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Advances in general Wiener-Hopf factorization

Abstract

Started in 1983-85 this research enjoyed a revival in 2015 by a paper entitled "Wiener-Hopf factorization through an intermediate space", in which an alternative to the cross factorization theorem was proposed that fits better with various applications.

In general, operator factorizations generate a certain middle space in a natural way that is related with important properties of the corresponding general or concrete Wiener-Hopf operator.

We report about this paper [S15], expose concerning applications [S17], and various consequences [BoeS16, S18].

Recent work

[S15] Wiener-Hopf factorization through an intermediate space.
Integral Equations Oper. Theory 82, 395-415 (2015).

[S17] A class of interface problems for the Helmholtz equation in R^n.
Math. Meth. Appl. Sciences 40, (2017), 391-403 (DOI 2015).

[BoeS16] On the symmetrization of general Wiener-Hopf operators.
J. Operator Theory 76, (2016), 335-349 (with Albrecht Böttcher).

[S18] On the reduction of general Wiener-Hopf operators.
In: Harm Bart et al. (eds.), Operator Theory, Analysis and the State Space Approach - The Rien Kaashoek Anniversary Volume. Operator Theory: Advances and Applications 271, 399-419. Birkhäuser/Springer, Basel 2018.

General Wiener-Hopf operators [DevShi69, S83, S85]

Overall assumptions:

$$X, Y \text{ are Banach spaces,}$$
$$P_1 \in \mathcal{L}(X)\,,\ P_2 \in \mathcal{L}(Y) \text{ are projectors,}$$
$$A \in \mathcal{L}(X, Y) \text{ is a bounded linear operator.}$$

Then the operator

$$W = P_2 A|_{P_1 X} = P_1 X \to P_2 Y \tag{1}$$

is referred to as a general Wiener-Hopf operator (WHO) in asymmetric space setting in contrast to the case where $X = Y$ and $P_1 = P_2$:

$$W = T_P(A) = PA|_{PX} = PX \to PX\,. \tag{2}$$

We shall also call W truncation of the underlying operator A.

Generalized inverses

Given $T \in \mathcal{L}(X,Y)$ an operator $T^- \in \mathcal{L}(Y,X)$ is said to be a generalized inverse of T if

$$T\ T^-\ T \quad = \quad T\,.$$

It is called a reflexive generalized inverse if $T^-\,T\,T^- = T^-$ holds additionally. In both cases T is said to be generalized invertible (since the existence of a generalized inverse implies the existence of a reflexive generalized inverse, replacing T^- by T^-TT^-).

It is well-known that T is generalized invertible if and only if its kernel and image are complemented subspaces of X and Y, respectively (which includes the cases of T to be Fredholm, one-sided invertible and others).

The knowledge of a generalized inverse yields solubility conditions and an explicit representation of the general solution of the operator equation $Tf = g$.

Cross factorization [S83, S85]

Let A be boundedly invertible. An operator triple A_-, C, A_+ with

$$\begin{aligned} A \quad &= \quad A_- \quad C \quad A_+ \\ &: \quad Y \leftarrow Y \leftarrow X \leftarrow X\,. \end{aligned} \tag{3}$$

is referred to as a cross factorization of A (with respect to X, Y, P_1, P_2), in brief CFn, if the factors $A_\pm$ are strong WH factors, i.e.,

$$\begin{aligned} A_+ \;\in\; \mathcal{GL}(X) \quad &, \quad A_- \;\in\; \mathcal{GL}(Y)\;, \\ A_+P_1X = P_1X \quad &, \quad A_-Q_2Y = Q_2Y\;, \end{aligned} \tag{4}$$

and the cross factor $C \in \mathcal{L}(X, Y)$ is boundedly invertible such that

$$C^{-1}P_2CP_1 \;\; \textit{is a projector (idempotent) in} \;\; \mathcal{L}(X). \tag{5}$$

The name cross factor (briefly CF) comes from the fact that it appears in operator factorizations and has following (equivalent) property: it splits the spaces X, Y both into four subspaces such that

$$
\begin{array}{ccccccccccc}
X & = & \overbrace{X_1 & \dot{+} & X_0}^{P_1 X} & \dot{+} & \overbrace{X_2 & \dot{+} & X_3}^{Q_1 X} \\
 & & \downarrow & & & C \; \swarrow\!\!\!\!\searrow & & & \downarrow \\
Y & = & \underbrace{Y_1 & \dot{+} & Y_2}_{P_2 X} & \dot{+} & \underbrace{Y_0 & \dot{+} & Y_3}_{Q_2 X} .
\end{array}
\tag{6}
$$

This means that C maps each X_j onto Y_j , j = 0, 1, 2, 3 , i.e., the complemented subspaces $X_0, X_1, ..., Y_3$ are images of corresponding projectors $p_0, p_1, ..., q_3$, namely

$$
\begin{aligned}
X_1 = p_1 X \;&=\; C^{-1} P_2 C P_1 X \quad , \quad X_0 = p_0 X \;=\; C^{-1} Q_2 C P_1 X \,, \\
X_2 = p_2 X \;&=\; C^{-1} P_2 C Q_1 X \quad , \quad X_3 = p_3 X \;=\; C^{-1} Q_2 C Q_1 X \,, \\
Y_1 \;= q_1 Y \;&=\; C P_1 C^{-1} P_2 Y \quad , \quad Y_2 \;= q_2 Y \;=\; C Q_1 C^{-1} P_2 Y \,, \\
Y_2 \;= q_1 Y \;&=\; C P_1 C^{-1} Q_2 Y \quad , \quad Y_3 \;= q_3 Y \;=\; C Q_1 C^{-1} Q_2 Y \,.
\end{aligned}
$$

The cross factorization theorem [S83, S85]

Let A be boundedly invertible. Then W is generalized invertible if and only if a cross factorization of A exists and, in this case, a formula for a reflexive generalized inverse of W is given by

$$W^{-} = A_{+}^{-1} P_1 C^{-1} P_2 A_{-}^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X\,. \tag{7}$$

Sufficiency is proved by inspection. Necessity is more complicated.

It yields the explicit determination of their kernels and complements of the images provided the factor inverses are known. More consequences on the Fredholm property, explicit presentation of solutions of the equation $Wf = g$ etc. are immediate.

A most important fact is the equivalence of W and $P_2 C|_{P_1 X}$, in brief $W \sim P_2 C|_{P_1 X}$, namely

$$W \quad = \quad P_2 A_{-}|_{P_2 Y} \;\; P_2 C|_{P_1 X} \;\; P_1 A_{+}|_{P_1 X} \quad = \quad E \;\, P_2 C|_{P_1 X} \;\, F \tag{8}$$

where E, F are isomorphisms (boundedly invertible linear operators).

WH factorization through an intermediate space

Now we study another type of factorization, which is quite different from the previous and more interesting for many applications.

$$\begin{array}{rcl} A & = & A_- \quad C \quad A_+ \\ & : & Y \leftarrow Z \leftarrow Z \leftarrow X\,. \end{array} \tag{9}$$

is referred to as a WH factorization through an intermediate space Z (with respect to X, Y, P_1, P_2) (in brief FIS), if the factors $A_\pm$ and C are linear and boundedly invertible in the above setting with an additional Banach space Z called intermediate space [CS95] and if there is a projector $P \in \mathcal{L}(Z)$ such that

$$A_+ P_1 X = PZ \quad , \quad A_- QZ = Q_2 Y \tag{10}$$

with $Q = I_Z - P$ and such that $C \in \mathcal{GL}(Z)$ and

$$C^{-1}PCP \text{ is a projector (idempotent).} \tag{11}$$

I.e., C splits the space Z twice into four subspaces:

$$
\begin{array}{ccccccccc}
Z & = & \overbrace{X_1 & \dot{+} & X_0}^{PZ} & \dot{+} & \overbrace{X_2 & \dot{+} & X_3}^{QZ} \\
 & & \downarrow & & & C \diagdown\!\!\!\!\diagup & & & \downarrow \\
 & = & \underbrace{Y_1 & \dot{+} & Y_2}_{PZ} & \dot{+} & \underbrace{Y_0 & \dot{+} & Y_3}_{QZ}
\end{array}
$$

where C maps each X_j onto Y_j.

Again $A_\pm$ are called strong WH factors and C is said to be a cross factor, now acting from a space Z into the same space Z.

By analogy to the cross factorization theorem, the following conclusion is straightforward, as well: A FIS of A implies a reflexive generalized inverse of W by putting

$$W^- = A_+^{-1} P C^{-1} P A_-^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X \, . \tag{12}$$

The inverse conclusion is not true in general, as we shall see later.

Unbounded FIS [S15]

Let $A \in \mathcal{L}(X, Y)$ be boundedly invertible.

A factorization $A = A_- C A_+$ is said to be an unbounded WH factorization through an intermediate (Banach) space Z (unbounded FIS), if the factors $A_\pm^{\pm 1}$ are densely defined injective linear operators in the above-mentioned spaces, P and $C \in \mathcal{L}(Z)$ have the same properties as before, and the operator

$$T = A_+^{-1} P C^{-1} P A_-^{-1} \quad : \quad Y \to X \tag{13}$$

is also densely defined and admits a bounded extension to the full space, in brief $T \in \mathcal{L}(Y, X)$.
I.e., the factorization holds with a cross factor $C \in \mathcal{GL}(Z)$ and with $A_\pm^{\pm 1}$ being densely defined injective linear operators with the above factor properties and with continuous extension in the sense of a FIS.

By analogy, an unbounded cross factorization may be defined.

If $C = I$, the previous factorizations are called canonical.

Full range factorization [S15]

Let $T \in \mathcal{L}(X, Y)$ and

$$T \quad = \quad L \ R \tag{14}$$

where $R \in \mathcal{L}(X, Z), L \in \mathcal{L}(Z, Y)$, X, Y, Z are Banach spaces, R is right invertible and L is left invertible. Then (13) is said to be a full range factorization (FRF) of T.

This notion is well-known from matrix theory as full rank factorization ($\dim Z = \operatorname{rank} T$). Evidently a FRF implies that

$$T^{-} \quad = \quad R^{-} \ L^{-}$$

is a reflexive generalized inverse of T provided $RR^{-} = I_Z = L^{-}L$. The intermediate space Z is isomorphic to the image (or range) of T and to any complement of the kernel of T, as well.

Main results of [S15]

Let $A \in \mathcal{L}(X, Y)$ be boundedly invertible, $W = P_2 A|_{P_1 X}$ as before.

Theorem 1. *W is invertible if and only if A admits a canonical FIS:*

$$\begin{aligned} A &= \quad A_- \quad A_+ \\ &: \quad Y \leftarrow Z \leftarrow X\,. \end{aligned} \tag{15}$$

Theorem 2. *The following assertions are equivalent:*
(i) *W is generalized invertible and $P_1 \sim P_2$ holds,*
(ii) *A admits a FIS.*

Herein the condition $P_1 \sim P_2$ is not redundant.

Theorem 3. *Every canonical unbounded FIS can be regarded as a canonical bounded FIS, by a change of the intermediate space.*

Corollary

The following statements are equivalent:

(j) *A admits a CFn (with respect to X, Y, P_1, P_2),*

(jj) *W admits a full range factorization,*

and moreover, if $P_1 \sim P_2$ holds,

(jjj) *A admits a FIS (with respect to X, Y, P_1, P_2).*

In all cases W is generalized invertible, a generalized inverse of W is given in terms af the factorization and a factorization of one type can be computed from any other (via W^-).

Remark

In 2015 the question appeared if $P_1 \sim P_2$ is also necessary for A to admit a FIS which was finally proved in [BoeS16] (filling a gap in the proof of Theorem 2 and correcting also a misprint in Corollary 2.9 of [S15]).

Example 1: Generalized or Φ-factorization

Let $\Gamma \in \mathbb{C}$ be a closed contour which divides $\mathbb{C} \cup \infty$ into two domains D_+ and D_- such that $0 \in D_+, \infty \in D_-$ and $\partial D_+ = \partial D_- = \Gamma$. $L^p_\pm \subset L^p(\Gamma)$ $(p > 0)$ are the spaces of functions which are boundary values of functions holomorphic in $D_\pm$ in the sense of Privalov (see [LitSpi87] for details). For simplicity we consider the unit circle $\Gamma = \Pi_0 = \{z \in \mathbb{C} : |z| = 1\}$.

A (right, generalized) factorization of $G \in L^\infty(\Gamma)^{n\times n}$ in $L^p, 1 < p < \infty$, relative to Γ is a representation in the form

$$G(z) \quad = \quad G_-(z)\ \Lambda(z)\ G_+(z) \quad , \quad z \in \Gamma \tag{16}$$

where $G_- \in L^p_-, G_+ \in L^q_+, G_-^{-1} \in L^q_-, G_+^{-1} \in L^p_+, q = p/(p-1)$ and the matrix function Λ has the form

$$\Lambda(z) \quad = \quad \mathrm{diag}\,(z^{\kappa_1}, ..., z^{\kappa_n}) \quad , \quad z \in \Gamma \tag{17}$$

where $z^{\kappa_1} \geq ... \geq z^{\kappa_n}$ are integers. Λ can be seen as a cross facor in the following context:

Consider the Toeplitz operator

$$T \quad = \quad P\,G\cdot|_{(L^p_+)^n} \tag{18}$$

in the space of vector functions $X = (L^p_+)^n, p \in (1,\infty)$, where $A = G\cdot$ denotes the multiplication operator and P the Riesz projection. T can be seen as an example for a general Wiener-Hopf operator W in symmetric setting $X = Y$, $P_1 = P_2 = P$.
Let us assume that $G \in \mathcal{G}L^\infty(\Gamma)^{n\times n}$ admits a (right, generalized) factorization in L^p [Sim68] for some $p \in (1,\infty)$. Then T is normally solvable (and moreover Fredholm) if and only if

$$K \;=\; G_+^{-1}\cdot\Lambda\,Q\,G_-^{-1} \quad \texttt{and} \quad K_1 \;=\; G_-\cdot P\,G_-^{-1} \tag{19}$$

are bounded in $(L^p)^n$. Then (15) is said to be a Φ-factorization of G [LitSpi87].
The Φ-factorization can be interpreted in the sense of an unbounded FIS (still with $X = Y$) and the formulas for a generalized inverse in terms of the factorization are applicable. We have $Z = \operatorname{im} A_+ = \operatorname{im} A_-^{-1}$ with the induced norm and $P, C \in \mathcal{L}(Z)$. This was pointed out already in [CasS95] where the nature of those spaces was studied.

Ex 2: WHOs in diffraction from plane screens

The following species appears particularly in problems of diffraction from plane screens ($n = 2$ or $n = 3$) such as the Sommerfeld diffraction problem [MS89], its various generalizations, see [CDS14] for instance, and other elliptic boundary value problems [Esk81, HW08, Wlo87], in the scalar case (pure Dirichlet, Neumann, or impedance conditions):

$$W_{\Phi,\Sigma} \quad = \quad r_\Sigma \mathcal{F}^{-1}\Phi \cdot \mathcal{F}|_{H^r_\Sigma} \quad : \quad H^r_\Sigma \to H^s(\Sigma) \tag{20}$$

where $\Sigma \subset \mathbb{R}^{n-1}$ is an open set (subset of a hyperplane in $\mathbb{R}^n$), $\mathcal{F}$ the $n-1$-dimensional Fourier transformation, $A = \mathcal{F}^{-1}\Phi \cdot \mathcal{F}$ a translation invariant operator, elliptic of order $r - s$, i.e., $\lambda^{s-r}\Phi \in \mathcal{G}L^\infty$ where $\lambda(\xi) = (|\xi|^2 + 1)^{1/2}, \xi \in \mathbb{R}^{n-1}, r, s \in \mathbb{R}$ for $n = 2$ (and $r, s \in \mathbb{R}^2$ for $n = 3$), and $H^r_\Sigma, H^s(\Sigma)$ Sobolev spaces of distributions supported on $\overline{\Sigma}$ or restricted to Σ, respectively, see [Esk81] for instance.

The operator (19) is not directly of the form of a general WHO (1) but equivalent, provided $\operatorname{int}\operatorname{clos}\Sigma = \Sigma$ holds and Σ has the strong extension property [HW08], i.e., there exists (for any $s \in \mathbb{R}$) an extension operator $E^s_\Sigma \in \mathcal{L}(H^s(\Sigma), H^s(\mathbb{R}^{n-1}))$ which is left invertible by restriction: $r_\Sigma E^s_\Sigma = I_{H^s(\Sigma)}$. Then we have

$$W_{\Phi,\Sigma} \quad = \quad r_\Sigma W \quad \sim \quad W \quad = \quad P_2 A|_{P_1 X} \tag{21}$$

where $X = H^r, Y = H^s, P_1$ is a projector in H^r onto H^r_Σ and P_2 is a projector in H^s along $\Sigma' = \mathbb{R}^{n-1} \setminus \overline{\Sigma}$, for instance $P_2 = E^s_\Sigma r_\Sigma$.

Concrete examples appear in diffraction theory where typically matrix operators with entries like (20) are relevant [CDS14, S14], considered now:

Ex 3: Interface problems in $\mathbb{R}^2$ [S86, S89]

An important variant of Example 2 is the WHO

$$W \quad = \quad r_+ \, A|_{P_1 X} \quad : \quad H_+^{1/2} \times H_+^{-1/2} \to H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+) \tag{22}$$

where $X = Y = H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$, $\Sigma = \mathbb{R}_+ =]0, \infty[$, $r_+ = r_{\mathbb{R}_+}$ and $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ with

$$\sigma_\lambda \quad = \quad \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \tag{23}$$

and $t(\xi) = (\xi^2 - k^2)^{1/2}$, $\xi \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

A canonical generalized factorization of $\sigma_\lambda = \sigma_{\lambda-}\,\sigma_{\lambda+}$ was derived with the help of Khrapkov's formulas and Daniele's trick (stimulated by the mixed Dirichlet-Neumann problem where $\lambda = -1$ [Raw81]), however covering a large class of "Sommerfeld diffraction problems" [S89].

The result is as follows:

$$
\begin{aligned}
\sigma_{\lambda+} &= (1-\lambda^{-1})^{-1/4} \begin{pmatrix} c_+ & -s_+\sqrt{\lambda}/t \\ -c_+\xi/\sqrt{\lambda} - s_+t/\sqrt{\lambda} & s_+\xi/t + c_+ \end{pmatrix} \\
\sigma_{\lambda-} &= (1-\lambda^{-1})^{-1/4} \begin{pmatrix} c_- - s_-\xi/t & -s_-\sqrt{\lambda}/t \\ -s_-t/\sqrt{\lambda} + c_-\xi/\sqrt{\lambda} & c_- \end{pmatrix}
\end{aligned}
\tag{24}
$$

where

$$
\begin{aligned}
c_\pm(\xi) &= \cosh[C \log \gamma_\pm(\xi)] \\
s_\pm(\xi) &= \sinh[C \log \gamma_\pm(\xi)] \\
\gamma_\pm(\xi) &= \frac{\sqrt{k \pm \xi} + i\sqrt{k \mp \xi}}{\sqrt{2k}} \quad , \quad \xi \in \mathbb{R} \\
C &= \frac{i}{\pi} \log \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \quad .
\end{aligned}
$$

Because of the asymptotic behavior of $\sigma_{\lambda\pm}$ at infinity, the corresponding factorization of $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ represents a canonical Wiener-Hopf factorization through a vector Sobolev space:

$$
\begin{aligned}
A_\lambda &= A_{\lambda-} \quad A_{\lambda+} = \mathcal{F}^{-1}\sigma_{\lambda-}\cdot\mathcal{F} \quad \mathcal{F}^{-1}\sigma_{\lambda+}\cdot\mathcal{F} \\
& H^{1/2}\times H^{-1/2} \leftarrow Z \leftarrow H^{1/2}\times H^{-1/2} \\
Z &= H^{\vartheta}(\mathbb{R}) \quad , \quad \vartheta = (\vartheta_1, \vartheta_2) = (\frac{1}{2}(1-\delta), \frac{1}{2}(\delta-1))
\end{aligned}
\tag{25}
$$

where $\delta = \Re e\, C = \frac{-1}{\pi}\arg\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \in]0,1]$.

Ex 4: Interface problems in $\mathbb{R}^n$, $n \geq 3$ [S17]

Consider the higher-dimensional case ($m = n-1 \geq 2$) where Σ is a half-space which is of particular interest in various applications:

$$X = Y = H^{1/2}(\mathbb{R}^m) \times H^{-1/2}(\mathbb{R}^m) \ , \ \ \Sigma = \mathbb{R}^m_+ = \mathbb{R}^{m-1} \times (0, \infty)$$

and $t(\xi) = (\xi_1^2 + ... + \xi_m^2 - k^2)^{1/2}$, $\xi = (\xi', \xi_m) \in \mathbb{R}^m$, we can consider the same factorization given by (47) replacing k by $(k^2 - \xi'^2)^{1/2}$, i.e., the previous factorization as to be parameter-dependent of $\xi' \in \mathbb{R}^{m-1}$. It turns out that the factorization can be seen as a canonical FIS of A where the intermediate space is an unisotropic vector Sobolev space

$$\begin{aligned} Z &= H^{\vartheta}(\mathbb{R}^m) \times H^{-\vartheta}(\mathbb{R}^m) && (26) \\ H^{\vartheta}(\mathbb{R}^m) &= \mathcal{F}(w_{\vartheta} L^2(\mathbb{R}^m)) \quad , \quad w_{\vartheta}(\xi) \ = \ (1 + |\xi'|^2)^{\vartheta_1/2} \, (1 + \xi_m^2)^{\vartheta_2/2} \\ \vartheta &= (\vartheta_1, \vartheta_2) \ = \ (\frac{1}{2}(\delta - 1), \frac{1}{2}(1 - \delta)) \, . \end{aligned}$$

Symmetrization of general WHOs [BoeS16]

Question 1 When is the operator W in (1) equivalent to a WHO $\tilde{W}$ in symmetric setting ($X = Y$, $P_1 = P_2$)? I.e. there exists a space Z, an operator $\tilde{A} \in G\mathcal{L}(Z)$, a projector $P \in \mathcal{L}(Z)$ and isomorphisms E, F such that

$$W \quad = \quad P_2 A|_{P_1 X} \quad = \quad E\ \tilde{W}\ F \quad = \quad E\, P \tilde{A}|_{PZ}\, F\,.$$

The answer depends heavily on all "parameters" X, Y, P_1, P_2, A and is particularly trivial for finite rank operators W or for separable Hilbert spaces X, Y. Hence we modify the question:

Question 2 When is the operator W of (1) equivalent to a WHO $\tilde{W}$ in symmetric setting (2), for any choice of $A \in G\mathcal{L}(X, Y)$?

Remark. This does not imply that E and F are independent of A, but has to do with factorizations of A. The answer of question 2 can be seen as a property of the space setting $X, Y, \operatorname{im} P_1, \ker P_2$, as we shall see.

Motivation

A strong motivation to study the operator (1) in an asymmetric space setting is given by the theory of pseudo-differential operators, which naturally act between Sobolev-like spaces of different orders; see Eskin's book 1973/81. Their symmetrization (lifting) by generalized Bessel potential operators was considered in [DudS93].

Furthermore, Toeplitz operators with singular symbols are another source of motivation for considering symmetrization [BoeS16].

Symmetrizable space settings

We call the setting X, Y, P_1, P_2 symmetrizable if there exist a Banach space Z, operators $M_+ \in G\mathcal{L}(X, Z)$ and $M_- \in G\mathcal{L}(Z, Y)$, and a projector $P \in \mathcal{L}(Z)$ such that

$$M_+(P_1 X) = PZ, \quad M_-(QZ) = Q_2 Y, \tag{27}$$

where $Q = I_Z - P$ and $Q_2 = I_Y - P_2$.

Note that the invertibility of M_+ and M_- in conjunction with (27) implies that the truncated operators

$$U_+ := M_+|_{P_1 X} : P_1 X \to PZ, \quad V_- := M_-|_{QZ} : QZ \to Q_2 Y, \tag{28}$$

are invertible.

Symmetrization of asymmetric WHOs

If the setting X, Y, P_1, P_2 is symmetrizable, then asymmetric WHOs may also be symmetrized: given an operator of the form (1), there is an operator $\widetilde{A} \in \mathcal{L}(Z)$ such that $A = M_- \widetilde{A} M_+$ and $W = V_+ \widetilde{W} U_+ = V_+ T_P(\widetilde{A}) U_+$. Indeed, we have $\widetilde{A} = M_-^{-1} A M_+^{-1}$, and since $PM_-^{-1} = PM_-^{-1} P_2$ and $PM_+ P_1 = M_+ P_1$, we get

$$
\begin{aligned}
V_+ \widetilde{W} U_+ &= (PM_-^{-1}|_{P_2 Y})^{-1}\, PM_-^{-1} A M_+^{-1}|_{PZ}\, (PM_+|_{P_1 X}) \\
&= (PM_-^{-1}|_{P_2 Y})^{-1}\, PM_-^{-1} P_2 A M_+^{-1}\, M_+|_{P_1 X} \\
&= P_2 A|_{P_1 X} \;=\; W\,.
\end{aligned}
$$

Thus, in the case of a symmetrizable setting,

$$W \sim T_P(\widetilde{A}).$$

Main result

Given two Banach spaces Z_1 and Z_2, we write $Z_1 \cong Z_2$ if the two spaces are isomorphic, that is, if there exists an operator $A \in \mathcal{GL}(Z_1, Z_2)$.

Theorem 4. *The following are equivalent:*

(i) *the setting X, Y, P_1, P_2 is symmetrizable,*
(ii) $P_1 X \cong P_2 Y$ *and* $Q_1 X \cong Q_2 Y$,
(iii) $P_1 \sim P_2$.

The theorem implies in particular that every setting given by two separable Hilbert spaces X, Y and two infinite-dimensional bounded projectors P_1, P_2 with isomorphic kernels is symmetrizable. Many examples from applications satisfy this condition.

Remark

From Theorem 4 we see that if $P_1 \sim P_2$, then

$$P_1X \times Q_2Y \cong P_1X \times Q_1X \cong P_1X \oplus Q_1X = X,$$
$$P_1X \times Q_2Y \cong P_2Y \times Q_2Y \cong P_2Y \oplus Q_2Y = Y,$$

and hence

$$X \cong P_1X \times Q_2Y \cong Y. \tag{29}$$

However, (29) does not imply that $P_1 \sim P_2$. A counterexample is provided by the setting $X = Y = \ell^2(\mathbb{Z})$,

$$P_1 : (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \mapsto (..., 0, 0, 0, x_1, x_2, ...),$$
$$P_2 : (..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \mapsto (..., 0, 0, x_0, 0, 0, ...).$$

Condition (29) holds because X, Y, P_1X, Q_2Y are infinite-dimensional separable Hilbert spaces, but P_1 and P_2 are clearly not equivalent.

On the reduction of general WHOs [S18]

Again we consider the general Wiener-Hopf operator (1)

$$W = P_2 A|_{P_1 X} \quad : \quad P_1 X \to P_2 Y$$

where

$$X, Y \text{ are Banach spaces,}$$
$$P_1 \in \mathcal{L}(X)\,,\ P_2 \in \mathcal{L}(Y) \text{ are projectors,}$$
$$A \in \mathcal{L}(X, Y) \text{ is a bounded linear operator.}$$

Basic questions

Under which conditions the general Wiener-Hopf operator W can be "equivalently reduced" to a "simpler" operator $\tilde{W}$ of the form (1) where at least one of the following conditions is satisfied:

$$X = Y\,, \tag{30}$$

$$P_1 = P_2\,, \tag{31}$$

$$A \in \mathcal{GL}(X, Y), \tag{32}$$

i.e., in case (32) holds, the underlying operator A is an isomorphism (boundedly invertible) or, moreover, a cross factor with respect to the setting X, Y, P_1, P_2.

Equivalent operators

Herein “equivalent reduction” of an operator T to an operator S is defined by the validity of an operator relation between T and S. We admit different kinds of relations, because the use of the term “equivalent reduction” is not unique in the literature (and sometimes not precisely indicated).

First of all we consider operator equivalence between Banach space operators in a most popular sense, written as $T \sim S$, recalling that this means: there exist isomorphisms E and F such that $T = ESF$. Hence we look for sufficient (and eventually necessary) conditions under which

$$W \sim \tilde{W} = \tilde{P}_2 \tilde{A}|_{\tilde{P}_1 \tilde{X}} \quad : \quad \tilde{P}_1 \tilde{X} \to \tilde{P}_2 \tilde{Y} \tag{33}$$

where the new setting $\tilde{X}, \tilde{Y}, \tilde{P}_1, \tilde{P}_2, \tilde{A}$ has certain prescribed properties.

Equivalence after extension

In various cases it is necessary to replace the relation $T \sim S$ by an equivalence after extension (EAE) relation abbreviated by

$$T \overset{*}{\sim} S \tag{34}$$

which stands for the fact that there exist additional Banach spaces Z_1, Z_2 and isomorphisms E and F acting between suitable spaces such that

$$\begin{pmatrix} T & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Z_2} \end{pmatrix} F \, . \tag{35}$$

This more general relation was intensively studied for instance in [BGK84, BT91, HorRan13].

On the other hand, a more special relation (rather than $T \sim S$) plays a fundamental role, namely that $W = \tilde{W}$ holds, which means that W itself can be written in a "simpler" or somehow more convenient form (33).

Transfer properties [S85, BT91, Cas98, S13, BoeS16]

The advantages of these relations are well-known: *"They allow to transfer (conclude) nice properties from one operator to another"*.

In applications it is important that inverses or generalized inverses of T and S can be computed from each other provided E, F or E^{-1}, F^{-1}, respectively, are known.

Various other relations appear in the literature and sometimes rather weak relations are used, e.g., defined by the transfer of the Fredholm property [BSD95] and/or other "transfer properties", see Section 2 in [S13]. Here we consider only relations which transfer at least the representation of generalized inverses.

Reduction to the truncation of an isomorphism

Theorem 5. *Let W be given by (1) be satisfied. Then there is a Banach space $\tilde{X}$, two projectors $\tilde{P}_1, \tilde{P}_2 \in \mathcal{L}(\tilde{X})$ and an* ***isomorphism*** *$\tilde{A} \in \mathcal{GL}(\tilde{X})$ such that*

$$W \sim \tilde{P}_2 \tilde{A}|_{\tilde{P}_1 \tilde{X}} \tag{36}$$

where $\tilde{P}_1 \tilde{X} \cong P_1 X$ and $\tilde{P}_2 \tilde{X} \cong P_2 Y$.

Remark Note that now $\tilde{A}$ is a mapping from a Banach space $\tilde{X}$ onto the same space. However, there are two in general different projectors $\tilde{P}_1, \tilde{P}_2$ involved which are not necessarily equivalent.

Example If $X = \ell^p$, $Y = \ell^q$ where $1 < p < q < +\infty$ and P_1, P_2 have finite rank, then $W = P_2 A|_{P_1 X}$ is generalized invertible for any $A \in \mathcal{L}(X, Y)$, however not representable as $W = P_2 \tilde{A}|_{P_1 X}$ with $\tilde{A} \in \mathcal{GL}(X, Y)$ since the two spaces are not isomorphic. It is not hard to see that $\tilde{X} = X \times Y$ helps to find operators A, P_1, P_2 that satisfy (36).

Extended symmetrization criterion

Theorem 6. *Given a space setting X, Y, P_1, P_2 such that $X \cong Y$. Then the following statements are equivalent:*

1. *The setting X, Y, P_1, P_2 is symmetrizable;*
2. $P_1 X \cong P_2 Y$ *and* $Q_1 X \cong Q_2 Y$;
3. $P_1 \sim P_2$;
4. *There exists an invertible WHO of the form (1) with $A \in \mathcal{GL}(X, Y)$;*
5. *There exists an invertible WHO of the form $Q_1 B|_{Q_2 Y} : Q_2 Y \to Q_1 X$ where $P_1 + Q_1 = I|_X$ and $P_2 + Q_2 = I|_Y$ and $B \in \mathcal{GL}(Y, X)$.*

Remarks

1. It is known (and easily verified) that the two associated WHOs $W = P_2A|_{P_1X}$ and $Q_1A^{-1}|_{Q_2Y}$ are matricially coupled [S83, BGK84] and therefore equivalent after extension [BGK84, BT91]. This has tremendous consequences in operator theory and important applications in mathematical physics [HorRan13, CDS14]. Only recently an explicit EAE relation between associated WHOs was given in the asymmetric case, which allows to derive a formula for a generalized inverse of W in a most convenient way from a formula of a generalized inverse of W_* see Formula (1.11) in [S17]:

$$\begin{pmatrix} W & 0 \\ 0 & I|_{Q_1X} \end{pmatrix} = \begin{pmatrix} -P_2A|_{Q_1X} & I|_{P_2Y} - P_2AQ_1A^{-1}|_{P_2Y} \\ I|_{Q_1X} & Q_1A^{-1}|_{P_2Y} \end{pmatrix} \tag{37}$$

$$\begin{pmatrix} W_* & 0 \\ 0 & I|_{P_2Y} \end{pmatrix} \begin{pmatrix} Q_2A|_{P_1X} & Q_2A|_{Q_1X} \\ P_2A|_{P_1X} & P_2A|_{Q_1X} \end{pmatrix}$$

$$: \quad Q_1X \times P_2Y \ \leftarrow \ Q_1X \times P_2Y \ \leftarrow \ P_2Y \times Q_2Y \ \leftarrow \ P_1X \times Q_1X$$

2. It is also known that the conditions of the previous Theorem characterize exactly the Banach space settings where the FIS Theorem holds: W is generalized invertible iff A admits a Wiener-Hopf factorization through an intermediate space.

A result involving equivalence after extension

We study the case where the setting X, Y, P_1, P_2 is not symmetrizable but it satisfies a weaker condition, namely $P_1 \overset{*}{\sim} P_2$ instead of $P_1 \sim P_2$.

Theorem 7. *Given a setting* X, Y, P_1, P_2 *let* $X = Y$, $P_1 \overset{*}{\sim} P_2$, *and* $\operatorname{im} P_1 \cap \ker P_2 = \{0\}$. *Then*

$$Z \;=\; \operatorname{im} P_1 \times \ker P_2 \;\cong\; \operatorname{im} P_1 \oplus \ker P_2 \;=\; X\,. \tag{38}$$

Hence the projector $P \in \mathcal{L}(X)$ *onto* $\operatorname{im} P_1$ *along* $\ker P_2$ *exists and*

$$W \;=\; P_2 A|_{P_1 X} \;\sim\; PA|_{PX} \tag{39}$$

holds for any $A \in \mathcal{L}(X)$.

Reduction of W by equivalence after extension

Final question: When is W reducible by EAE to a truncation of a cross factor? (In a case where A is not necessarily invertible.)

Theorem 8. *Let W be given by (1). Then W is generalized invertible if and only if*

$$W \overset{*}{\sim} W_C = \tilde{P}_2 C|_{\tilde{P}_1 \tilde{X}} : \tilde{P}_1 \tilde{X} \to \tilde{P}_2 \tilde{Y} \tag{40}$$

with a suitable space setting $\tilde{X}, \tilde{Y}, \tilde{P}_1, \tilde{P}_2$ and a cross factor C.

For further results and details see [S18].

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APPENDIX

Proofs of main results

Constructing a FRF of W from a FIS of A

Proof of Theorem 1

Sufficiency. As in many other cases, this step is just a verification of the formula for the inverse. If we have a canonical FIS, then

$$\begin{aligned} WW^- &= P_2A_-A_+|_{P_1X}\, A_+^{-1}PA_-^{-1}|_{P_2Y} = P_2A_-A_+P_1\, A_+^{-1}PA_-^{-1}|_{P_2Y} \\ &= P_2A_-\, PA_-^{-1}|_{P_2Y} = I|_{P_2Y}\,. \end{aligned} \tag{41}$$

Similarly we see that $W^-W = I|_{P_1X}$.

Necessity. Let W be invertible. We identify A with an equivalent operator matrix and factor this straightforwardly

$$\begin{aligned} A \quad &\sim \quad \tilde{A} = \begin{pmatrix} P_2A|_{P_1X} & P_2A|_{Q_1X} \\ Q_2A|_{P_1X} & Q_2A|_{Q_1X} \end{pmatrix} : P_1X \times Q_1X \to P_2Y \times Q_2Y \\ &= \begin{pmatrix} I|_{P_2X} & 0 \\ Q_1A^{-1}|_{P_2X} & Q_1A^{-1}|_{Q_2X} \end{pmatrix}^{-1} \begin{pmatrix} P_2A|_{P_1X} & P_2A|_{Q_1X} \\ 0 & I|_{Q_1X} \end{pmatrix} \\ &= \tilde{A}_-\, \tilde{A}_+ : P_1X \times Q_1X \to P_2Y \times Q_1X \to P_2Y \times Q_2Y. \end{aligned} \tag{42}$$

With the above-mentioned identification of the direct sum $P_1X\dot{+}Q_1X$ and the product space $P_1X \times Q_1X$ (in the algebraic and topological sense) we obtain a factorization of $A = A_-\,A_+$ through $Z = P_2Y \times Q_1X$ because the invertibility of W implies that (dropping the tildes)

$$A_+ \quad = \quad \begin{pmatrix} I_{P_2Y} & P_2A|_{Q_1X} \\ 0 & I_{Q_1X} \end{pmatrix} \quad \begin{pmatrix} I_{P_2Y}A|_{P_1X} & 0 \\ 0 & I_{Q_1X} \end{pmatrix} \tag{43}$$

is invertible. The calculation

$$\begin{aligned} A_-^{-1} \quad &= \quad A_+ \begin{pmatrix} P_1A^{-1}|_{P_2X} & P_1A^{-1}|_{Q_2X} \\ Q_1A^{-1}|_{P_2X} & Q_1A^{-1}|_{Q_2X} \end{pmatrix} \\ &= \quad \begin{pmatrix} I|_{P_2X} & 0 \\ Q_1A^{-1}|_{P_2X} & Q_1A^{-1}|_{Q_2X} \end{pmatrix} \end{aligned}$$

shows that $A = A_-A_+$ where A_- is invertible, as well. Finally, the factor properties of $A_\pm$ are obvious from the foregoing formulas. $\square$

This direct proof, say, has an alternative contained in the following proof, by reduction to a symmetric WHO.

Proof of Theorem 2

Sufficiency. If a FIS is given, we define

$$W^{-} = A_{+}^{-1} P C^{-1} P A_{-}^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X\,. \tag{44}$$

Now we verify $WW^{-}W = W$ by calculations similar to the previous and with the help of diagram (4) and by analogy to the calculations in case of a CFn, see [S85], p. 27-29.

The factor properties of A_{+} imply $P_1 \sim P$ and the factor properties of A_{-} imply $P_2 \sim P$, therefore $P_1 \sim P_2$ is necessarily satisfied.

Necessity. Since $P_1 \sim P_2$, we can confine ourselves to the symmetric case where $P_2 = P_1$ by splitting an isomorphism from A which maps P_2Y onto P_1X and Q_2Y onto Q_1X. Note that two bounded projectors in Banach spaces are equivalent if and only if their kernels are isomorphic and their co-kernels are isomorphic, as well [BT91]. Hence consider an operator of the form $W = P_1A|_{P_1X}$. This can be considered as an element of the form $w = pap$ in the unital ring $\mathcal{R} = \mathcal{L}(X)$ where p is idempotent and a invertible.

From [S85] we know that the regularity of w [Neu36], i.e., existence of an element $v \in \mathcal{R}$ with $pvp = v$ and $wvw = w$ implies a ring cross factorization given by, e.g.,

$$\begin{aligned} a &= a_- \, c \, a_+ \qquad (45) \\ &= [e + qav][a - ava + w + a(p - vw)a^{-1}(p - wv)a] \\ &\qquad \cdot \, [e + vaq - (p - vw)a^{-1}(p - wv)a] \, , \end{aligned}$$

see formula (6.7a) in [S85]. This can be interpreted as a cross factorization of A which coincides with a FIS through the intermediate space $Z = X$ in this symmetric space setting ($X = Y$).

Non-redundance. We give an example where W is generalized invertible, but A does not admit a FIS in a case where $P_1 \sim P_2$ is violated. Let $X = Y = \mathbb{R}^3, P_1x = (x_1, 0, 0), P_2 = (x_1, x_2, 0)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. If $A \in \mathcal{L}(\mathbb{R}^3)$ is any invertible operator (real linear transformation), W has finite rank (namely rank 0 or 1) and it is therefore generalized invertible, but obviously $P_1 \sim P_2$ is not satisfied, because their kernels are not isomorphic. □

Proof of Theorem 3

Starting with a canonical unbounded FIS $A = A_- A_+$ we put $Z_1 = \operatorname{im} A_+|_{\operatorname{dom} A_+}$ equipped with the norm induced by X:

$$\|z\|_{Z_1} = \|A_+^{-1} z\|_X \tag{46}$$

and define Z by taking the closure of Z_1 which is obviously isomorphic to the Banach space X. Now $Z_2 = \operatorname{im} A_-^{-1}|_{\operatorname{dom} A_-^{-1}}$ yields the same result since $X \cong Y$ because of the assumption that A is invertible. Namely:

$$\|z\|_{Z_1} = \|A_+^{-1} z\|_X \sim \|A A_+^{-1} z\|_Y = \|A_- z\|_Y \tag{47}$$

in the sense of equivalent norms. □

Proof of the Corollary

The only missing step is the conclusion that the generalized invertibility of T yields a full range factorization of T. Hence, let T^- be a generalized inverse of $T \in \mathcal{L}(X, Y)$ and

$$\operatorname{Rst} T \quad : \quad X \;\to\; \operatorname{im} T \tag{48}$$

the image restricted operator, considered as an operator acting not into Y but onto $\operatorname{im} T = TX$. Then

$$\begin{array}{ccccccccc} T & = & T\,T^-\,T & = & & (T\,T^-)|_{\operatorname{im} T} & & \operatorname{Rst} T & \\ & & & & Y & \longleftarrow & \operatorname{im} T & \longleftarrow & X \end{array} \tag{49}$$

represents obviously a full range factorization through $Z = \operatorname{im} T$. Another one would be

$$\begin{array}{ccccccccc} T & = & T\,T^-\,T & = & & T|_{X_1} & & \operatorname{Rst}(T^-\,T) & \\ & & & & Y & \longleftarrow & X_1 & \longleftarrow & X \end{array} \tag{50}$$

where the intermediate space $X_1 = \operatorname{im} T^-$ is a complement of the kernel of T. $\square$

Constructing a FRF of W from a FIS of A

We study the question: How can a FIS be employed to construct a FRF of W in a more direct (constructive) way than via a generalized inverse? In general the construction of a FRF of W is a difficult task and not much treated in the literature, see [S83] where a so-called weak factorization was used and the complicated interaction between the two factors was pointed out.

Looking at the symmetric situation $W = PA|_{PX}$, $A \in \mathcal{GL}(X)$, $P^2 = P \in \mathcal{L}(X)$, a weak factorization $A = B_- B_+$ is characterized by

$$B_\pm \in \mathcal{GL}(X) \quad , \quad B_+ P = PB_+ P \quad , \quad PB_- = PB_- P \, , \tag{51}$$

i.e., B_+ maps PX into PX and B_- maps the complement QX into QX. This yields

$$W = PB_- B_+|_{PX} = PB_-|_{PX} \, PB_+|_{PX} = W_- W_+ \tag{52}$$

where W_- is right invertible and W_+ is left invertible. I.e., we do not have a FRF and the consequences in general are poor.

However, in more special situations, the two operators W_- and W_+ commute. It happens typically in the case of classical Toeplitz and Wiener-Hopf operators. Looking again at Example 1 we observe that the (reduced) WHO

$$T \quad = \quad PC|_{PX} \quad = \quad P\,\mathrm{diag}\,(z^{\kappa_1}, ..., z^{\kappa_n})|_{PX} \tag{53}$$

has also this property: Writing

$$T \quad = \quad T_- T_+ \quad = \quad P\,\mathrm{diag}\,(z^{\kappa_1^-}, ..., z^{\kappa_n^-})|_{PX}\, P\,\mathrm{diag}\,(z^{\kappa_1^+}, ..., z^{\kappa_n^+})|_{PX} \tag{54}$$

where $\kappa_j^+ = \max\{\kappa_j, 0\}$, $\kappa_j^- = \min\{\kappa_j, 0\}$, we see that T_- and T_+ commute. So we arrive at the conclusion that any Φ-factorization of a measurable matrix function can be easily transformed into a FRF of W, which can be also seen as a consequence of the general version:

Corollary. Let W be given as before and $A = A_- C A_+$ be a FIS where $PC|_{PX} = \operatorname{diag}(T_1, ..., T_n)$ and all T_j are one-sided invertible. Then a FRF of W is given by

$$W \quad = \quad (P_2 A_- C_+|_{PZ})\,(PC_- A_+|_{P_1 X} \qquad (55)$$

with $PC_+|_{PZ}$ right invertible and $PC_+|_{PZ}$ left invertible.
Note that the knowledge of a CFn instead of a FIS does not suffice because the commutativity of the two middle factors is needed.