivalent** if there exist isomorphisms E, F such that $T = E\,S\,F$. In this case it often suffices to consider only one type, say $T = PA + Q$ and to refer to analogy in several other cases.

How it appears

In connection with **singular integral operators** of Cauchy type, **Riemann boundary value problems** and **factorization theory** the following well known formula occurs (see GF71, Section 5.1 and Pro74, Section 2.2.1, for instance):

$$\begin{aligned} PAP + Q &= (I - PAQ)(PA + Q) = \\ &= (I - PAQ)(I + QA^{-1}P)(P + QA^{-1}Q)A\,. \end{aligned} \tag{1.2}$$

In particular it shows the equivalence between $PAP + Q$, $PA + Q$ and $P + QA^{-1}Q$, and allows many further conclusions. The formula is known from various different sceneries in matrix, ring and operator theory and has received several names such as **Schur complement identity, Kozak formula, Sherman-Morrison-Woodbury formula** and has even roots in the works of **Sylvester**, see for instance BoeSil06,Car86,Ceb67,S85,Syl51.

Brief outlook

In the present article we consider an **extension** of formula (1.2) (and related formulas) to an **asymmetric space setting** where $A \in \mathcal{GL}(X, Y)$ is an isomorphism acting between different Banach spaces. It produces **heavy formulas**, but a very clear and useful strategy for many purposes exposed subsequently:

- A more direct proof of the **Cross Factorization Theorem**, connected with further results for general Wiener-Hopf operators;
- a more direct proof of the **Bart-Tsekanovskii Theorem** including an extension of that result with the equivalence to the existence and construction of a special form of the equivalence after extension relation;
- **applications** to the solution of **boundary and transmission problems** that appear in problems of diffraction of time-harmonic waves from plane screens in $\mathbb{R}^3$.

Basic idea

Hence the key idea of this paper is **to give a proper meaning** to terms such as $P_2A + Q_1$ where $A \in \mathcal{L}(X,Y)$, $P_1 = P_1^2 = I - Q_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ which does not make sense immediately when the two operators P_2A and Q_1 do not map into the same space. Thinking of

$$U \;=\; \begin{pmatrix} P_2A & 0 \\ 0 & Q_1 \end{pmatrix} \;:\; X \times X \to Y \times X$$

we have a proper definition but find the difficulty that T can hardly be interpreted as an invertible operator. So we switch to

$$T \;=\; \begin{pmatrix} P_2A|_{P_1X} & P_2A|_{Q_1X} \\ 0 & I|_{Q_1X} \end{pmatrix} \;:\; P_1X \times Q_1X \to P_2Y \times Q_1X \tag{1.3}$$

as generalization of the paired operator $PA + Q$. Now it **acts in an asymmetric space setting**, with a restricted image space in contrast to the previous operator U.

2 A generalization of the basic formulas

Formula (1.2) ”can be written” in **matrix form** as

$$\begin{pmatrix} PAP & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P & -PAQ \\ 0 & Q \end{pmatrix} \begin{pmatrix} PAP & PAQ \\ 0 & Q \end{pmatrix} =$$

$$= \begin{pmatrix} P & -PAQ \\ 0 & Q \end{pmatrix} \begin{pmatrix} P & 0 \\ QA^{-1}P & Q \end{pmatrix} \circ \tag{2.1}$$

$$\circ \begin{pmatrix} P & 0 \\ 0 & QA^{-1}Q \end{pmatrix} \begin{pmatrix} PAP & PAQ \\ QAP & QAQ \end{pmatrix}.$$

For comparison we recall (1.2):

$$\begin{aligned} PAP + Q &= (I - PAQ)(PA + Q) = \\ &= (I - PAQ)(I + QA^{-1}P)(P + QA^{-1}Q)A\,. \end{aligned}$$

Precise identification

More precisely, (1.2) and (2.1) are **equivalent** via the **decomposition operator**

$$\iota \ : \ X \ = \ PX \oplus QX \ \ \to \ \ PX \times QX \quad , \quad x \mapsto \begin{pmatrix} Px \\ Qx \end{pmatrix} \tag{2.2}$$

where $PX \oplus QX$ denotes the **direct sum** and $PX \times QX$ denotes the **topological product space**, which are isomorphic (as normed spaces). Namely we have

$$PAP + Q \ = \ \iota^{-1} \begin{pmatrix} PAP & 0 \\ 0 & Q \end{pmatrix} \iota \ = \ (P\,,\,Q) \begin{pmatrix} PAP & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} . \tag{2.3}$$

Hence (2.1) can be understood simply as an operator acting in $PX \times QX$ whilst (2.3) is acting in $X = PX \oplus QX$, strictly speaking. Their identification is useful to avoid very heavy formulas (***Convention 1***).

Note that so far all these formulas hold for elements of a unital ring $\mathcal{R}$, as well, provided A is invertible and P, Q are idempotent with $P + Q = I$.

Now the first operator PAP in (2.1) (as acting in PX and not in X) is often denoted as ***general Wiener-Hopf operator*** (WHO) and written in the form DevShi69,Shi64

$$W \;=\; T_P(A) \;=\; PA|_{PX} \;:\; PX \to PX\,. \tag{2.4}$$

It is also referred to as abstract WHO, projection of A, truncation of A or compression of A, to mention a few notations for the same thing BoeSil06,Ceb67,GK73,Pel03,Pro74,S85. Other terms in (2.1) are identified similarly, e.g., PAQ as $PA|_{QX}$ etc.

Note that (2.4) stands for an **operator restricted not only in the domain but in the image space**, as well (***Convention 2***). These notational conventions are used for brevity and tradition, in order to avoid rather heavy formulas.

Asymmetric WHOs

A general Wiener-Hopf operator* (WHO) *in asymmetric space setting has the form S85

$$W \;=\; P_2A|_{P_1X} \;:\; P_1X \to P_2Y \tag{2.5}$$

provided $A \in \mathcal{L}(X,Y)$, $P_1 = P_1^2 \in \mathcal{L}(X)$ and $P_2 = P_2^2 \in \mathcal{L}(Y)$.

The two conventions are applied by analogy: W is briefly written as P_2AP_1 (in certain operator matrices) and considered as a mapping into P_2Y, not as a mapping into Y.

Two operator relations

Two bounded linear operators S, T acting in Banach spaces are said to be ***matrically coupled***, if there exist operator matrices such that

$$\begin{pmatrix} T & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & S \end{pmatrix}^{-1} \tag{2.6}$$

where the stars stand for suitable bounded linear operators (in the corresponding Banach spaces). Secondly, S and T are ***equivalent after extension***, if there exist operators E and F such that

$$\begin{pmatrix} T & 0 \\ 0 & I_1 \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_2 \end{pmatrix} F \tag{2.7}$$

where I_1, I_2 are identity operators and E, F are isomorphisms acting between suitable Banach spaces.

The first relation is called a ***matrical coupling relation*** (MCR), the second an ***equivalence after extension relation*** (EAER).

They were introduced in BGK84 and gained great attention in system theory and other applications, cf. BGK84,BGKR00 and CS98.

Paired operators acting between different Banach spaces

We are here only interested in a generalization of the operators

$$PA + Q\ ,\ \ AP + Q\ ,\ \ P + QA\ ,\ \ P + AQ\,. \tag{2.8}$$

Definition 2.1 *Consider a* ***basic space setting***

$$\begin{aligned} &X, Y \ \text{ are Banach spaces}\,, \\ &P_1 \in \mathcal{L}(X)\,, P_2 \in \mathcal{L}(Y) \ \text{ are projectors}\,, \\ &Q_1 = I - P_1\,, Q_2 = I - P_2\,, \end{aligned} \tag{2.9}$$

and let $A \in \mathcal{L}(X, Y)$*.*

Then each of the following operators is referred to as a **paired operator in an asymmetric space setting** *(PAO):*

$$\begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} : P_1X \times Q_1X \to P_2Y \times Q_1X$$

$$\begin{pmatrix} P_2AP_1 & 0 \\ Q_2AP_1 & Q_2 \end{pmatrix} : P_1X \times Q_2Y \to P_2Y \times Q_2Y$$

$$\begin{pmatrix} P_1 & 0 \\ Q_2AP_1 & Q_2AQ_1 \end{pmatrix} : P_1X \times Q_1X \to P_1X \times Q_2Y$$

$$\begin{pmatrix} P_2 & P_2AQ_1 \\ 0 & Q_2AQ_1 \end{pmatrix} : P_2Y \times Q_1X \to P_2Y \times Q_2Y$$

where we apply Convention 1 and Convention 2, as well.

Remark 2.2 ***Any upper or lower triangular operator matrix with an identity in the main diagonal can be seen as a PAO*** *can be seen as a PAO and vice versa. Hence this formulation can serve as an alternative definition. However we preferred here to see the analogy with classical PAOs in the notation more clearly.*

A first result

Lemma 2.3 *Let* $X = X_1 \oplus X_2\,,\ Y = Y_1 \oplus Y_2$ *be Banach spaces,* P_1, Q_1, P_2, Q_2 *the corresponding projectors onto* $X_1, X_2\,,\ Y_1, Y_2$*, resp., and let* $A \in \mathcal{GL}(X, Y)$*.*

Then the two ***associated asymmetric WHOs***

$$W = P_2 A|_{X_1} \,:\, X_1 \to Y_1 \tag{2.10}$$

$$W_* = Q_1 A^{-1}|_{Y_2} \,:\, Y_2 \to X_2$$

are ***equivalent after extension****. An EAER between the two operators is explicitly given by the following formula:*

$$\begin{pmatrix} W & 0 \\ 0 & I|_{Q_1X} \end{pmatrix} = \begin{pmatrix} -P_2A|_{Q_1X} & I|_{P_2Y} - P_2AQ_1A^{-1}|_{P_2Y} \\ I|_{Q_1X} & Q_1A^{-1}|_{P_2Y} \end{pmatrix} \tag{2.11}$$

$$\begin{pmatrix} W_* & 0 \\ 0 & I|_{P_2Y} \end{pmatrix} \begin{pmatrix} Q_2A|_{P_1X} & Q_2A|_{Q_1X} \\ P_2A|_{P_1X} & P_2A|_{Q_1X} \end{pmatrix}$$

$$: \quad Q_1X \times P_2Y \;\leftarrow\; Q_1X \times P_2Y \;\leftarrow\; P_2Y \times Q_2Y \;\leftarrow\; P_1X \times Q_1X\,.$$

Proof. Generalizing the symmetric analogue (2.1) we write

$$\begin{pmatrix} P_2AP_1 & 0 \\ 0 & Q_1 \end{pmatrix} = \begin{pmatrix} P_2 & -P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} = \tag{2.12}$$

$$= \begin{pmatrix} P_2 & -P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} P_2 & 0 \\ Q_1A^{-1}P_2 & Q_1 \end{pmatrix} \begin{pmatrix} P_2 & 0 \\ 0 & Q_1A^{-1}Q_2 \end{pmatrix} \begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ Q_2AP_1 & Q_2AQ_1 \end{pmatrix}.$$

A straightforward verification shows that the identity is correct and the matrix factorization in the last line maps the spaces like

$$\begin{matrix} Y_1 & & Y_1 & & Y_1 & & Y_1 & & X_1 \\ \times & \leftarrow & \times & \leftarrow & \times & \leftarrow & \times & \leftarrow & \times \\ X_2 & & X_2 & & X_2 & & Y_2 & & X_2 \end{matrix} \;. \tag{2.13}$$

By Convention 1 we have a factorization of the extended WHO into a composition of bounded operators where the first two factors and the last one are obviously invertible and the remainder factor is an extension of the associated WHO W_*. Formula (2.12) can be regarded as an **EAER between $T = W$ and $S = W_*$** (which are matrically coupled).

Remarks 2.4 *Some common properties of the two* ***associated WHOs*** W ***and*** W_* *in an asymmetric setting were already discovered by the author in 1983 such as the fact that they are* ***one-sided invertible only simultaneously****, see Proposition 2 in S83.*

But the EAER (2.12) implies much more, namely that the two operators have ***isomorphic kernels and cokernels****, see Proposition 1 in BT91. This yields that they (as every two operators fulfilling an EAER) belong to the* ***same regularity class*** *of normally solvable linear operators in Banach spaces, such as the class of invertible, Fredholm, or generalized invertible operators etc., cf. a classification in CS98, S85, which is just recalled here as a convenient vehicle for clear and efficient formulations later on.*

Classification of bounded linear operators in Banach spaces

Abbreviating $\alpha(T) = \dim\ker T, \beta(T) = \dim\ \mathrm{coker}\, T$, we distinguish the following **regularity classes of operators**:

	$\alpha(T) = 0$	$\alpha(T) < \infty$	$\ker T$ complem.	$\ker T$ closed
$\beta(T) = 0$	bdd. invertible	right inv. Fredholm	right invertible	surjective
$\beta(T) < \infty$	left inv. Fredholm	Fredholm	right regulariz.	semi-Fred. $\mathcal{F}_-$
$\mathrm{im}\, T$ complem.	left invertible	left regulariz.	generalized invertible	no name
$\mathrm{im}\, T$ closed	injective	semi-Fred. $\mathcal{F}_+$	no name	normally solvable

Moreover the knowledge of an **explicit (generalized) inverse, (Fredholm) pseudoinverse or regularizer** of W implies a formula for a corresponding one of W_* and vice versa CS98. That is evident from an EAER, but not from a MCR, see BGKR00, Corollary 5.13, for comparison.

3 An alternative proof of the Cross Factorization Theorem

Assume a **basic setting** where X, Y are Banach spaces, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ projectors, $Q_1 = I_X - P_1$, $Q_2 = I_Y - P_2$ and let $A \in \mathcal{GL}(X, Y)$. Then (an operator triple A_-, C, A_+ with)

$$\begin{aligned} A \quad &= \quad A_- \quad C \quad A_+ \\ &: \quad Y \leftarrow Y \leftarrow X \leftarrow X \,. \end{aligned} \tag{3.1}$$

is referred to as a ***cross factorization*** of A (with respect to X, Y, P_1, P_2) S85, in brief CFn, if the factors $A_\pm$ and C possess the following properties:

$$\begin{aligned} A_+ \;\in\; \mathcal{GL}(X) \quad &, \quad A_- \;\in\; \mathcal{GL}(Y)\ , \\ A_+ P_1 X = P_1 X \quad &, \quad A_- Q_2 Y = Q_2 Y\ , \end{aligned} \tag{3.2}$$

and $C \in \mathcal{L}(X, Y)$ **splits the spaces** X, Y both **into four subspaces** such that

$$
\begin{array}{ccccccc}
X & = & \overbrace{X_1 \oplus X_0}^{P_1 X} & \oplus & \overbrace{X_2 \oplus X_3}^{Q_1 X} & & \\
 & & \downarrow \qquad\qquad & C \swarrow\!\!\!\!\searrow & \qquad\qquad \downarrow & & \\
Y & = & \underbrace{Y_1 \oplus Y_2}_{P_2 Y} & \oplus & \underbrace{Y_0 \oplus Y_3}_{Q_2 Y} . & &
\end{array}
\tag{3.3}
$$

$A_\pm$ are called ***strong WH factors*** (or ***plus/minus factors***), C is said to be a ***cross factor***, since it maps a part of P_1X onto a part of Q_2Y ($X_0 \to Y_0$) and a part of Q_1X onto a part of P_2Y ($X_2 \to Y_2$), which are all complemented subspaces, namely the images of corresponding projectors $p_0, p_1, ..., q_3$, given by

$$
\begin{array}{lllll}
X_1 = p_1 X & = C^{-1} P_2 C P_1 X & , & X_0 = p_0 X & = C^{-1} Q_2 C P_1 X , \\
X_2 = p_2 X & = C^{-1} P_2 C Q_1 X & , & X_3 = p_3 X & = C^{-1} Q_2 C Q_1 X , \\
Y_1 = q_1 Y & = C P_1 C^{-1} P_2 Y & , & Y_2 = q_2 Y & = C Q_1 C^{-1} P_2 Y , \\
Y_0 = q_0 Y & = C P_1 C^{-1} Q_2 Y & , & Y_3 = q_3 Y & = C Q_1 C^{-1} Q_2 Y .
\end{array}
\tag{3.4}
$$

An auxiliary result: recognition of a cross factor

Lemma 3.1 *Let $C \in \mathcal{GL}(X,Y)$ be of the form*

$$\begin{aligned} C &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \qquad (3.5) \\ &= \begin{pmatrix} P_2CP_1 & P_2CQ_1 \\ Q_2CP_1 & Q_2CQ_1 \end{pmatrix} \;:\; P_1X \times Q_1X \to P_2Y \times Q_2Y \end{aligned}$$

(applying Convention 1) and let $W = P_2C|_{P_1X}$ be generalized invertible. Then ***C is a cross factor*** *with respect to X, Y, P_1, P_2* ***if and only if the operators*** *$p_0, p_1, ..., q_3$ mentioned in (3.4)* ***are idempotent****.*

The proof is an interpretation of the mapping diagram (3.3)

The following theorem is mainly known from S85, and here extended by another (the last) statement. The proof of the necessity part of the first statement is new and more compact in comparison with S85, pp. 117–119, at last as a consequence of the PAO notion.

The Cross Factorization Theorem

Theorem 3.2 *Let X, Y, P_1, P_2 be a basic space setting and A be an isomorphism. Then $W = P_2 A|_{P_1 X}$ is* ***generalized invertible*** *if and only if A admits a* ***cross factorization****. In this case, a formula for a* ***reflexive generalized inverse of W is explicitly given by the reverse order law***

$$W^- = A_+^{-1} P_1 C^{-1} P_2 A_-^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X \,. \tag{3.6}$$

Conversely, if V is a reflexive generalized inverse of W, then (applying our conventions) a ***cross factorization is explicitly given by the formulas***

$$\begin{aligned}
A &= A_- \; C \; A_+ \quad : \quad P_1 X \times Q_1 X \to P_2 Y \times Q_2 Y, \qquad (3.7)\\
A_- &= \begin{pmatrix} P_2 & 0 \\ Q_2 A P_1 V & Q_2 \end{pmatrix},\\
C &= \begin{pmatrix} W & P_2(A - AVP_2A)Q_1 \\ Q_2(A - AVP_2A)P_1 & S \end{pmatrix},\\
S &= Q_2(A - AVP_2A + A(P_1 - VWP_1)A^{-1}(P_2 - WVP_2)A)Q_1,\\
A_+ &= \begin{pmatrix} P_1 & (VP_2A - (P_1 - VWP_1)A^{-1}(P_2 - WVP_2)A)Q_1 \\ 0 & Q_1 \end{pmatrix}.
\end{aligned}$$

Proof. If A admits a cross factorization $A = A_- C A_+$, the operator defined by (3.6) is shown to satisfy $WW^-W = W$ and $W^-WW^- = W^-$ **by verification**. For details see S85, p. 29.

The inverse conclusion is only known in the symmetric case, see S85, p. 117–118. Here we present a **similar new poof** in the asymmetric case, with the help of Lemma 2.3 and Lemma 3.1. Now concretely, if $WVW = W$ and $VWV = V$ hold, it is not hard to prove that the first line of (3.7) is an identity. Moreover $A_\pm$ are easily recognized as strong plus/minus factors (see (3.2)).

It remains to prove that C as defined in (3.7) is a cross factor. To this end we first define the projectors $p_0, p_1, ..., q_3$ by (instead of (3.4))

$$p_1 = VW \ , \ p_0 = P_1 - VW \ , \ p_2 = Q_1 A^{-1}(P_2 - WV)A \ , \ p_3 = Q_1 - p_2 \,, \tag{3.8}$$

$$q_1 = WV \ , \ q_2 = P_2 - WV \ , \ q_0 = A(P_1 - VW)A^{-1}Q_2 \ , \ q_3 = Q_2 - q_0 \,.$$

Obviously $p_2 Q_1 = p_2$, $Q_2 q_0 = q_0$ are satisfied and (hence) all items are idempotent.

Second we prove (in brief) that C can be written as

$$\begin{aligned} C &= q_1(A - AVA)p_1 + q_0(A - AVA)p_0 + q_2(A - AVA)p_2 + q_3(A - AVA)p_3 \\ &= q_1\, A\, p_1 \;+\; q_0\, A\, p_0 \;+\; q_2\, A\, p_2 \;+\; q_3\, V_*\, p_3\,. \end{aligned} \tag{3.9}$$

where

$$V_* \;=\; Q_2(A - AVA)Q_1\,, \tag{3.10}$$

see analogous computations for the symmetric case in S85, p. 118, which needs some calculations and where the two cited lemmas (in symmetric setting) played a crucial role. This implies the mapping properties of C in the diagram (3.3).

The foregoing two direct verifications include already the **mutual calculation of a reflexive generalized inverse of W and a CFn of A**.

Some consequences

The previous results allow further interpretations. We summarize some of them:

Corollary 3.3 *Consider a basic setting X, Y, P_1, P_2 and $A \in \mathcal{GL}(X,Y)$. Then the following statements are equivalent:*

1. *The WHO* $W = P_2 A|_{P_1 X}$ *is generalized invertible,*
2. $W = P_2 A|_{P_1 X} \sim P_2 C|_{P_1 X}$ *where C is a cross factor,*
3. $W = P_2 A|_{P_1 X} = P_2 C|_{P_1 X}$ *where C is a cross factor,*
4. *A admits a CFn,*
5. *The WHO* $W_* = Q_1 A^{-1}|_{Q_2 Y}$ *is generalized invertible,*
6. $W_* = Q_1 A^{-1}|_{Q_2 Y} = Q_1 C_*|_{Q_2 Y}$ *where C_* is a cross factor.*

Decomposition Theorem for generalized invertible WHOs

Corollary 3.4 *Again consider a basic setting* X, Y, P_1, P_2 *and* $A \in \mathcal{GL}(X, Y)$. *If* V *is a reflexive generalized inverse of* $W = P_2 A|_{P_1 X}$, *then the spaces* X *and* Y *are decomposable as shown in the diagram* (3.3) *with*

$$\begin{aligned}
X_0 &= A_+ \ker W &&= A_+ (I - VW) P_1 X &&= C^{-1} Q_2 C P_1 X \\
X_3 &= Q_1 A_+ \operatorname{im} W_* &&= Q_1 A_+ \; W_* V_* Q_1 X &&= C^{-1} Q_2 C Q_1 X \\
Y_1 &= P_2 A_-^{-1} \operatorname{im} W &&= P_2 A_-^{-1} \; W V P_2 Y &&= C P_1 C^{-1} P_2 Y \\
Y_0 &= A_-^{-1} \ker W_* &&= A_-^{-1} (I - V_* W_*) Q_2 Y &&= C P_1 C^{-1} Q_2 Y
\end{aligned}$$

where C *is given by* (3.7) *and* $V_* = Q_2(A - AVP_2A)|_{Q_1 X}$, *which is a reflexive generalized inverse of* W_* (cf. Formula (2.3) in S85).

This is useful for concrete results in applications.

4 An alternative proof of the Bart-Tsekanovsky Theorem

H. Bart and V.E. Tsekanovskii published the following theorem in 1991, see BT91.

Theorem 4.1 *Given two bounded linear operators $T : X_1 \to Y_1$ and $S : Y_2 \to X_2$ acting in Banach spaces, the relations* (2.6) *and* (2.7) *between S and T are equivalent, i.e., the two operators are* ***matrically coupled*** *if and only if they are* ***equivalent after extension***.

Recalling the definitions (2.6) of a MCR and (2.7) of an EAER:

$$\begin{pmatrix} T & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & S \end{pmatrix}^{-1} \qquad , \qquad \begin{pmatrix} T & 0 \\ 0 & I_1 \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_2 \end{pmatrix} F$$

The original proof by Bart and Tsekanovsky

The step from (2.6) to (2.7) was proved already in 1984 BGK84. It can also be regarded as an interpretation of Formula (2.12) as shown below.

The inverse conclusion of how to construct an MCR from an EAER between T and S is not at all evident. A **prompt (maybe surprising) solution** was given in BT91, in the proof of Theorem 1, namely: If T, S satisfy the EAER (2.7), then one can verify the following MCR where $E_{11}, ..., F_{22}^{-}$ are the elements of E, F, E^{-1}, F^{-1}:

$$\begin{pmatrix} T & -E_{11} \\ F_{11} & F_{12}E_{21} \end{pmatrix} = \begin{pmatrix} F_{12}^{-}E_{21}^{-} & F_{11}^{-} \\ -E_{11}^{-} & S \end{pmatrix}^{-1}. \qquad (4.1)$$

Some questions

Clearly the solution is not unique and the variety of possible MRCs depends on the two given operators. Hence some further questions appear:

- Is it possible to determine **all MCRs** from one of them?
- How can one find such a formula (of one or all MCRs) **constructively** from an EAER?
- What does this have to do with the **idea of paired operators** as realized in (2.12)?

The following result provides a partial answer to these questions.

EAERs of special form

Theorem 4.2 *Given any EAER* (2.7) *between the two bounded linear operators $T : X_1 \to Y_1$ and $S : Y_2 \to X_2$, we obtain* ***another EAER*** *of the form* (2.11) (resulting from the Kozak formula), *i.e., in the notation of Bart and Tsekanowskii of the special form*

$$\begin{pmatrix} T & 0 \\ 0 & I_1 \end{pmatrix} = \begin{pmatrix} * & * \\ I_1 & * \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} * & * \\ T & * \end{pmatrix} \tag{4.2}$$

by putting $I_1 = I_{X_2}$, $I_2 = I_{Y_1}$ and

$$\begin{pmatrix} T & 0 \\ 0 & I_1 \end{pmatrix} = \begin{pmatrix} \alpha E_{11} & I_2 - E_{11}E_{11}^- \\ I_1 & -\beta E_{11}^- \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \beta F_{11} & F_{12}E_{21} \\ T & -\alpha E_{11} \end{pmatrix} \tag{4.3}$$

where $\alpha, \beta \in \mathbb{C}$, $\alpha\beta = 1$.

Proof. Given (2.6) with the above notation, one has just to solve the **sudoku** (4.2) ... which is straightforward linear algebra.

Proof of the BT Theorem based upon formula (2.12)

The step from (2.6) to (2.7), i.e., MCR $\Rightarrow$ EAER can be regarded as an interpretation of formula (2.12). Recall

$$\begin{pmatrix} P_2AP_1 & 0 \\ 0 & Q_1 \end{pmatrix} = \begin{pmatrix} P_2 & -P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} =$$
$$= \begin{pmatrix} P_2 & -P_2AQ_1 \\ 0 & Q_1 \end{pmatrix} \begin{pmatrix} P_2 & 0 \\ Q_1A^{-1}P_2 & Q_1 \end{pmatrix} \begin{pmatrix} P_2 & 0 \\ 0 & Q_1A^{-1}Q_2 \end{pmatrix} \begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ Q_2AP_1 & Q_2AQ_1 \end{pmatrix}.$$

Namely, if T and S are matrically coupled, we can identify the matrix on the left of (2.6) with an invertible operator matrix A, further T with P_2AP_1 and S with $Q_1A^{-1}Q_2$. The identity (2.12) tells us that $T \overset{*}{\sim} S$.

EAER $\Rightarrow$ MCR

The step from (2.7) to (2.6) runs as follows: An EAER (2.7) between T and S implies another one of the form (4.3), say with $\alpha = \beta = 1$, because of Lemma 1.4. This can be re-written by a permutation of the lines and rows as

$$\begin{pmatrix} T & 0 \\ 0 & I_1 \end{pmatrix} = \begin{pmatrix} I_2 - E_{11}E_{11}^{-1} & E_{11} \\ -E_{11}^{-1} & I_1 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} T & -E_{11} \\ F_{11} & F_{12}E_{21} \end{pmatrix}. \tag{4.4}$$

It has the form of (2.12), if we identify the last matrix with

$$A = \begin{pmatrix} T & -E_{11} \\ F_{11} & F_{12}E_{21} \end{pmatrix}. \tag{4.5}$$

Hence we have the MCR (2.6) with the identifications of (2.10).

Extension of the BT Theorem

Corollary 4.3 *Let* $T : X_1 \to Y_1$ *and* $S : Y_2 \to X_2$ *be bounded linear operators in Banach spaces. Then the following statements are equivalent:*

1. T *and* S *are matrically coupled;*

2. T *and* S *are equivalent after extension;*

3. T *and* S ***satisfy a special EAER*** *of the form* (4.2).

Corollary 4.4 *EAERs of the special form* (4.2) *are reflexive, symmetric and transitive, hence represent (for its own) an equivalence relation in the genuine mathematical sense.*

5 Further alternative formulas

The first line of formula (1.2) can be seen as a relation between the general WHO $W = PA|_{PX}$ in (2.4) and a paired operator $(PA + QB)$ where $B = I$. The second line of (1.2) holds if A is invertible. Alternatively one can consider another relation with a paired operator of the form $AP + BQ$ (again $B = I$):

$$PAP + Q = (AP + Q)(I - QAP)\,. \tag{5.1}$$

We obtain instead of (1.2), (2.1), (2.12) the following formulas. Firstly

$$PAP + Q = A(P + QA^{-1}Q)(I + PA^{-1}Q)(I - QAP) \tag{5.2}$$

which is also well-known from GF71,Pro74. In matrix form it may be written as

$$\begin{aligned}\begin{pmatrix} PAP & 0 \\ 0 & Q \end{pmatrix} &= \begin{pmatrix} PAP & 0 \\ QAP & Q \end{pmatrix} \begin{pmatrix} P & 0 \\ -QAP & Q \end{pmatrix} = \\ &= \begin{pmatrix} PAP & PAQ \\ QAP & QAQ \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & QA^{-1}Q \end{pmatrix} \begin{pmatrix} P & PA^{-1}Q \\ 0 & Q \end{pmatrix} \begin{pmatrix} P & 0 \\ -QAP & Q \end{pmatrix}.\end{aligned} \tag{5.3}$$

This identity can be seen as a special case of the factorization

$$\begin{pmatrix} P_2AP_1 & 0 \\ 0 & Q_2 \end{pmatrix} = \begin{pmatrix} P_2AP_1 & 0 \\ Q_2AP_1 & Q_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ -Q_2AP_1 & Q_2 \end{pmatrix} = \tag{5.4}$$

$$\begin{pmatrix} P_2AP_1 & P_2AQ_1 \\ Q_2AP_1 & Q_2AQ_1 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & Q_1A^{-1}Q_2 \end{pmatrix} \begin{pmatrix} P_1 & P_1A^{-1}Q_2 \\ 0 & Q_2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ -Q_2AP_1 & Q_2 \end{pmatrix}$$

with a mapping scheme which is now in the last line given by (cf. (2.12))

$$\begin{matrix} Y_1 & & X_1 & & X_1 & & X_1 & & X_1 \\ \times & \leftarrow & \times & \leftarrow & \times & \leftarrow & \times & \leftarrow & \times \\ Y_2 & & X_2 & & Y_2 & & Y_2 & & Y_2 \end{matrix} . \tag{5.5}$$

Hence we obtain analogous results as before.

6 Further relations with paired operators

The combination of the BT Theorem with the paired operators reasoning lead us to the following results.

Proposition 6.1 *Let X be a Banach space, $P, Q \in \mathcal{L}(X)$ be complementary projectors, and $A \in \mathcal{L}(X)$.* ***The following operators are matrically coupled:***

$$\begin{array}{rcl} T_P(A) = PA|_{PX} & : & PX \to PX \\ AP+Q\,,\, PA+Q\,,\, PAP+Q & : & X \to X \end{array}$$

If A is boundedly invertible, each of these four operators is matrically coupled with each of the following four operators:

$$\begin{array}{rcl} T_Q(A^{-1}) = QA^{-1}|_{QX} & : & QX \to QX \\ A^{-1}Q+P\,,\, QA^{-1}+P\,,\, QA^{-1}Q+P & : & X \to X\,. \end{array}$$

Hence we obtain analogous results using the above-mentioned operator relations.

7 Applications

The **main applications** of the BT Theorem consist roughly speaking in the conclusion: **If S and T satisfy a MCR, then we can transfer nice properties from S to T and vice versa.** Examples can be found in various fields ranging from abstract algebraic settings S85 to concrete applications in diffraction theory, for instance. Here we outline a few of them. The first two are known from CS98 and S83, respectively. They coincide partly with the previous results, after an identification of the operators $T = P_2 A|_{P_1 X}$ etc. as done in Section 1. However the outcome is considered to deserve independent interest due to their different notation and concrete applications, cf. corresponding remarks in BGK84a.

Theorem 7.1 *Let S and T satisfy a MCR (2.6). Then the two operators belong to the* ***same regularity class*** *in the sense of the classification in Remark 1.5.3.*

Proof. (Sketch) The BT Theorem implies that the two matrically coupled operators are equivalent after extension which yields the statement after substitution according to Corollary 5.4.

Mutual computation of gen. inverses of associated WHOs

Theorem 7.2 *Let S and T be bounded linear operators in Banach spaces which are matrically coupled, i.e., $T = W = P_2A|_{P_1X}$ and $S = W_* = Q_1A^{-1}|_{Q_2Y}$ in the above notation. Further let V be a generalized inverse of W, i.e., $WVW = W$. Then a* ***generalized inverse of W_* is given by***

$$V_* \quad = \quad Q_2(A - AP_1VP_2A)|_{Q_1X}. \tag{7.1}$$

Remarks 7.3 *1. By symmetry we obtain that*

$$V \quad = \quad Q_2(A^{-1} - A^{-1}Q_2V_*Q_1A^{-1})|_{P_2Y} \tag{7.2}$$

is a generalized inverse of W provided $W_*V_*W_* = W_*$.

2. The operations (7.1) and (7.2) are inverse to each other in the sense that the step from V_* to V leads back to V if V_* was calculated by (7.1). This fact again results from the paired operators reasoning.

8 Examples from diffraction theory

A concrete realization of the MCR (2.6) can be found in the theory of diffraction of electromagnetic or acoustic waves from plane screens in $\mathbb{R}^3$. For precise formulation we need some preparation. Given a proper open subset $\Sigma \subset \mathbb{R}^2$ with $\operatorname{int}\operatorname{clos}\Sigma = \Sigma$, we consider the domain Ω defined by

$$\begin{aligned} &\Omega = \mathbb{R}^3 \setminus \Gamma \\ &\Gamma = \overline{\Sigma} \times 0 = \{x = (x_1, Y_1, 0) \in \mathbb{R}^3 : x' = (x_1, Y_1) \in \overline{\Sigma}\}. \end{aligned} \tag{8.1}$$

Problems of diffraction from a plane screen Γ are often formulated in terms of (or reduced to) the solution of the **three-dimensional Helmholtz equation** in Ω with **Dirichlet or Neumann conditions** on Γ, briefly written as

$$\begin{array}{rclll} \left(\Delta + k^2\right) u & = & 0 & \text{in} & \Omega \\ Bu & = & g & \text{on} & \Gamma = \partial\Omega. \end{array} \tag{8.2}$$

Modelling BVPs

Herein k is the **wave number** and we assume that $\Im m\, k > 0$. B stands for the **boundary operator**, taking the trace or normal derivative of u on Γ (in the same direction on both banks of Γ). We think of the **weak formulation** looking for $u \in \mathcal{H}^1(\Omega)$, i.e., u is defined in Ω and its restriction $u|_{\Omega^*} \in H^1(\Omega^*)$ is a weak solution of the **Helmholtz equation in any special Lipschitz subdomain** $\Omega^* \subset \Omega$, see CDS14,ENS11,ENS14 for details. **Boundary data** $g \in H^{1/2}(\Sigma)$ (in the Dirichlet problem) or $g \in H^{-1/2}(\Sigma)$ (in the Neumann problem) **are arbitrarily given** (to study continuous dependence from the data in well-posed problems) and one looks for the **resolvent operator** as the **inverse to the boundary (trace or normal derivative) operator**

$$B \quad = \quad \mathcal{H}^1(\Omega) \longrightarrow H^{\pm 1/2}(\Sigma). \tag{8.3}$$

Boundary ΨDO for the Dirichlet problem

Representation formulas for a function $u \in \mathcal{H}^1(\Omega)$ in the half-spaces $\{x \in \mathbb{R}^3 : \pm x_3 > 0\}$ yield that, in case of the **Dirichlet problem**, B **is equivalent to** a boundary pseudo-differential operator which has the form of **a general WHO**

$$W = P_2 A_t^{-1}|_{\operatorname{im} P_1} : H_\Sigma^{-1/2} \to \operatorname{im} P_2 \subset H^{1/2}(\mathbb{R}^2). \tag{8.4}$$

Herein $A_t = \mathcal{F}^{-1} t \cdot \mathcal{F} : H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$ is a convolution operator with Fourier symbol $t(\xi) = (\xi^2 - k^2)^{1/2}$, P_1 is a projector in $H^{-1/2}$ with $\operatorname{im} P_1 = H_\Sigma^{-1/2}$ denoting the subspace of distributions supported on $\overline{\Sigma}$, and P_2 a projector in $H^{1/2}$ with $\ker P_2 = H_{\Sigma'}^{1/2}$, the subspace of functions supported on the closure $\overline{\Sigma'}$ of the complementary screen $\Sigma' = \mathbb{R}^2 \setminus \overline{\Sigma}$.

Boundary ΨDO for the Neumann problem

By analogy we obtain an operator for the **Neumann problem** due to the complementary screen that has the form

$$W_* = Q_1 A_t|_{\operatorname{im} Q_2} : H^{1/2}_{\Sigma'} \to \operatorname{im} Q_1 \subset H^{-1/2}(\mathbb{R}^2) \tag{8.5}$$

with projectors $Q_1 = I - P_1$ and $Q_2 = I - P_2$ complementary to the previous. The operators (8.4) and (8.5) meet the (abstract) situation of (2.10), they are **matrically coupled** and consequently **the inverses can be computed from each other** according to Theorem 6.2 and Remark 6.3.5.

The "geometric perspective" of general WHOs

Theorem 8.1 *Let $A \in \mathcal{L}(X,Y)$ be boundedly invertible and $W = P_2 A|_{P_1 X}$ defined as in (2.10). Then W* ***is invertible if and only if*** *AP_1X and $Q_2Y = (I - P_2)Y$* ***are complemented subspaces*** *of Y. In this case,* ***the inverse of W is given by***

$$W^{-1} \quad = \quad A^{-1}\Pi|_{P_2 Y} \quad = \quad P_2 Y \to P_1 X \tag{8.6}$$

where $\Pi \in \mathcal{L}(Y)$ is the projector onto AP_1X along Q_2Y.

See MS87,San88,San89.

Resolvent operators for convex cones

Example 1 *Concrete applications can be found in the screen diffraction problems described before. In the* ***simplest case****, we consider the* ***wave number*** $k = i$*, as a model problem. Hence* $t(\xi) = (\xi^2 + 1)^{1/2}$ *and we take* $A = A_t$*. Then the above-mentioned* ***projectors are orthogonal****, since*

$$H^{1/2}_{\Sigma} \perp A_t^{-1} H^{-1/2}_{\Sigma'} \quad , \quad H^{-1/2}_{\Sigma} \perp A_t H^{1/2}_{\Sigma'}$$

for any admissible Σ.

Now let Σ *be a* ***convex cone****, thus intersection of two half-planes* Σ_1 *and* Σ_2*, say. In case of the Dirichlet problem, the* ***orthogonal projector*** $P_{1,\Sigma}$ *in* $H^{-1/2}$ *onto* $H^{-1/2}_{\Sigma}$ *can be represented as the* ***infimum of the two orthogonal projectors*** P_{1,Σ_1} *in* $H^{-1/2}$ *onto* $H^{-1/2}_{\Sigma_1}$ *and* P_{1,Σ_2} *onto* $H^{-1/2}_{\Sigma_2}$*, i.e.,*

$$P_{1,\Sigma} \quad = \quad P_{1,\Sigma_1} \wedge P_{1,\Sigma_2} \quad = \quad \prod_{j=1}^{\infty} (P_{1,\Sigma_1} P_{1,\Sigma_2})^j \tag{8.7}$$

in the sense of strong convergence, see Hal82, Problem 96.

Resolvent operators for more general screens

Example 2 *For the* ***complementary screen****, this argument is not directly applicable. However, one can determine the resolvent with formula* (8.6)*, further the resolvent for the Neumann problem and the complementary screen with formula* (7.1)*, and even the orthogonal projector* $P_{1,\Sigma'}$ *in* $H^{1/2}$ *onto* $H^{1/2}_{\Sigma'}$ *for the complementary screen. More details and further results can be found in CDS14,MS87,S12. Finally it has been shown that, with this technique, diffraction problems can be explicitly solved for* ***domains*** Σ ***which belong to the set algebra generated by half-planes****, i.e., for sets of the form*

$$\Sigma \quad = \quad \mathrm{int} \bigcup_{j=1,\ldots,m} \mathrm{clos}\, \Sigma_j$$

where Σ_j *are intersections of finite sets of half-planes.*

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