Lecture 1: Modular forms and Topology

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Outline

- Background
  - Stable homotopy groups of spheres
  - Cohomology theories
  - Elliptic curves and modular forms
- What is TMF?
  - Elliptic cohomology
  - Definition of TMF
  - Relationship to modular forms
- Computational Applications of TMF
  - Hurewicz image
  - v_2-self maps
  - Greek letter elements
- Geometry
  - Witten genus
  - Derived algebraic geometry

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  - Derived algebraic geometry

Central problem in algebraic topology: compute $\pi_i(S^n)$

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$
5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

$\pi_i(S^n)$

$n \downarrow$ / $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
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8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

$\pi_{<k}(S^k)$ — $\pi_k(S^k)$ — $\pi_{>k}(S^k)$

$\pi_i(S^n)$

	$i \rightarrow$ 1	2	3	4	5	6	7	8	9	10	11	12
$n \downarrow$ 1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
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7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0
8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

- Mostly torsion
- only $\pi_n S^n$, $\pi_{4n-1} S^{2n}$ contain $\mathbb{Z}$ summands

$\pi_i(S^n)$

$n \downarrow$ \ $i \rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12
1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
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8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

Values stabilize along diagonals:

$$\pi_{n+k}(S^k) = \pi_{n+k+1}(S^{k+1}) \text{ for } k \gg 0$$

$\pi_i(S^n)$

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1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0
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8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0

π^s_0 π^s_1 π^s_2 π^s_3 π^s_4

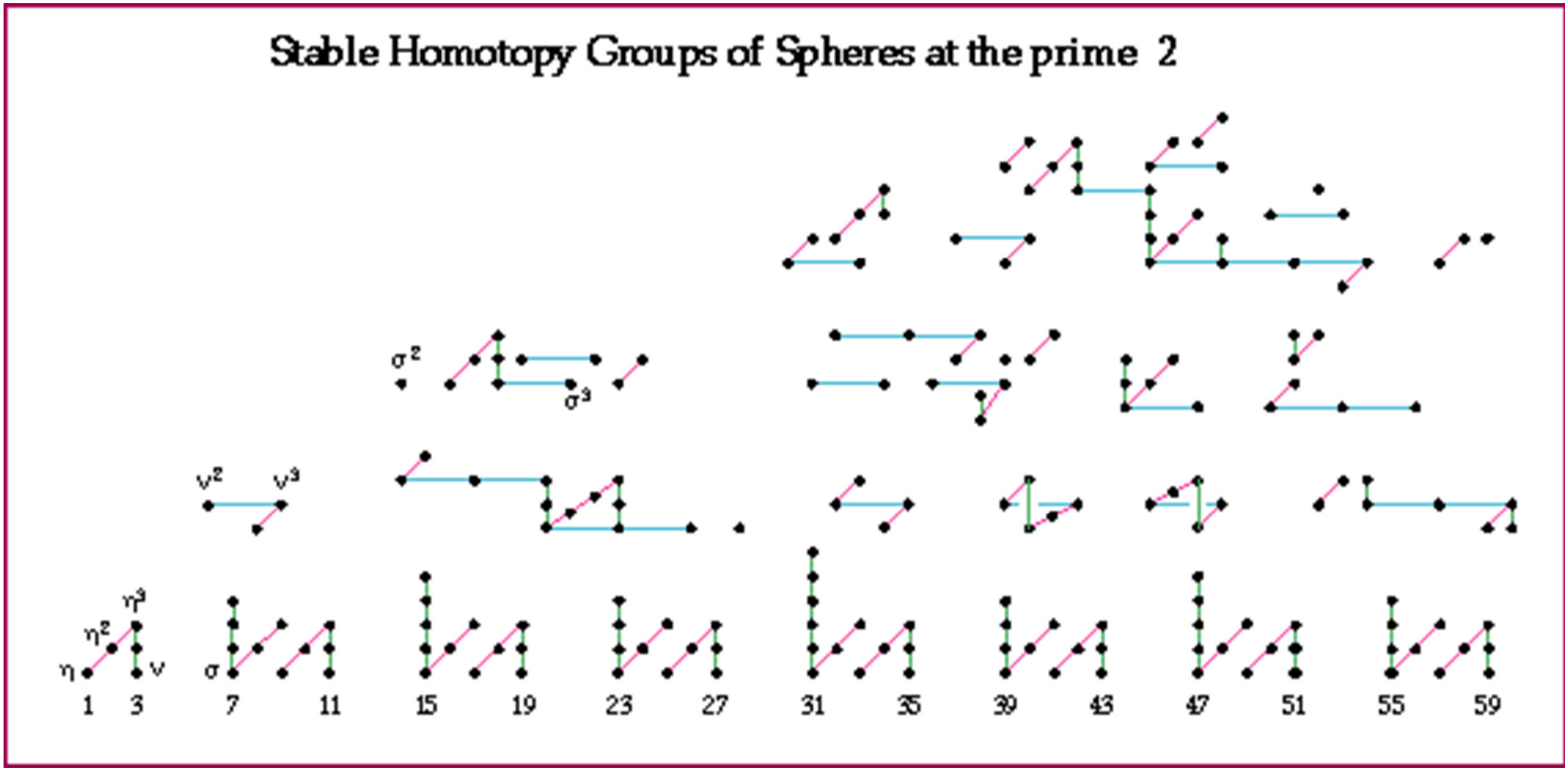
Stable homotopy groups:

$$\pi_n^s := \lim_{k\to\infty} \pi_{n+k}(S^k) \quad \text{(finite abelian groups for } n > 0\text{)}$$

Primary decomposition:

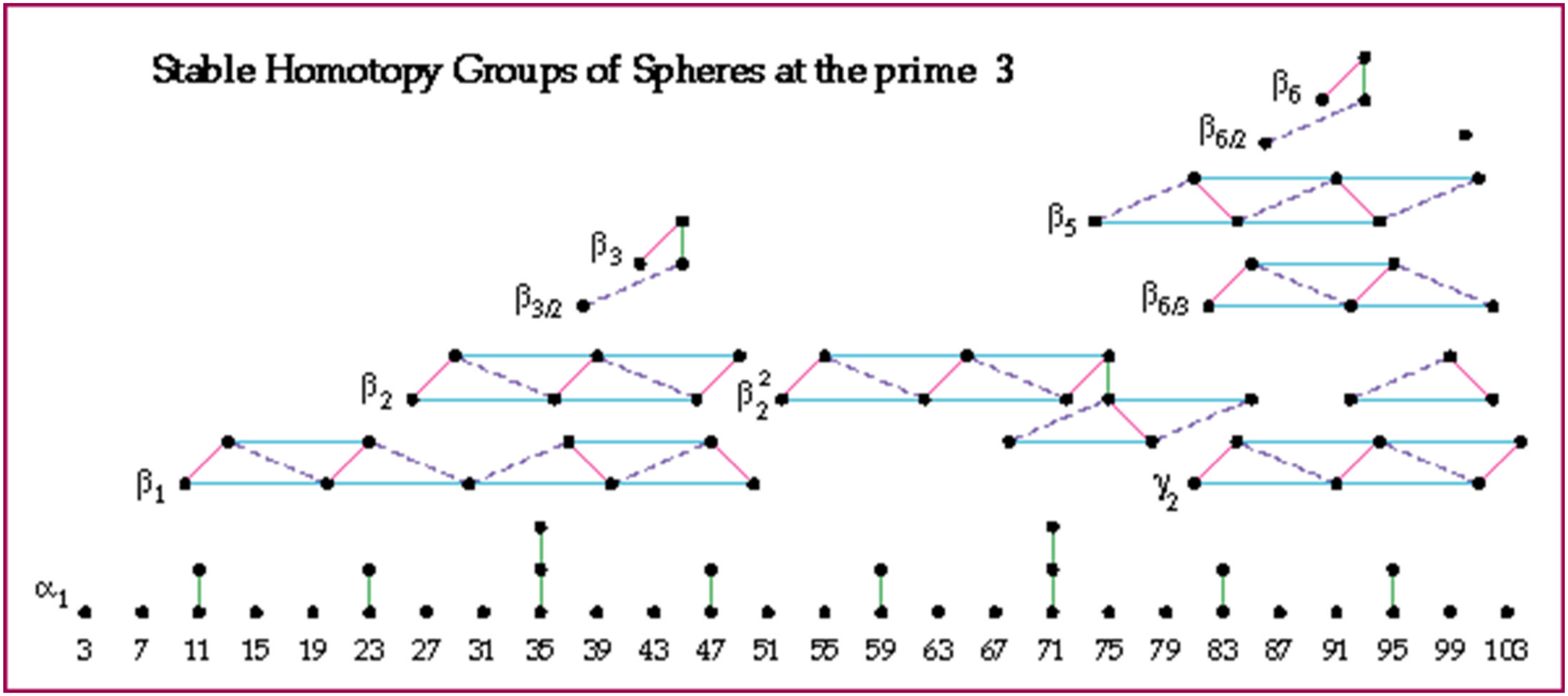
$$\pi_n^s = \bigoplus_{p\ prime} (\pi_n^s)_{(p)}$$

e.g.: $\pi_3^s = \mathbb{Z}_{24} = \mathbb{Z}_8 \oplus \mathbb{Z}_3$



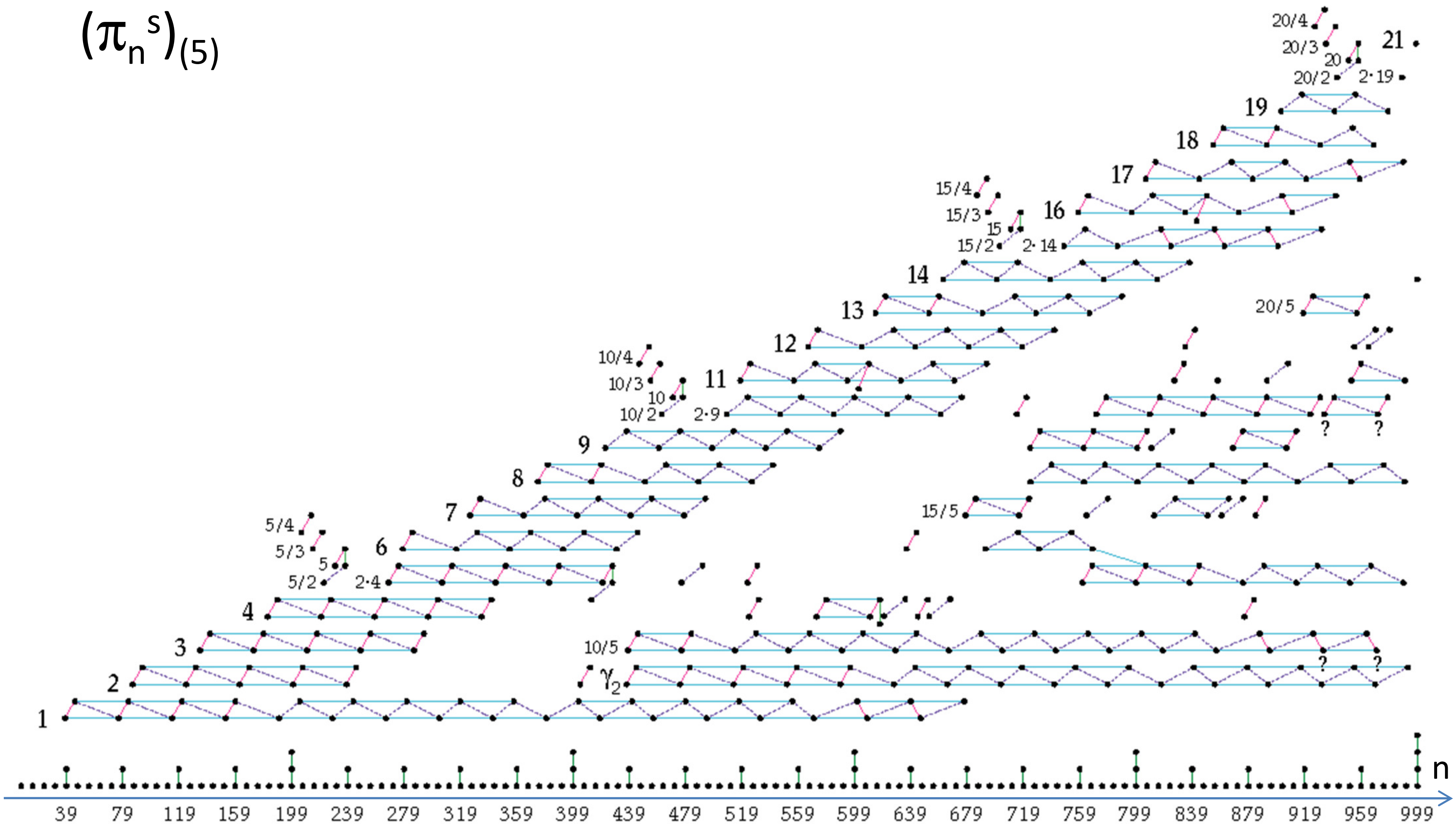
- Each dot represents a factor of 2, vertical lines indicate additive extensions

  e.g.: $(\pi_3^s)_{(2)} = \mathbb{Z}_8$, $(\pi_8^s)_{(2)} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- Vertical arrangement of dots is arbitrary, but meant to suggest patterns



Stable Homotopy Groups of Spheres at the prime 3
β6
β6/2
β5
β3
β6/3
β3/2
β2
β2 2
β1
γ2
α1
3 7 11 15 19 23 27 31 35 39 43 47 51 55 59 63 67 71 75 79 83 87 91 95 99 103

$(\pi_n^s)_{(5)}$



21
20/4
20/3
20
20/2
2·19
19
18
17
16
15/4
15/3
15
15/2
2·14
14
13
12
11
10/4
10/3
10
10/2
2·9
9
8
7
6
5/4
5/3
5
5/2
2·4
4
3
2
1
20/5
15/5
10/5
γ_2
?
n
39 79 119 159 199 239 279 319 359 399 439 479 519 559 599 639 679 719 759 799 839 879 919 959 999

Chromatic theory

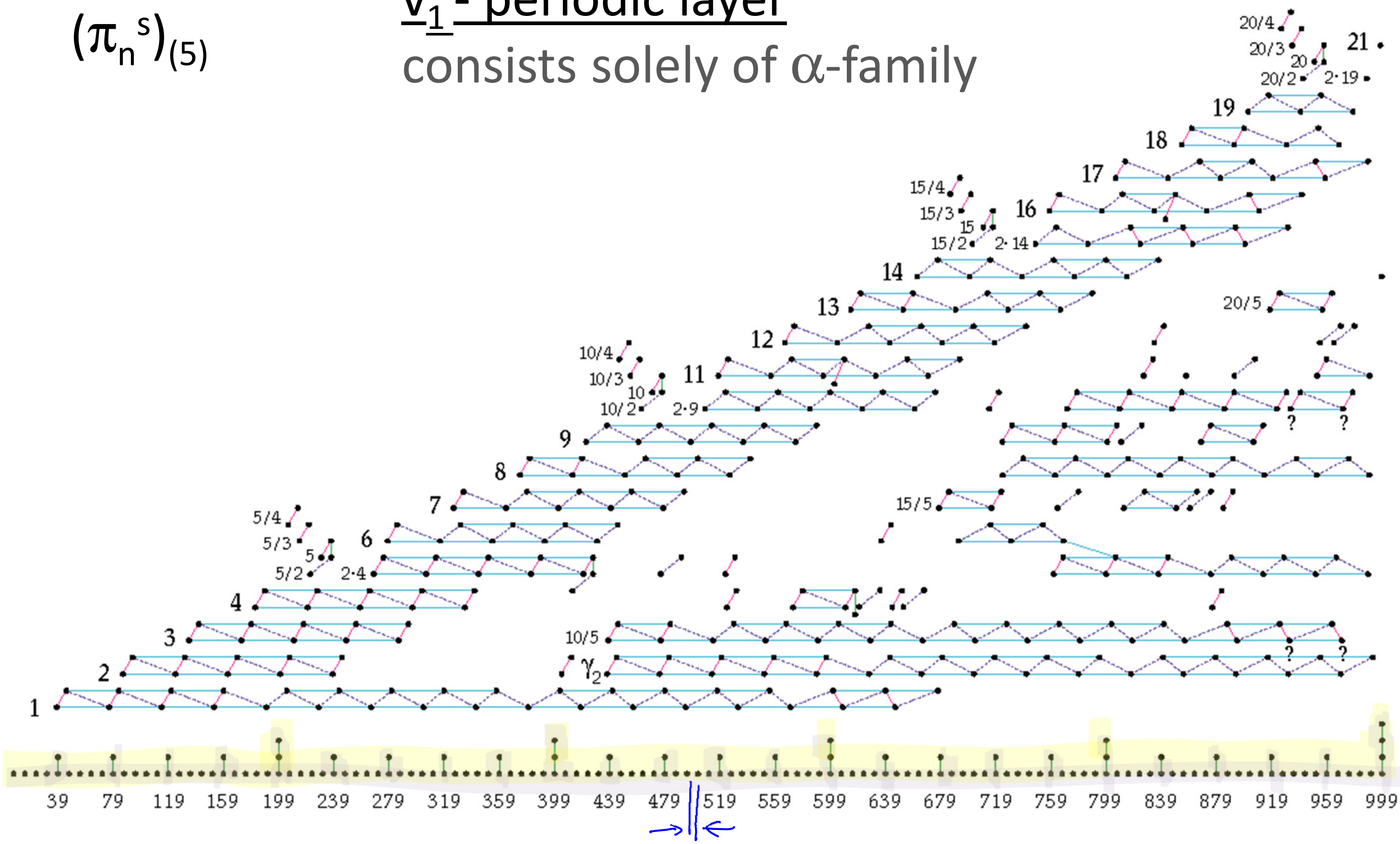
- $(\pi_k^s)_{(p)}$ is built out of *chromatic layers*
- The elements of the n^{th} layer fit into periodic families (*v_n – periodicity*)
- Important such families are the "Greek letter families" (α, β, γ...)
- The generic period in the n^{th} chromatic layer is $2(p^n-1)$
- It is likely no human will know all of the stable homotopy groups of spheres, but it is possible to completely compute a chromatic layer

v_1-periodic:	completely understood	(α – family)
v_2-periodic:	subject of recent work	(β – family)
v_3 and higher:	virtually unknown	(γ – family and higher)

$(\pi_n{}^s)_{(5)}$

v_1 - periodic layer

consists solely of α-family



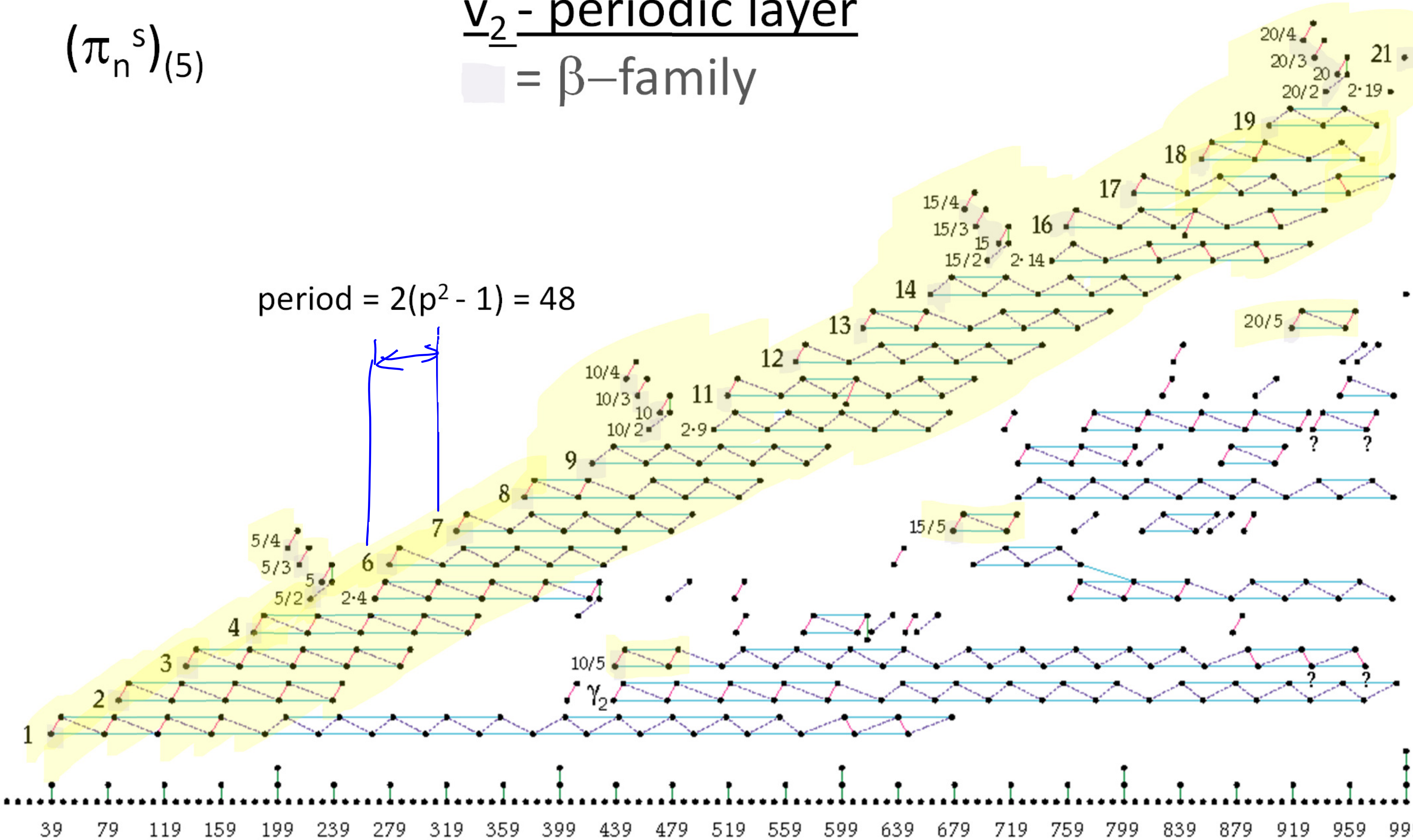
period = 2(p-1) = 8

$(\pi_n^s)_{(5)}$

v_2 - periodic layer

= β–family

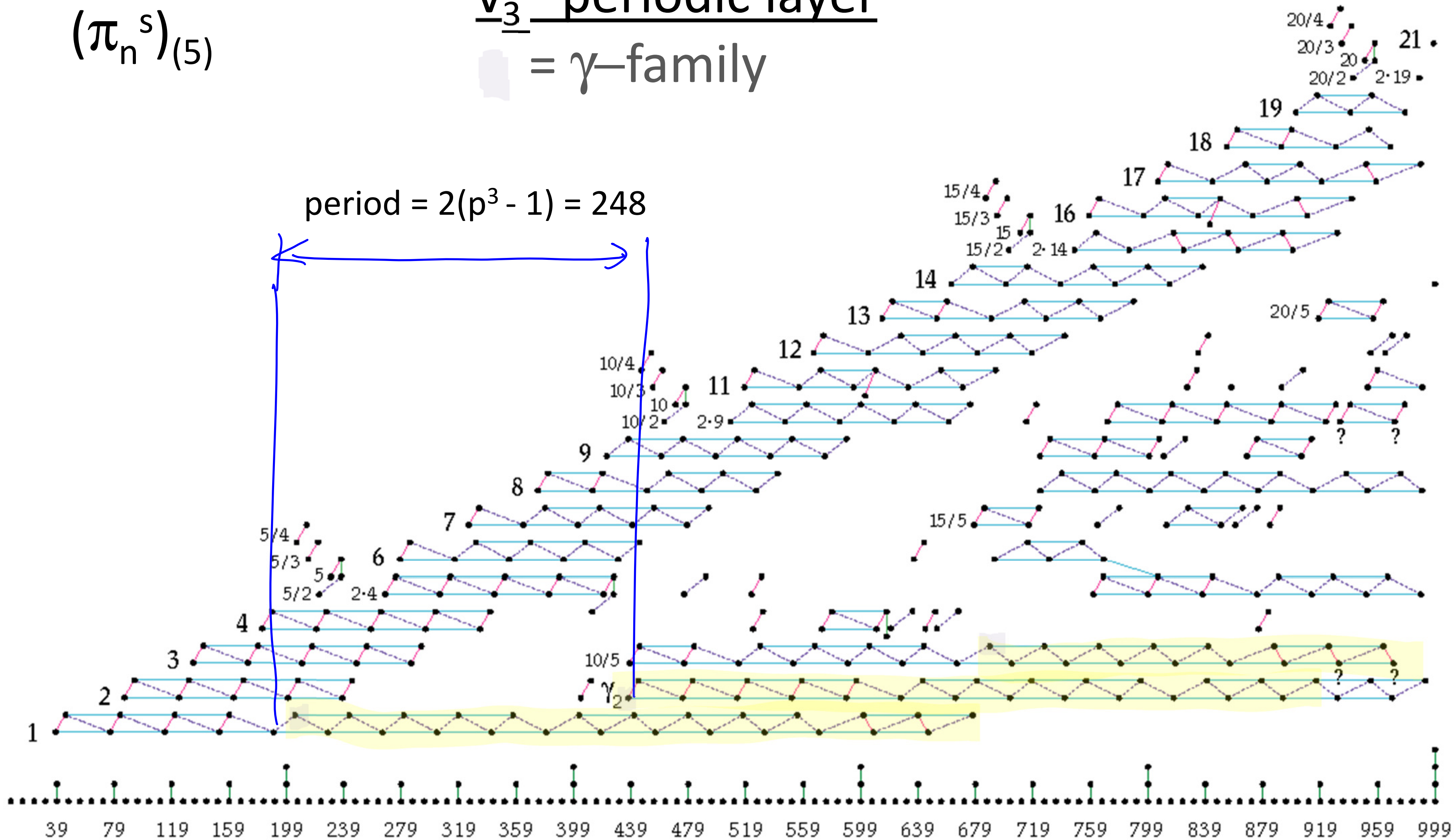
period = $2(p^2 - 1) = 48$



$(\pi_n^s)_{(5)}$

v_3 - periodic layer

= γ–family



Cohomology theories

- Use homology/cohomology to study homotopy
- A *cohomology theory* is a contravariant functor

$$E: \{\text{Topological spaces}\} \longrightarrow \{\text{graded ab groups}\}$$
$$X \longrightarrow E^*(X)$$

- Homotopy invariant: $f \simeq g \Rightarrow E(f) = E(g)$
- Excision: $Z = X \cup Y$ (CW complexes)

$$\cdots \to E^*(Z) \to E^*(X) \oplus E^*(Y) \to E^*(X \cap Y) \to$$

Cohomology theories

- Cohomology theories are representable by *spectra:*
  - A sequence of pointed spaces $\{\underline{E_n}\}$ so that $E^n(X) = [X, \underline{E_n}]$.
  - Consequence of excision: $\underline{E_n} \simeq \Omega \underline{E_{n+1}}$

- Homotopy groups:

$$\pi_n(E) \coloneqq \pi_{n+k}(\underline{E_k}) = E^{-n}(pt)$$

(Note, in the above, n may be negative)

Cohomology theories

- Example: singular cohomology
  - $E^n(X) = H^n(X)$
  - $\underline{H}_n = K(\mathbb{Z}, n)$
  - $\pi_n(H) = \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & \text{else.} \end{cases}$

- Example: K-theory
  - $K^0(X) = K(X) =$ Grothendieck group of $\mathbb{C}$-vector bundles over X.
  - $\underline{K}_{2n} = BU \times \mathbb{Z}, \qquad \underline{K}_{2n+1} = U.$
  - $\pi_n K = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

Hurewicz Homomorphism

- A spectrum E is a (commutative) *ring spectrum* if its associated cohomology theory has "cup products"

$E^*(X)$ is a graded commutative ring

- Such spectra have a *Hurewicz homomorphism*:

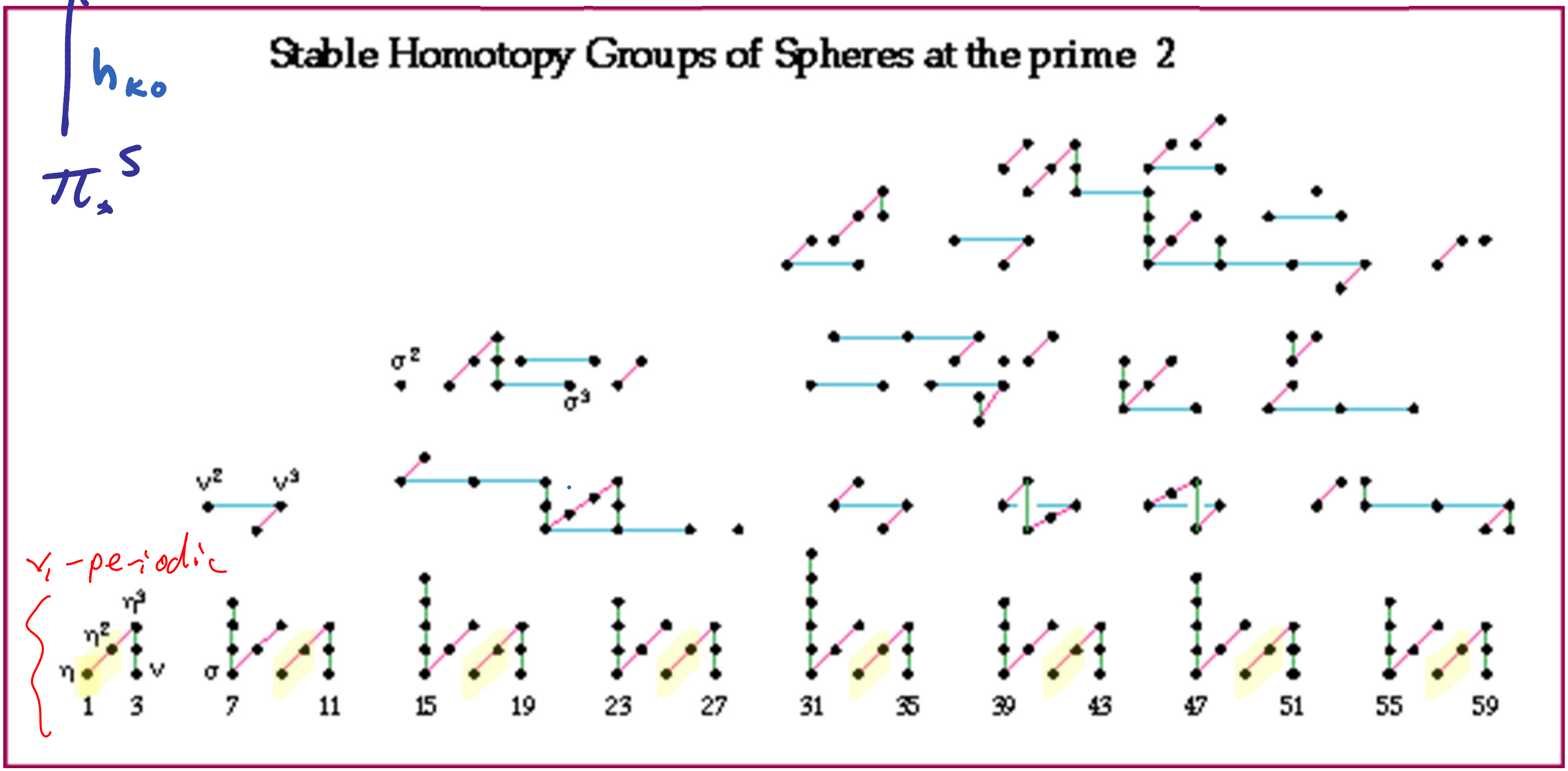
$$h_E: \pi_*^S \to \pi_* E$$

Example: H detects $\pi_0^S = \mathbb{Z}$.

Example: KO (real K-theory)

$$\pi_* KO = \mathbb{Z} \;\; \mathbb{Z}/2 \;\; \mathbb{Z}/2 \;\; 0 \;\; \mathbb{Z} \;\; 0 \;\; 0 \;\; 0 \;\; \mathbb{Z} \;\; \mathbb{Z}/2 \;\; \mathbb{Z}/2 \;\; 0 \;\; \mathbb{Z} \ldots$$

$$\pi_*^S \xrightarrow{h_{KO}} \pi_* KO$$



Chern classes and formal groups

- A ring spectrum E is said to be *complex orientable* if complex vector bundles are orientable in E-cohomology (have a Thom class)
- If E is complex orientable, it has *Chern classes*

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \rightsquigarrow \quad c_n^E \in E^{2n}(X)$$

- The *formal group* is the formal power series

$$F_E(x,y) \in \pi_*(E)[[x,y]]$$

defined by the relation on line bundles:

$$c_1^E(L \otimes L') = F_E(c_1^E(L), c_1^E(L'))$$

Chern classes and formal groups

- Example: $E = H$

$$F_E(x, y) = x + y \quad \text{(additive)}$$

- Example: $E = K$

$$F_E(x, y) = x + y + xy \quad \text{(multiplicative)}$$

(power series expansion of multiplication near 1 in the multiplicative group $\mathbb{G}_m$)

Topological modular forms and elliptic cohomology: the rough idea

K-theory:

$\mathbb{G}_m$ (Multiplicative group) $\rightsquigarrow$ (Formal nbhd of 1) Formal gp $\rightsquigarrow$ K

Elliptic cohomology:

C (Elliptic Curve) $\rightsquigarrow$ (Formal nbhd of identity) Formal Gp $\rightsquigarrow$ E_C

Topological modular forms and elliptic cohomology: the rough idea

- A *modular form* *f* associates to each elliptic curve a number

$$C \longmapsto f(C) \in \mathbb{C}$$

- The cohomology theory of *Topological Modular Forms* (TMF) consists of the following association: a cohomology class

$$\alpha \in TMF^n(X)$$

associates to every elliptic curve C a cohomology class in its associated elliptic cohomology theory:

$$C \longmapsto \alpha(C) \in E_C^n(X)$$

Elliptic Curves and modular forms: a brief review

- An elliptic curve over a ring R is a genus 1 curve over R (with a marked point)

- An elliptic curve over $\mathbb{C}$ is always of the form

$$\mathbb{C}/\Lambda$$

for some lattice $\Lambda \subset \mathbb{C}$.

- Elliptic curves are groups

(with identity the marked point)

- An elliptic curve has an associated formal group

$$F_c(x, y) \in R[\![x, y]\!]$$

(obtained by taking power series expansion of multiplication law at the identity)

Elliptic Curves and modular forms: a brief review

A *modular form* (of weight *k*) over *R* is a rule f which assigns to each tuple (C, v, R') with

- $R' =$ an R-algebra
- $C =$ an elliptic curve over R'
- $v =$ a non-zero tangent vector at the identity of C

an element:

$$f(C, v) \in R'$$

such that:

$$f(C, \lambda v) = \lambda^k f(C, v), \quad \lambda \in (R')^\times$$

Let $[M_k]_R$ denote the space of modular forms of weight k over R

Elliptic Curves and modular forms: a brief review

"High-brow perspective": sections of a line bundle

$$\begin{array}{ccc} \omega & & \omega_C = \mathrm{Lie}^* C \\ \downarrow & & \downarrow \\ \mathcal{M}_{ell} & \ni & C \end{array}$$

$$[M_k]_{\mathbb{Z}} = H^0\left(\mathcal{M}_{ell}; \omega^{\otimes k}\right)$$

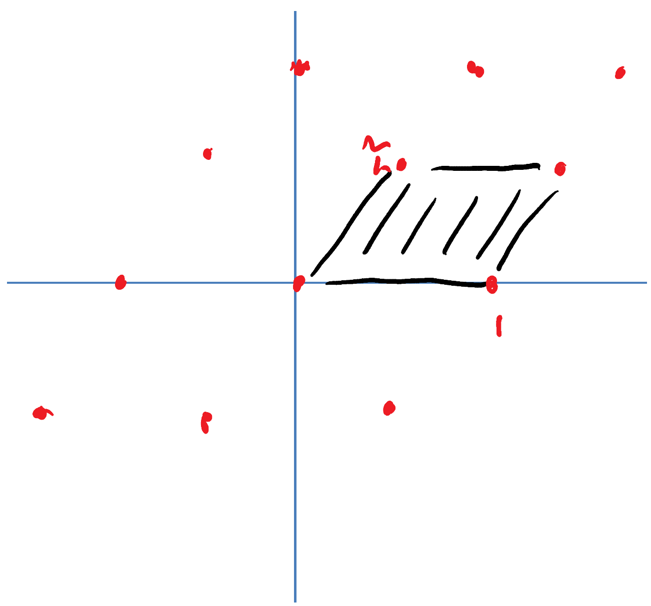
$\mathcal{M}_{ell}$ = Moduli space of elliptic curves

Elliptic Curves and modular forms: a brief review

"Low-brow perspective": functions on the upper half-plane

Over the complex numbers, every elliptic curve is isomorphic to

$$C_\tau = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}} \qquad \tau \in \mathcal{H}$$



$$v = 1 \longrightarrow v = \frac{1}{c\tau+d}$$

$$C_\tau \cong C_{\tau'}$$

$$\iff$$

$$\tau' = \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Elliptic Curves and modular forms: a brief review

If $R \subseteq \mathbb{C}$, a modular form $f \in [M_k]_R$ gives a holomorphic function on $\mathcal{H}$

$$f(\tau) = f(C_\tau, 1)$$

We therefore have:

$$f(\tau) = \frac{1}{(c\tau+d)^k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Elliptic Curves and modular forms: a brief review

Taking the matrix:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$$

we have

$$f(\tau) = f(\tau+1)$$

Thus f admits a Fourier expansion (q expansion)

$$f(\tau) = \sum a_n q^n \qquad q := e^{2\pi i \tau}$$

We also require $a_n = 0$ for $n < 0$. (f defined over R => $a_n \in$ R)

Elliptic Curves and modular forms: a brief review

Example: Eisenstein series: $E_{2k} \in [M_{2k}]_{\mathbf{Q}}$

$$E_{2k}(\tau) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(c\tau+d)^{2k}}$$

$$= -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \Big(\sum_{d|n} d^{2k-1} \Big) q^n$$

Example:

$$c_4 := 240 E_4$$

$$c_6 := -504 E_6$$

$$\Delta := \frac{c_4^3 - c_6^2}{1728} \quad (*)$$

$$[M_*]_{\mathbb{Z}} = \mathbb{Z}[c_4, c_6, \Delta] / (*)$$

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Elliptic Cohomology theories

Def: An Elliptic spectrum is a tuple

$$(E_C, C, \alpha)$$

Where:

- E_C is a commutative ring spectrum
- $\pi_* E = R[u, u^{-1}], \quad |u| = 2, \ \ R = \pi_0 E.$
- C is an elliptic curve over R.
- $\alpha: F_C \to F_E$ is an isomorphism of formal groups

Topological Modular Forms

Unfortunately, not every elliptic curve has an associated elliptic cohomology theory. However…

Thm (Goerss-Hopkins-Miller)

There exists a sheaf of commutative ring spectra $\mathcal{O}_{ell}$ on the etale site of $\mathcal{M}_{ell}$.

$$\mathcal{O}_{ell}\left(\begin{array}{c} \mathrm{Spec}(R) \\ \downarrow c \\ \mathcal{M}_{ell} \end{array}\right) = E_c$$

This theorem functorially associates elliptic cohomology theories to elliptic curves which are etale over $\mathcal{M}_{ell}$.

Topological Modular Forms

- Should think of $\mathcal{O}_{ell}$ as a topological version of the sheaf

$$\omega^{\otimes *} = \bigoplus_{k \in \mathbb{Z}} \omega^{\otimes k}$$

- Define

$$TMF := \Gamma \mathcal{O}_{ell}$$

- Analogous to $[M_*]_{\mathbb{Z}} = \Gamma \omega^{\otimes *}$

TMF is the "mother of all elliptic cohomology theories"

Topological Modular Forms

- There is a descent spectral sequence:

$$H^s\left(\mathcal{M}_{ell};\omega^{\otimes t}\right) \Rightarrow \pi_{2t-s}TMF$$

- Edge homomorphism:

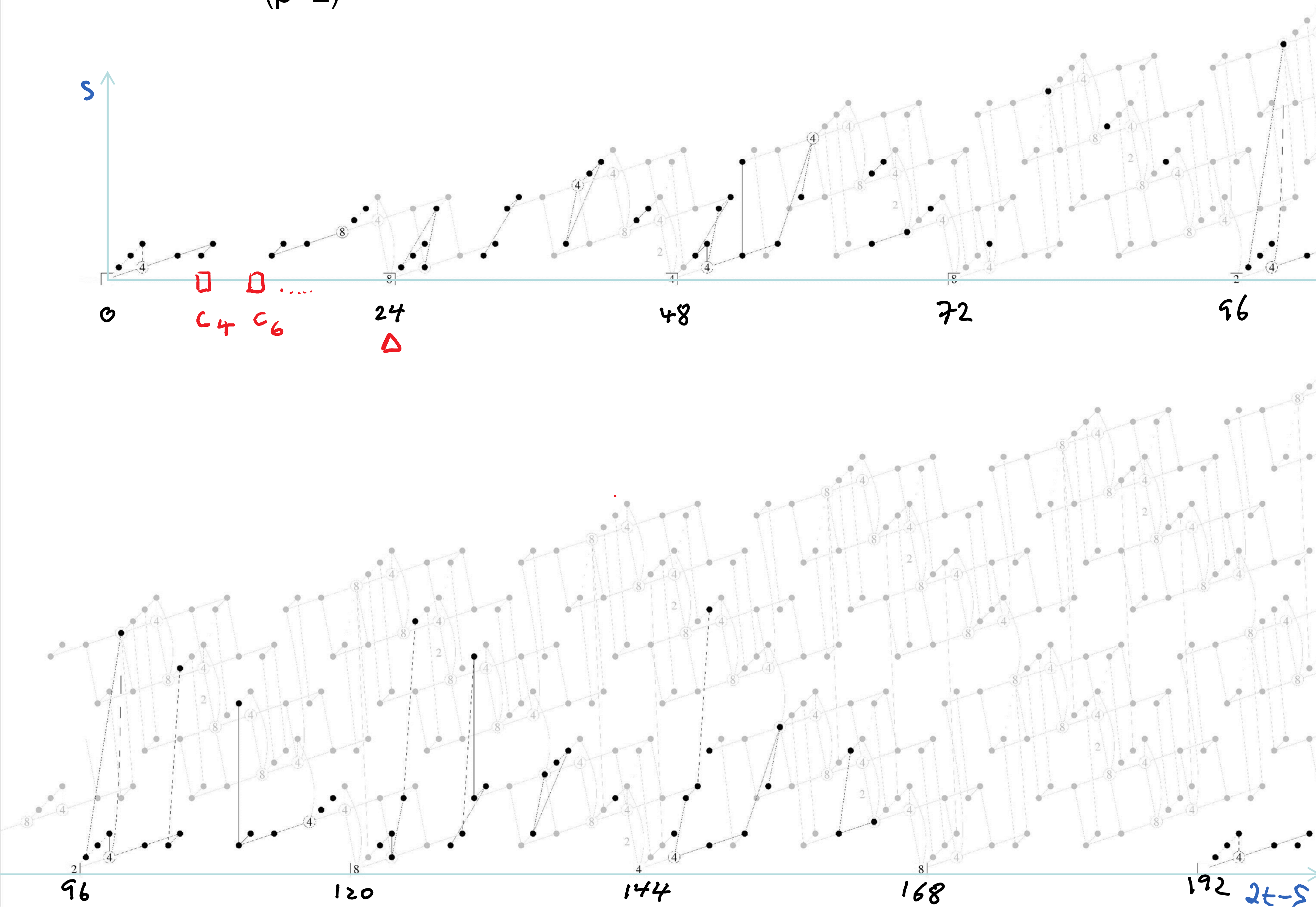
$$\pi_{2k}TMF \to [M_k]_{\mathbb{Z}}$$

(rationally this is an iso)

- $\pi_* TMF$ has a bunch of 2 and 3-torsion, and the descent spectral sequence is highly non-trivial at these primes.

The decent spectral sequence for TMF
(p=2)

$$H^s(\mathcal{M}_{ell}; \omega^{\otimes t}) \Rightarrow \pi_{2t-s} TMF$$



Outline

- Background ✓
  - Stable homotopy groups of spheres
  - Cohomology theories
  - Elliptic curves and modular forms
- What is TMF? ✓
  - Elliptic cohomology
  - Definition of TMF
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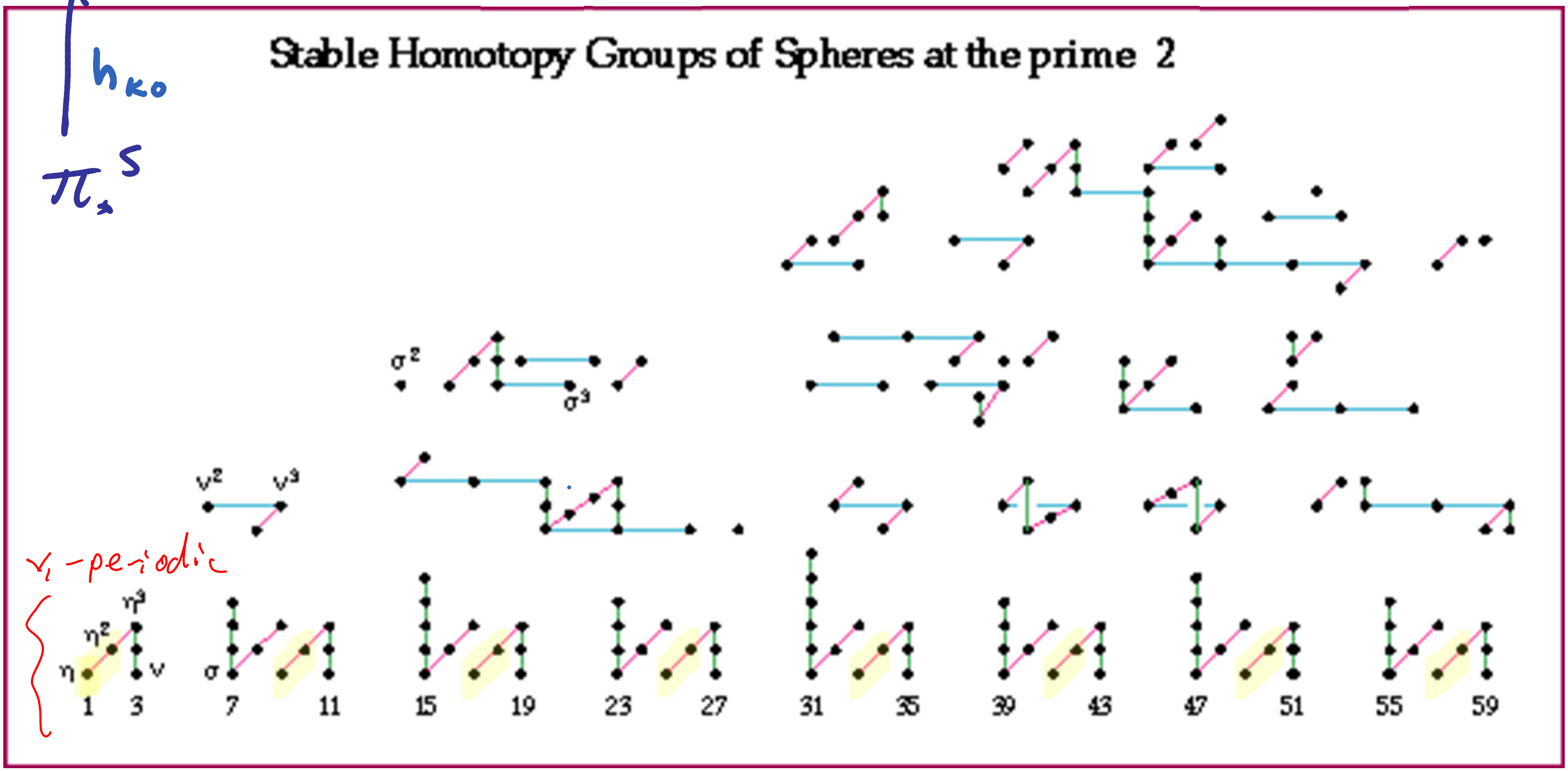
- Computational Applications of TMF
  - Hurewicz image
  - v_2-self maps
  - Greek letter elements
- Geometry
  - Witten genus
  - Derived algebraic geometry

Recall: the 2-torsion in real K-theory detects interesting classes in π_*^s via Hurewicz

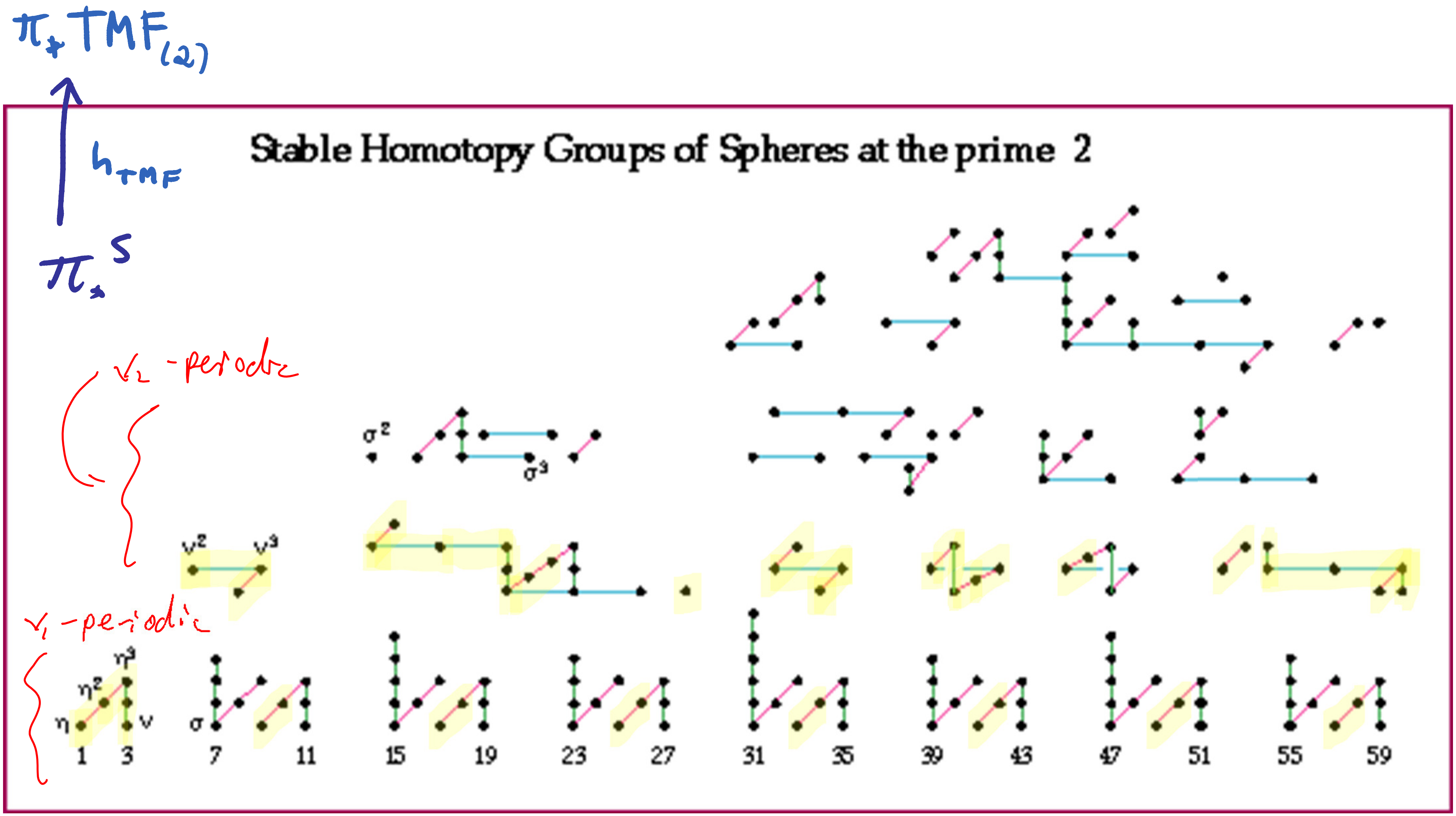
$\pi_* KO = \mathbb{Z}\ \ \mathbb{Z}/2\ \ \mathbb{Z}/2\ \ 0\ \ \mathbb{Z}\ \ 0\ \ 0\ \ 0\ \ \mathbb{Z}\ \ \mathbb{Z}/2\ \ \mathbb{Z}/2\ \ 0\ \ \mathbb{Z} \ldots$

$\uparrow h_{KO}$

π_*^s



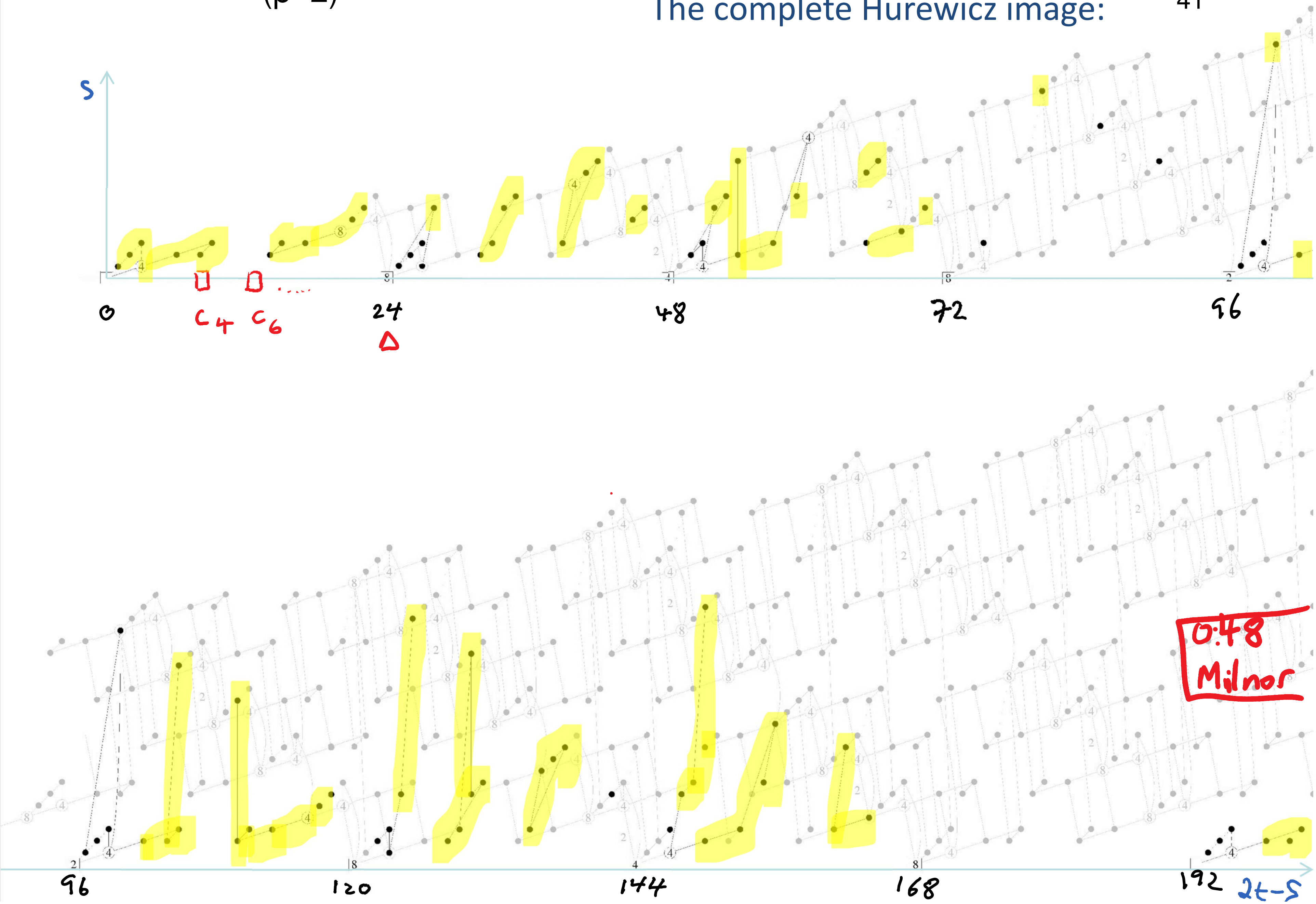
Hurewicz image of TMF (p = 2)



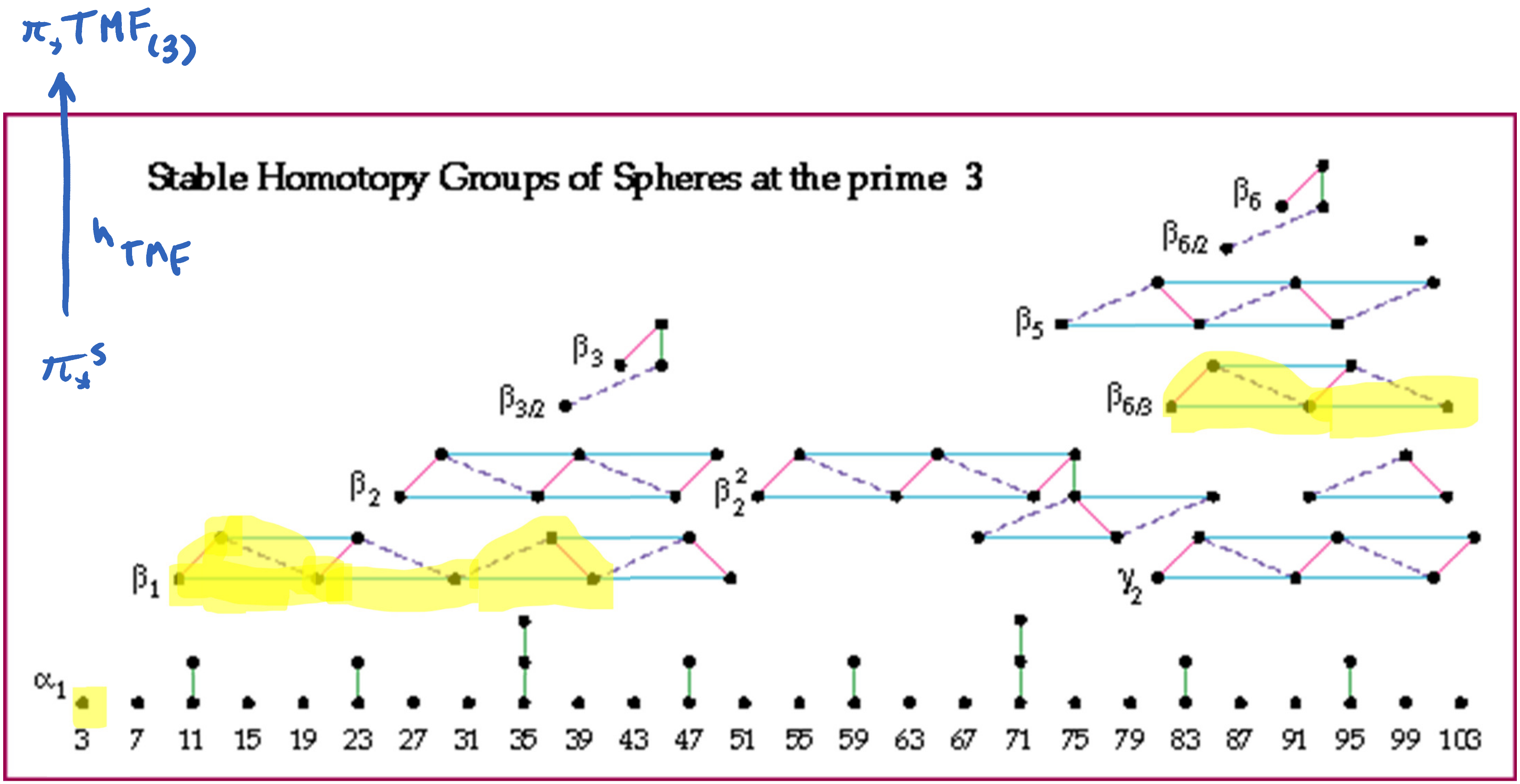
The decent spectral sequence for TMF
(p=2)

Work in progress: (B-Hopkins-Mahowald)

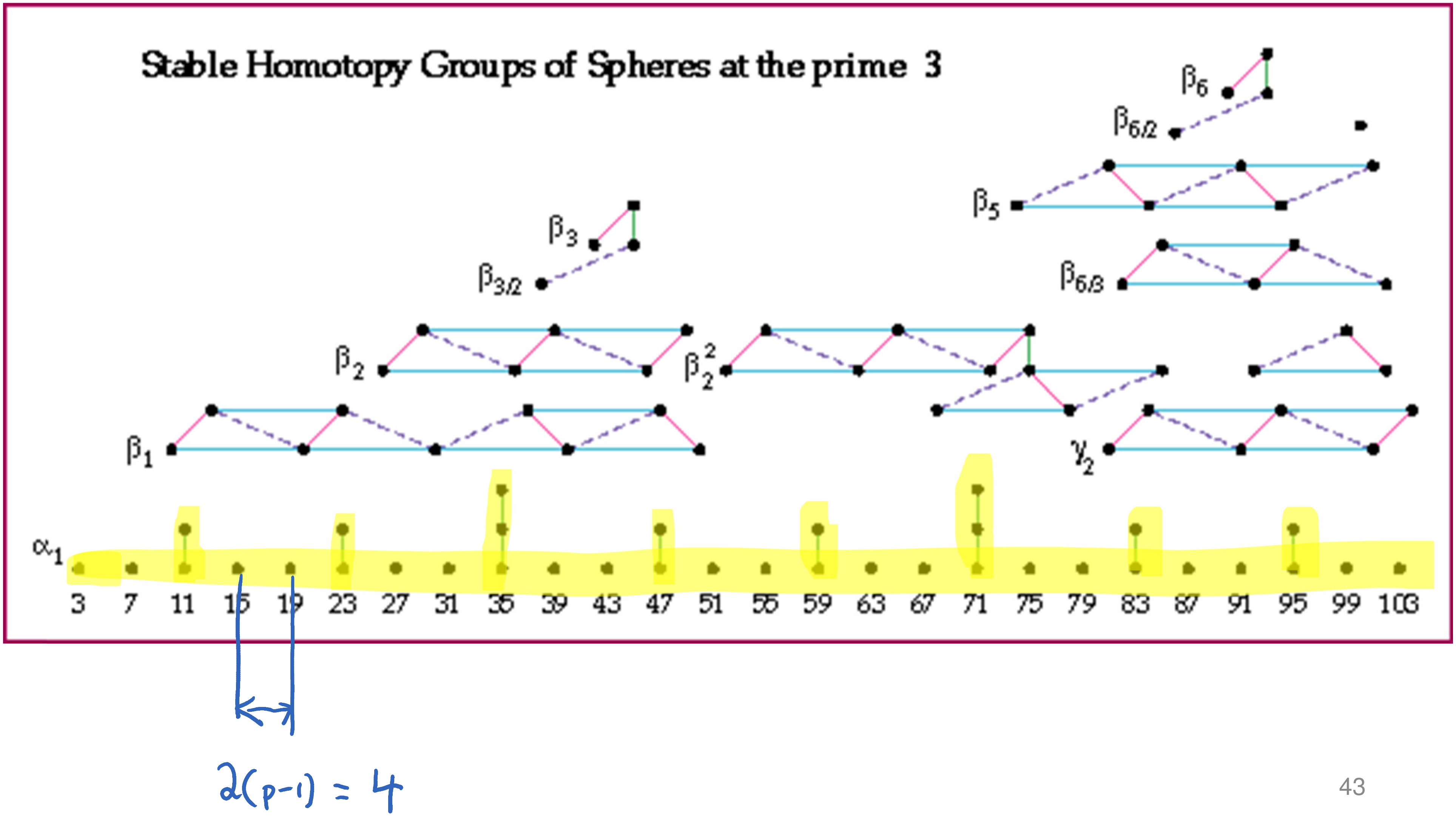
The complete Hurewicz image:



Hurewicz image of TMF (p = 3)

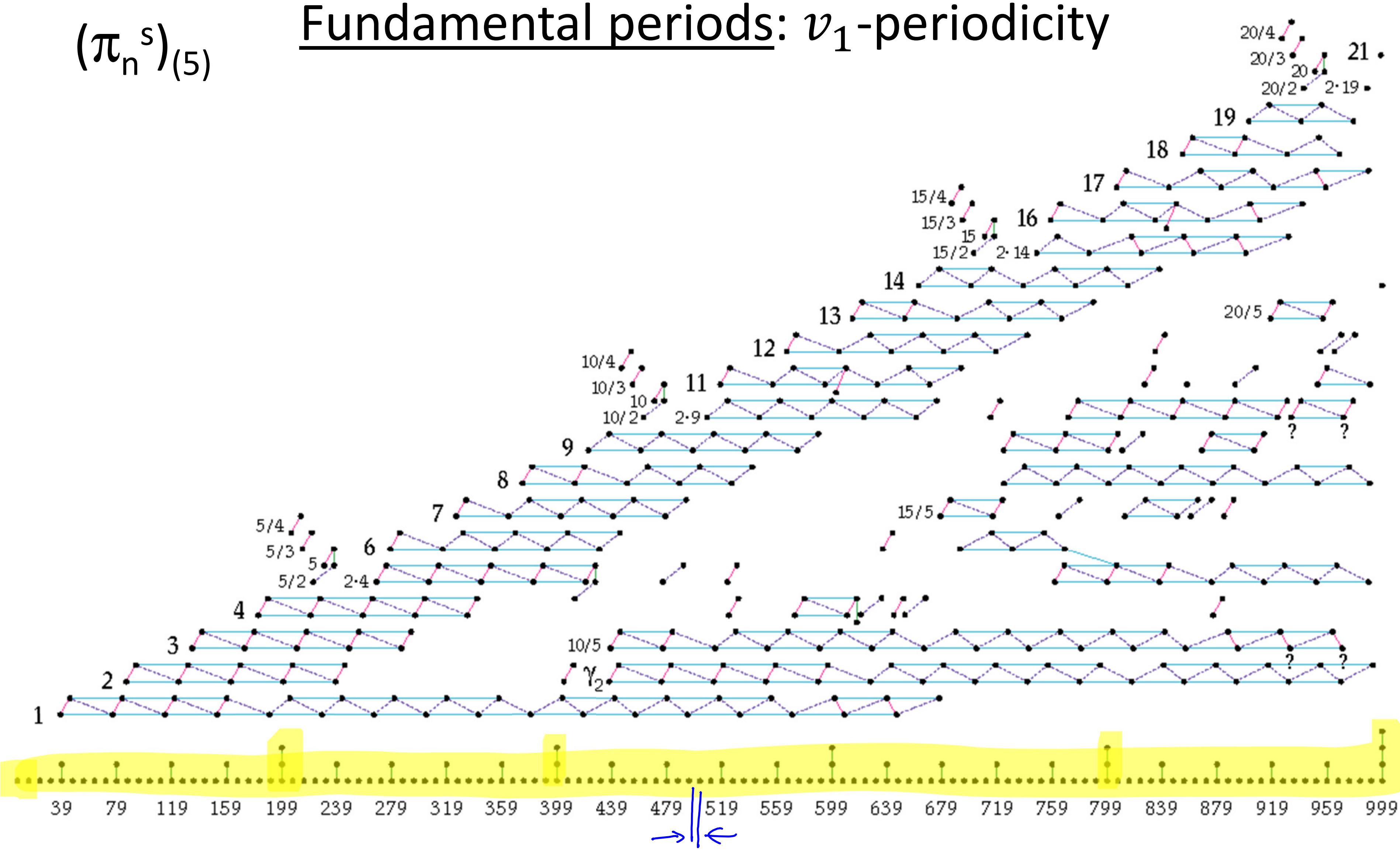


Fundamental periods: v_1-periodicity



$(\pi_n{}^s)_{(5)}$

Fundamental periods: v_1-periodicity

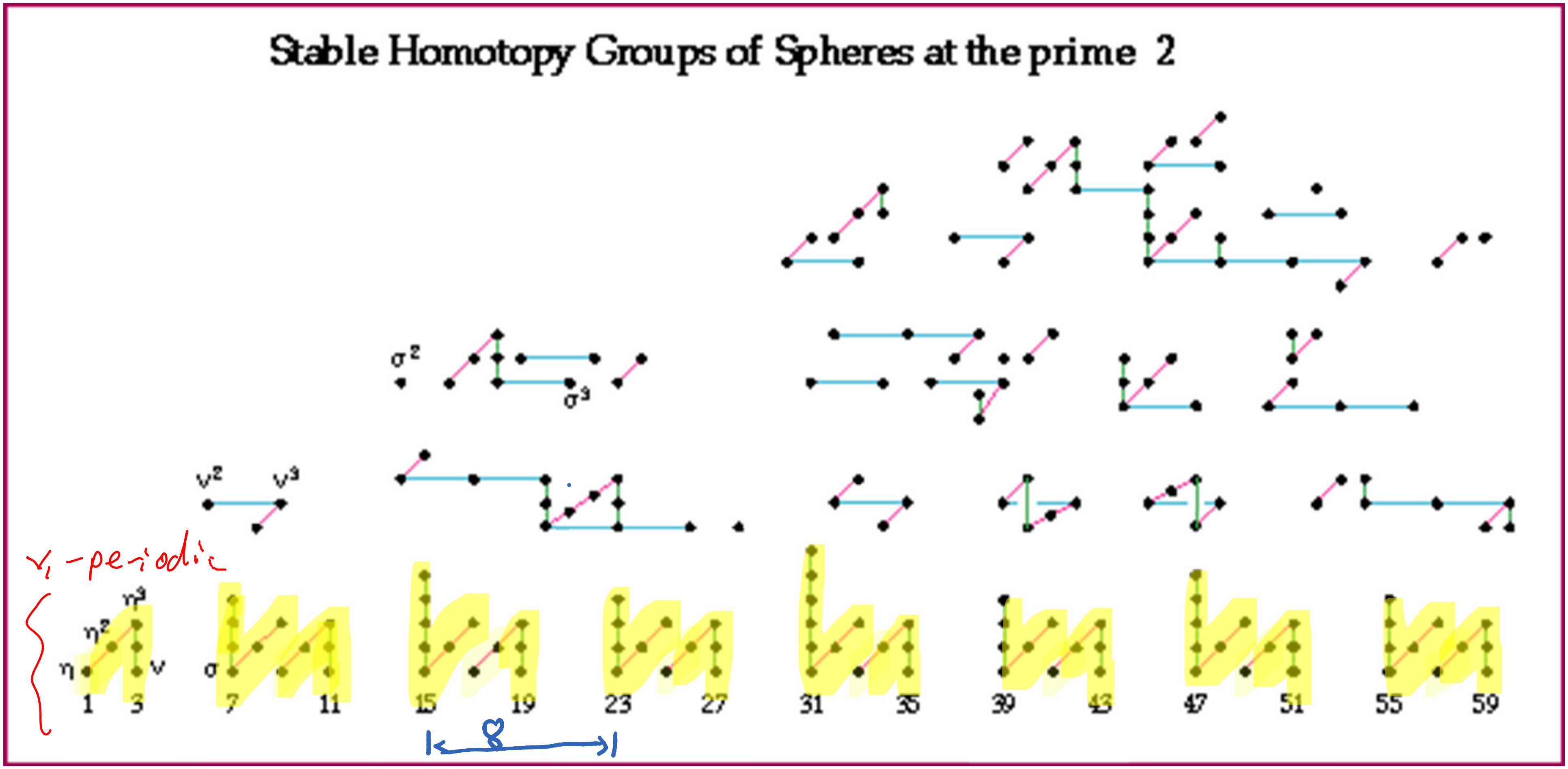


period = 2(p-1) = 8

Similarly for p > 5: the fundamental v_1-period is 2(p-1)

Fundamental periods: v_1-periodicity

Anomaly at p=2: period $= 8 \neq 2(p-1)$



Fundamental periods: v_1-periodicity

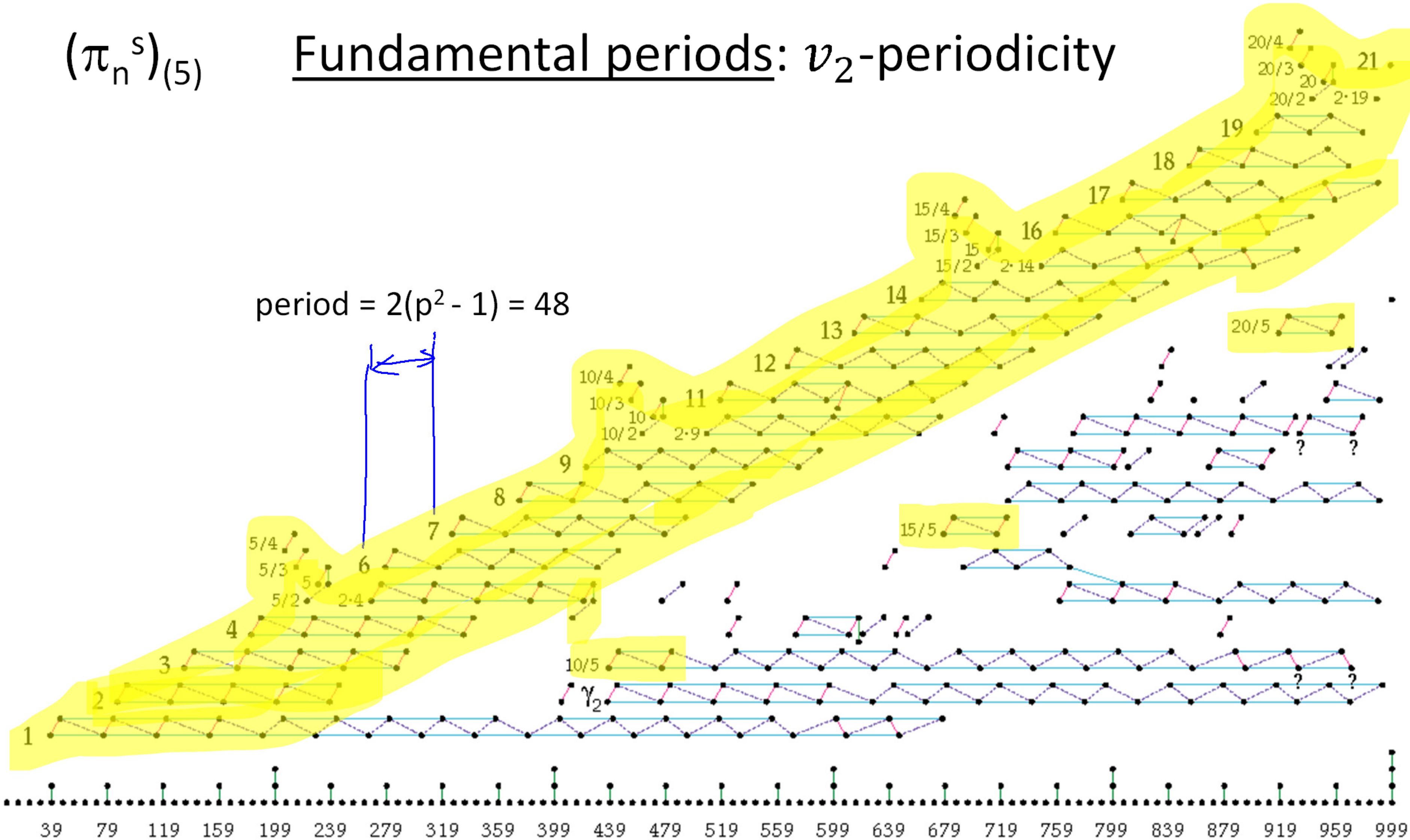
Anomaly at p=2: period $= 8 \neq 2(p-1)$

This anomaly is "explained" by the 8-fold periodicity of KO at the prime 2:

$$\pi_* KO = \mathbb{Z}\ \mathbb{Z}_2\ \mathbb{Z}_2\ 0\ \mathbb{Z}\ 0\ 0\ 0\ \mathbb{Z}\ \mathbb{Z}_2\ \mathbb{Z}_2\ 0\ \mathbb{Z}\ 0\ 0\ 0\ \ldots$$

$(\pi_n^s)_{(5)}$ **Fundamental periods**: v_2-periodicity

period = $2(p^2 - 1) = 48$



Similarly for $p > 5$: the fundamental v_2-period is $2(p^2 - 1)$

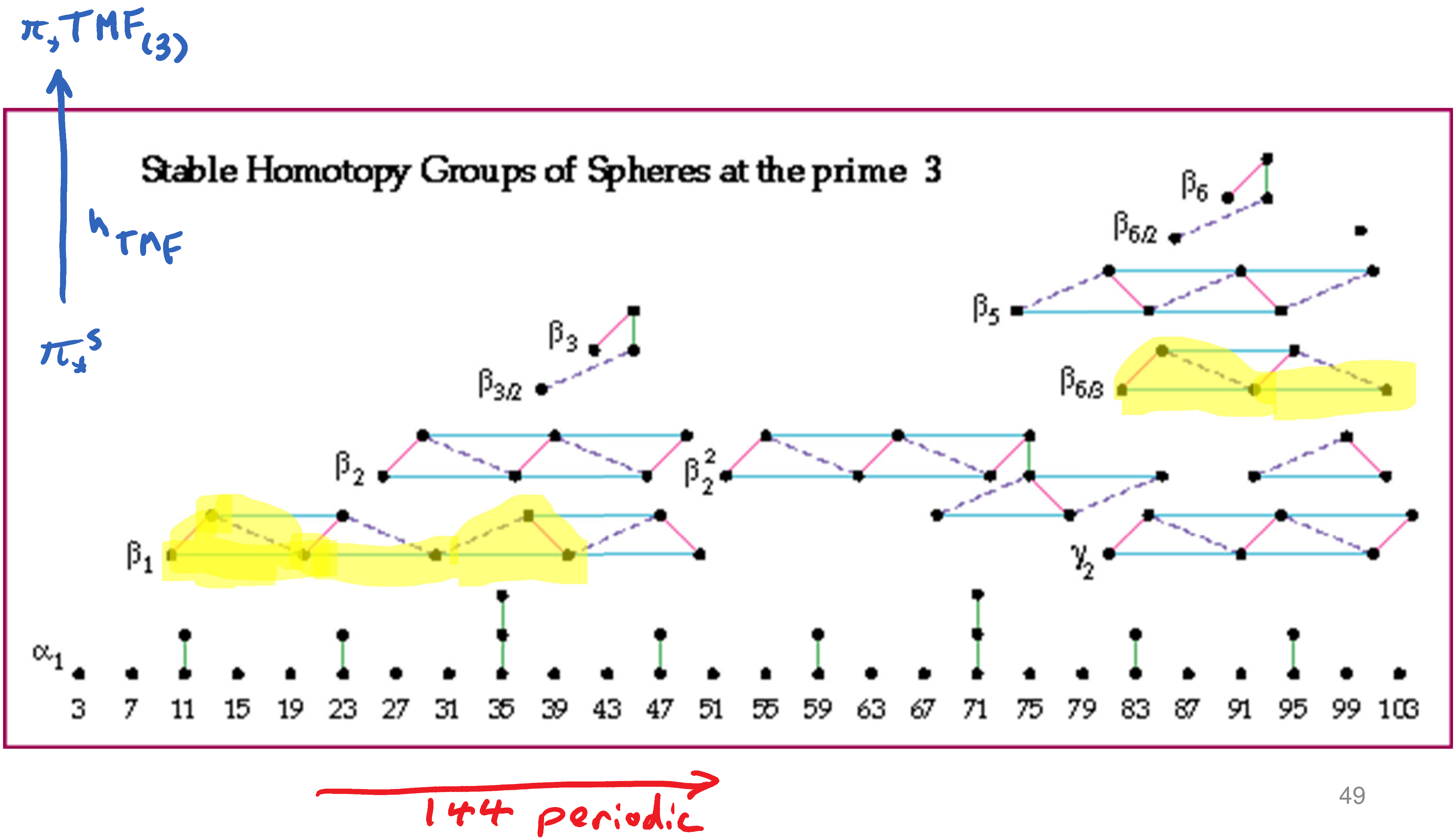
Fundamental periods: v_2-periodicity

$\pi_* TMF_{(3)}$ is 144-periodic

Theorem: (B-Pemmaraju)

The fundamental period for v_2-periodic homotopy at the prime 3 is 144.

Fundamental periods: v_2-periodicity



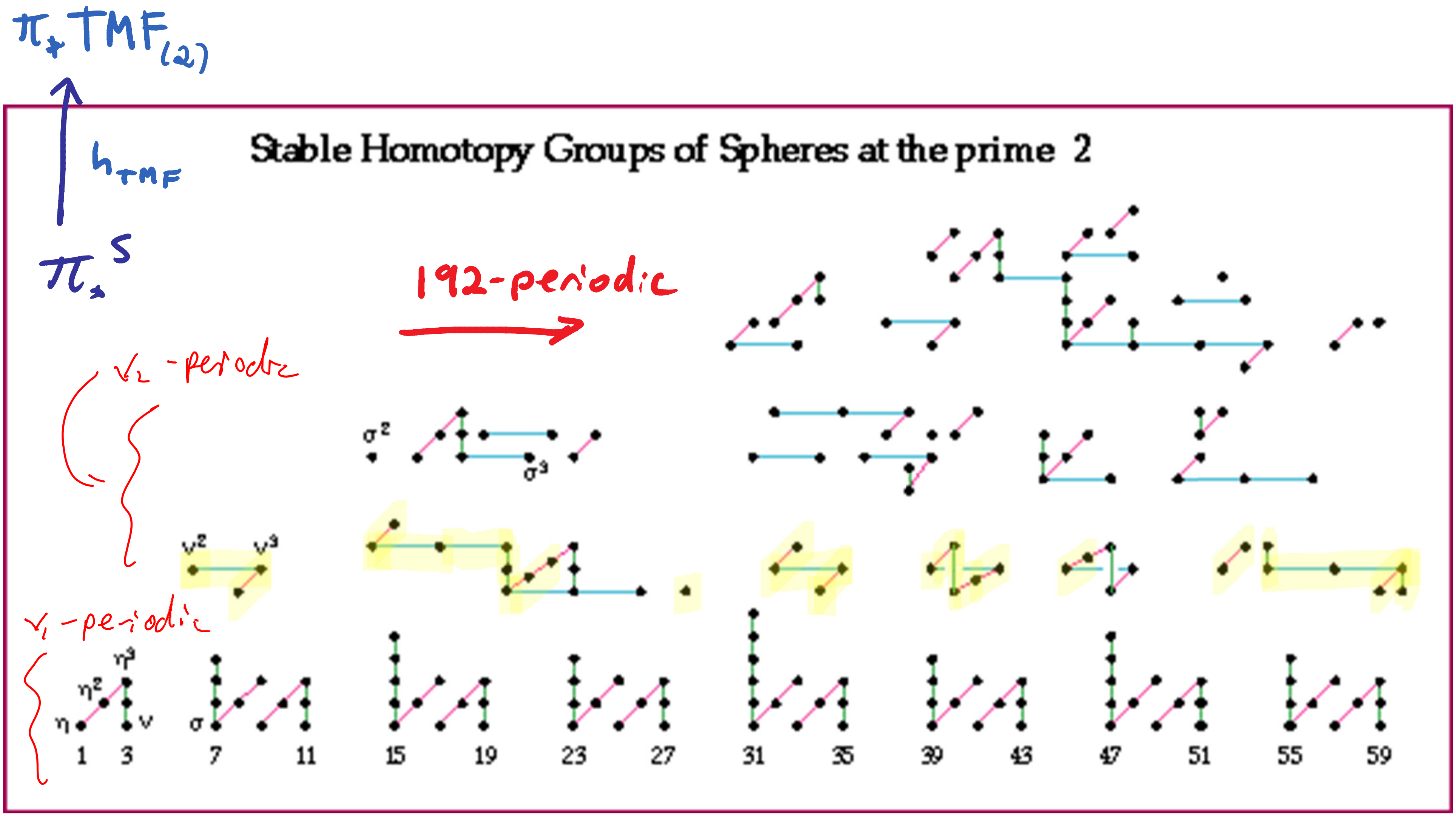
Fundamental periods: v_2-periodicity

$\pi_* TMF_{(2)}$ is 192-periodic

Theorem: (B-Hill-Hopkins-Mahowald)

The fundamental period for v_2-periodic homotopy at the prime 2 is 192.

Fundamental periods: v_2-periodicity



J-spectrum and α-family

Fix ℓ to be a prime which topologically generates $(\mathbb{Z}_p^\wedge)^\times$

$((\mathbb{Z}_p^\wedge)^\times/\{\pm 1\}$ if p = 2)

Define J to be the homotopy fiber

$$J = \text{fiber}\left(KO_p^\wedge \xrightarrow[\psi^\ell - 1]{} KO_p^\wedge \right)$$

The J-theory Hurewicz homomorphism detects much more.

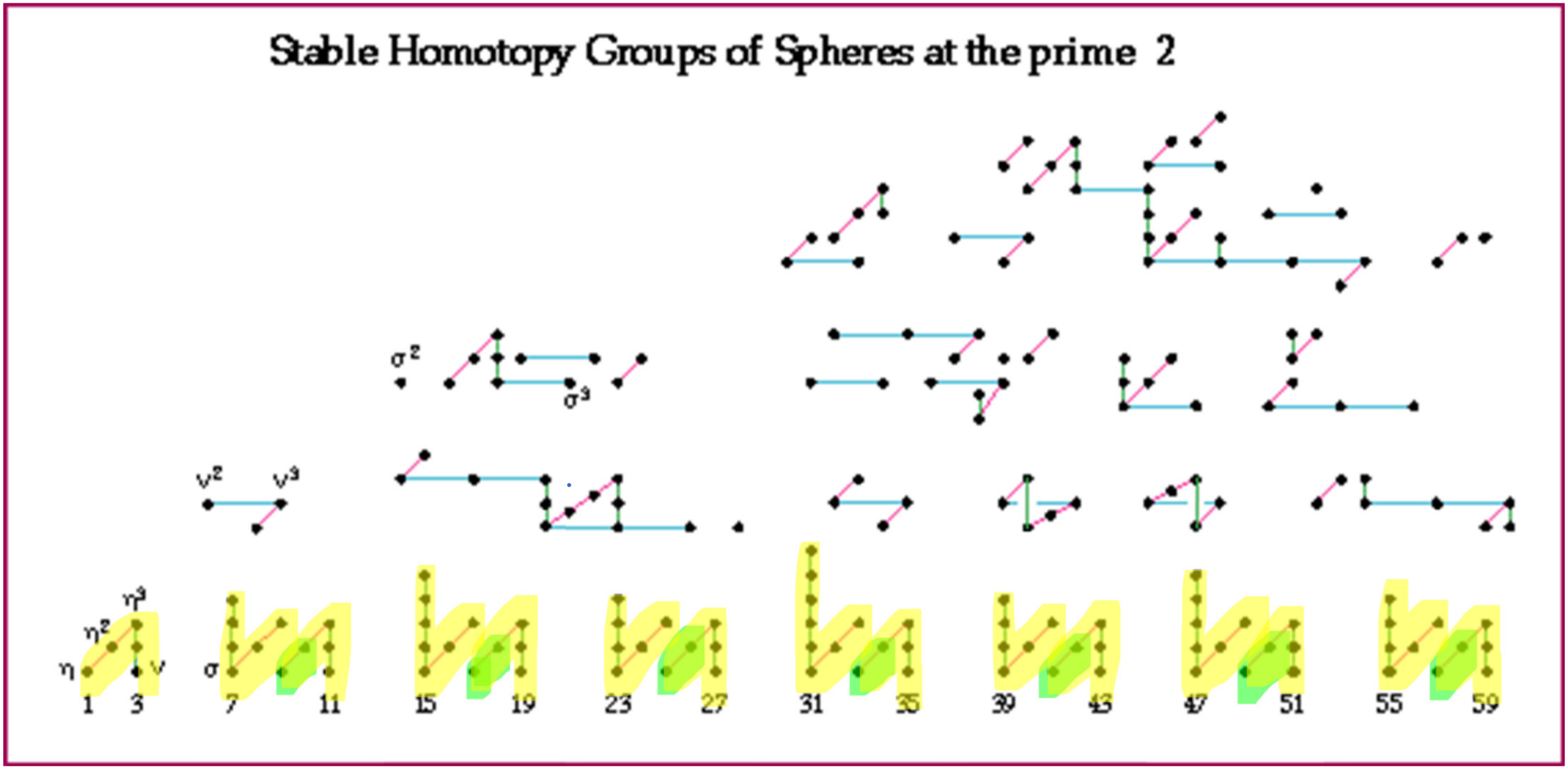
$$\pi_*^S \to \pi_* J$$

(J homo?)

= detected by KO Hurewicz

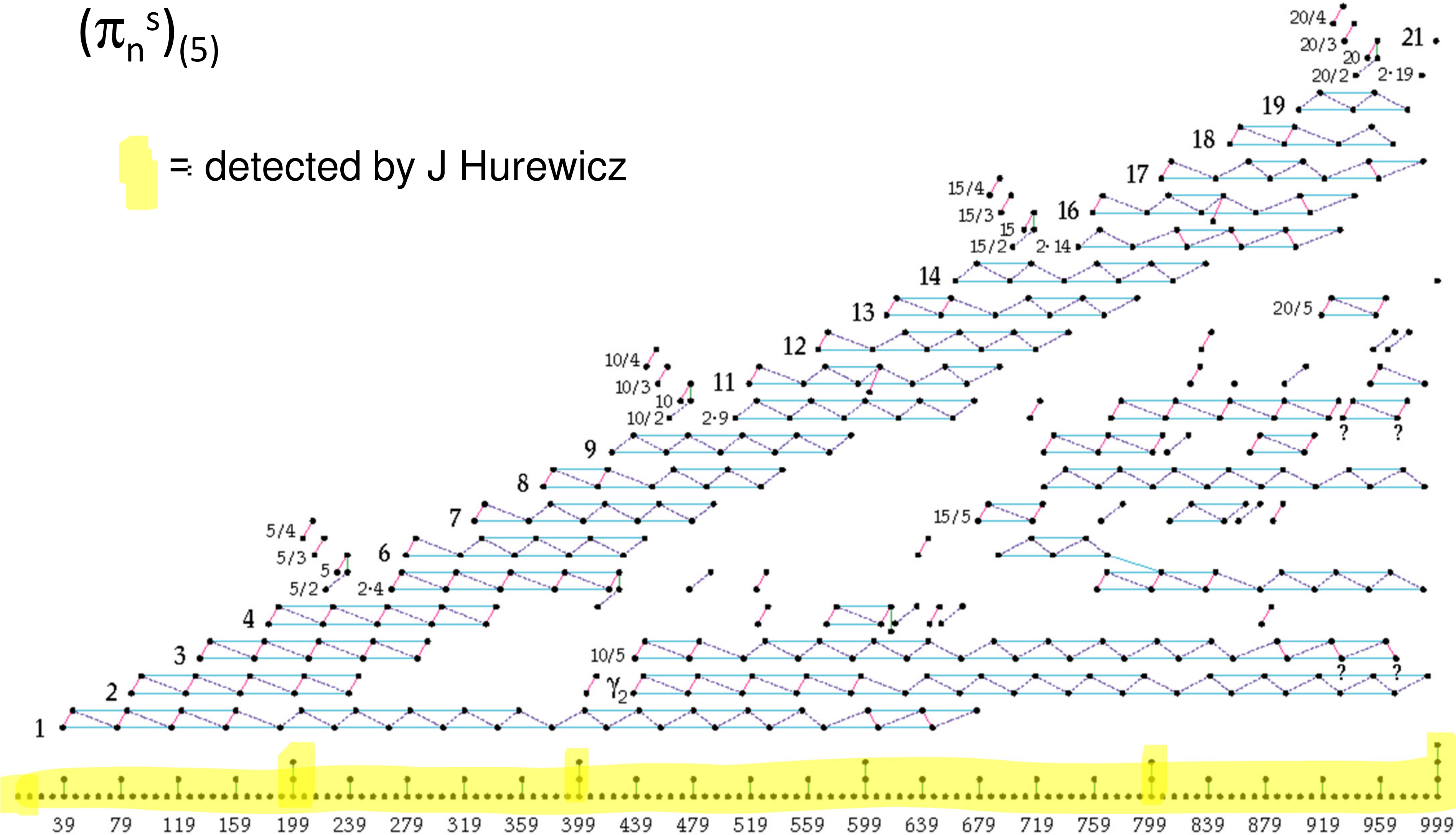
= detected by J Hurewicz

J detects *all* v_1-periodic homotopy

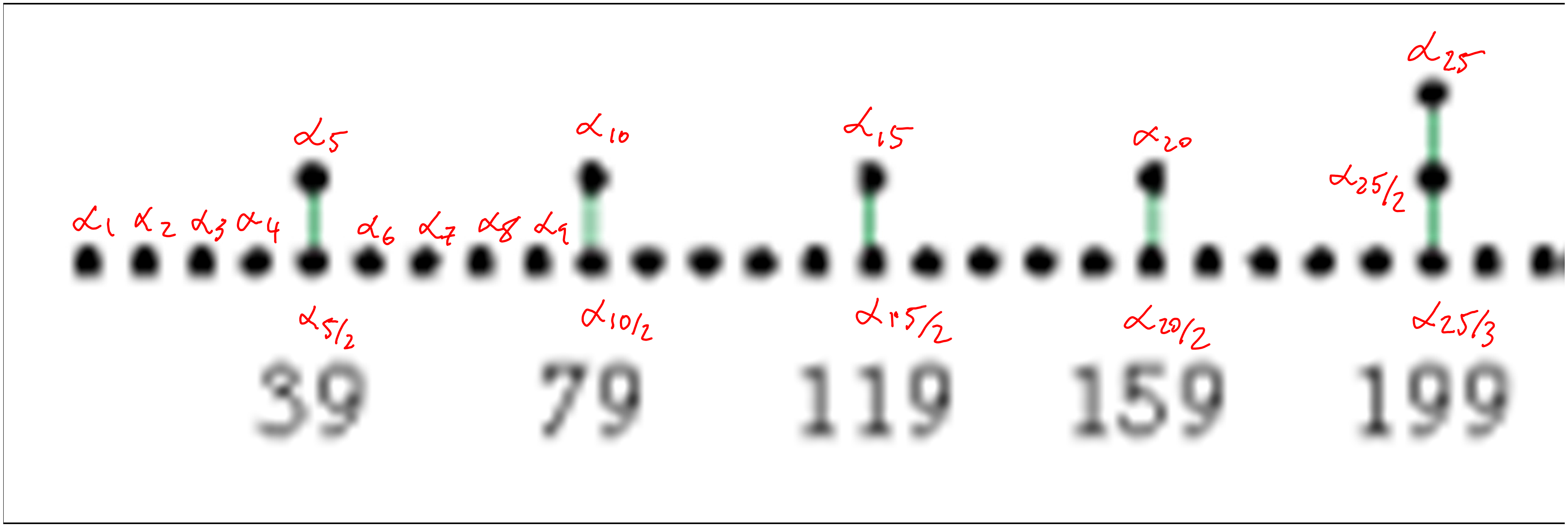


$(\pi_n^{\,s})_{(5)}$

= detected by J Hurewicz



Greek letter notation: the α-family



$\alpha_{i/j}$ is p^j-torsion

$(\alpha_i := \alpha_{i/1})$

Relationship to Bernoulli numbers

n	0	1	2	4	6	8	10	12	14	16	18	20
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

Thm (Adams) $p > 2$

$\alpha_{i/j}$ exists $\iff$ $p^j \mid \text{denom}\left(\frac{B_n}{n}\right)$

$n = (p-1)i$

Key points

- $\psi^\ell - 1$ acts by multiplication by $\ell^{2k} - 1$ on $\pi_{4k}KO = \mathbb{Z}$
- Thm(Lipshitz-Sylvester)

  $(\ell^k - 1)\frac{B_k}{k}$ is p-integral, and not p-divisible if $(p-1)|k$

An analog of J for TMF:

$$Q(\ell) := \underleftarrow{\mathrm{holim}} \left(TMF \rightrightarrows \begin{array}{c} TMF_0(\ell) \\ \times \\ TMF \end{array} \rightrightarrows\!\!\to TMF_0(\ell) \right)$$

NB: $TMF_0(\ell)$ is a version of TMF for the congruence subgroup $\Gamma_0(\ell) < SL_2(\mathbb{Z})$

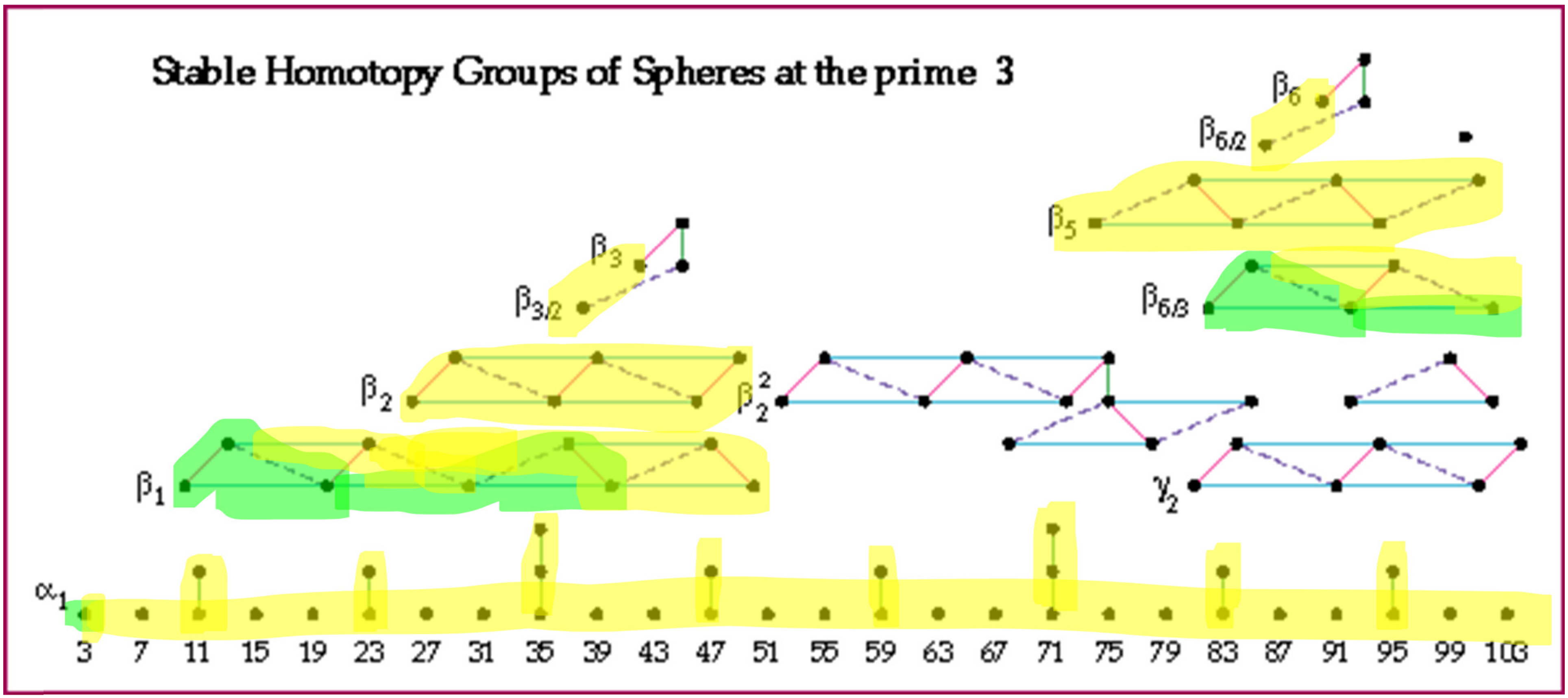
The $Q(\ell)$-theory Hurewicz homomorphism detects much more.

$$\pi_*^s \to \pi_* Q(\ell)$$

1:10 $Q(\ell)$

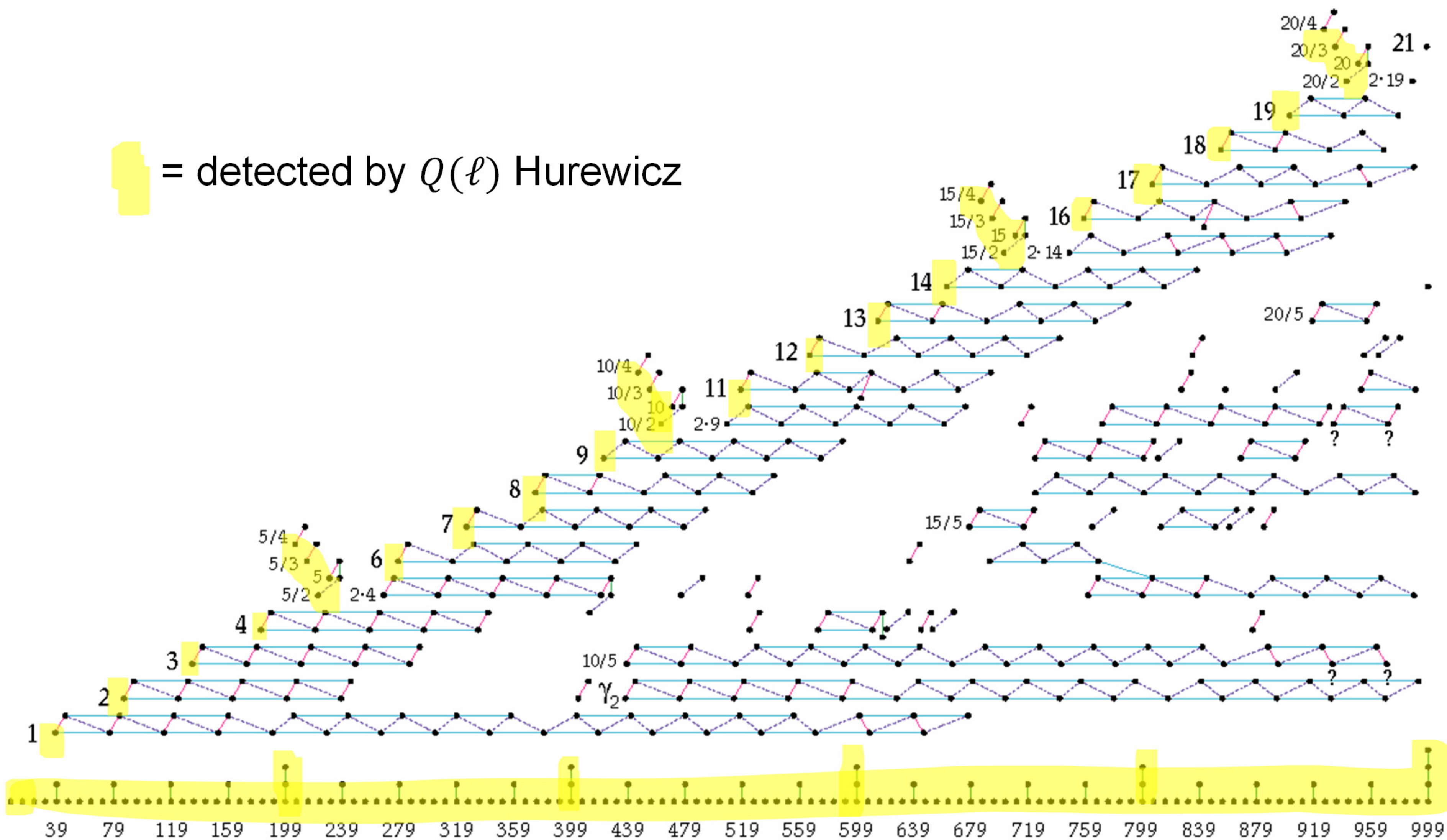
= detected by TMF Hurewicz

= detected by $Q(\ell)$ Hurewicz

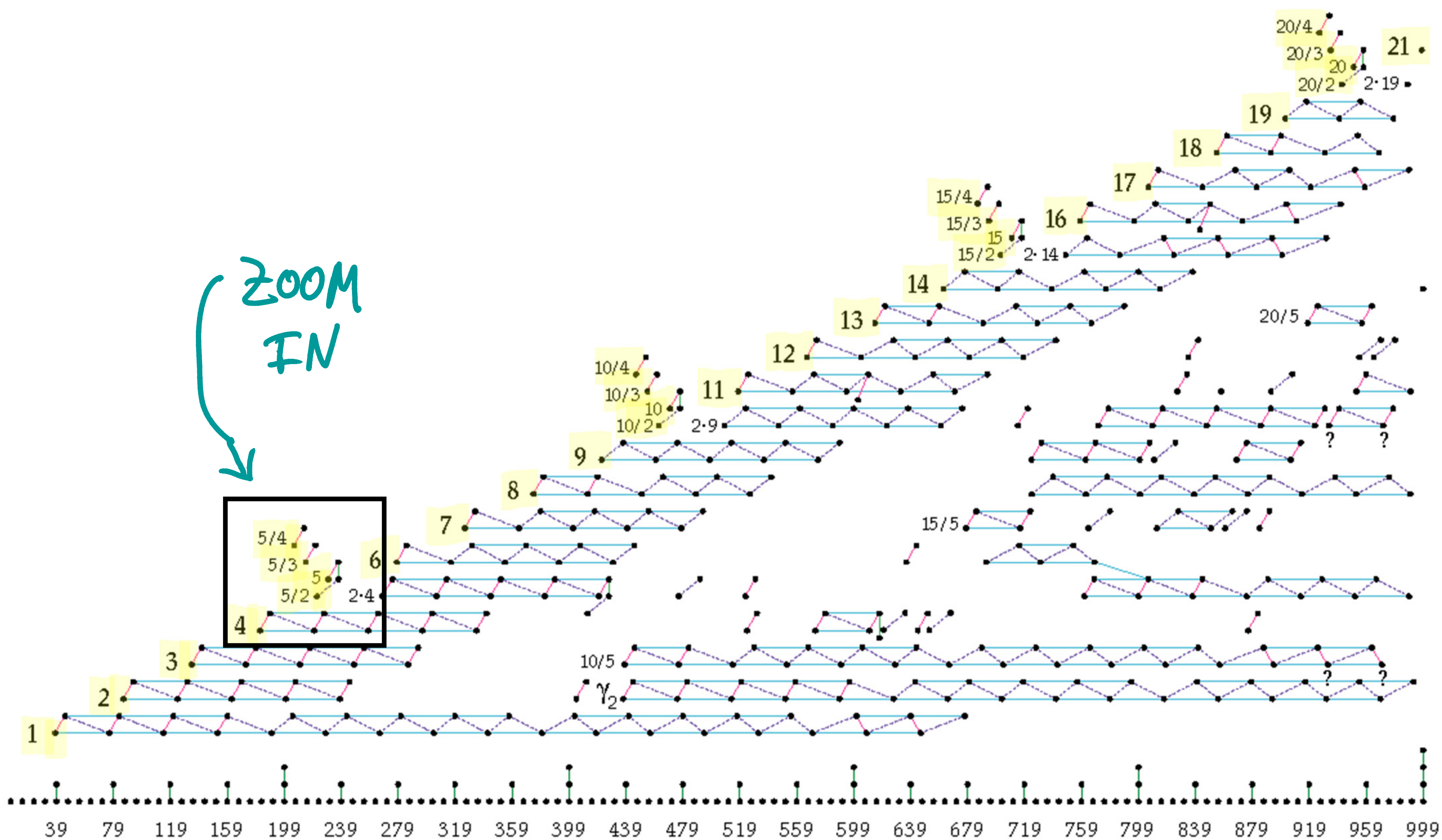


$(\pi_n^s)_{(5)}$

= detected by $Q(\ell)$ Hurewicz



$(\pi_n^s)_{(5)}$


ZOOM IN
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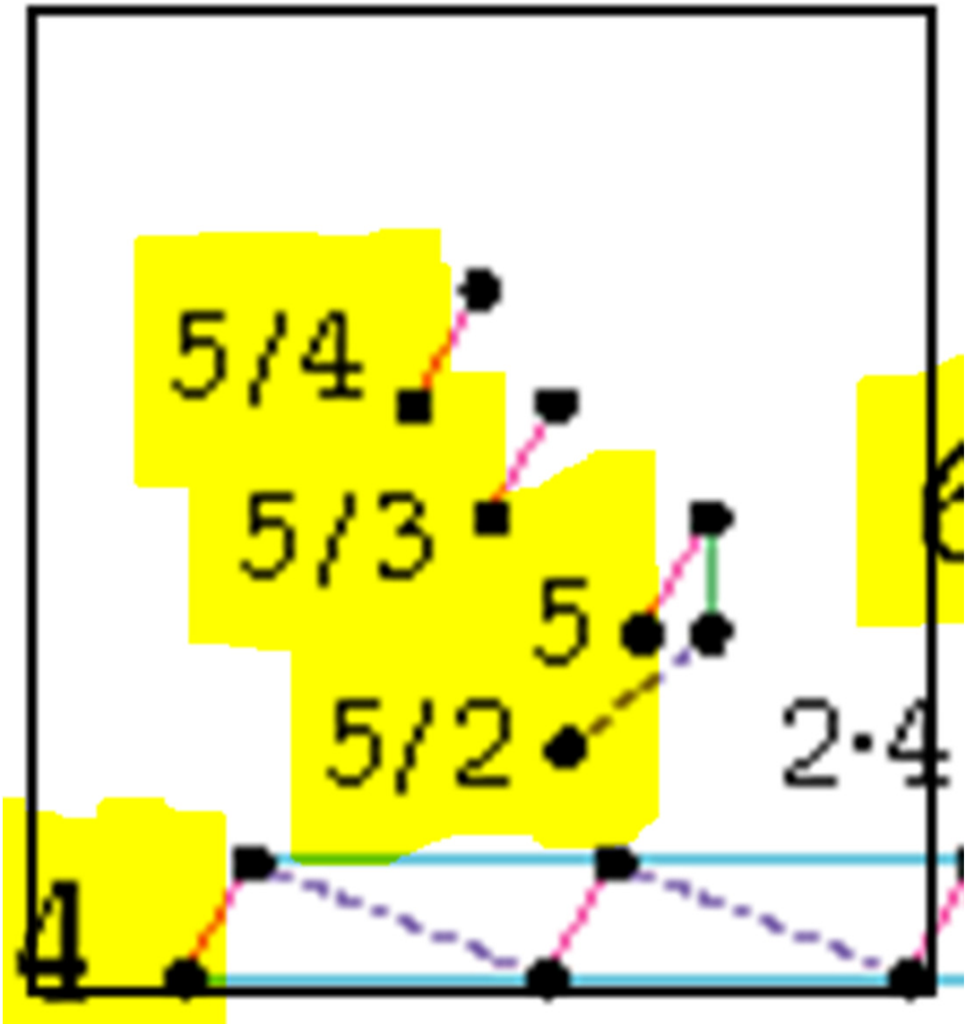
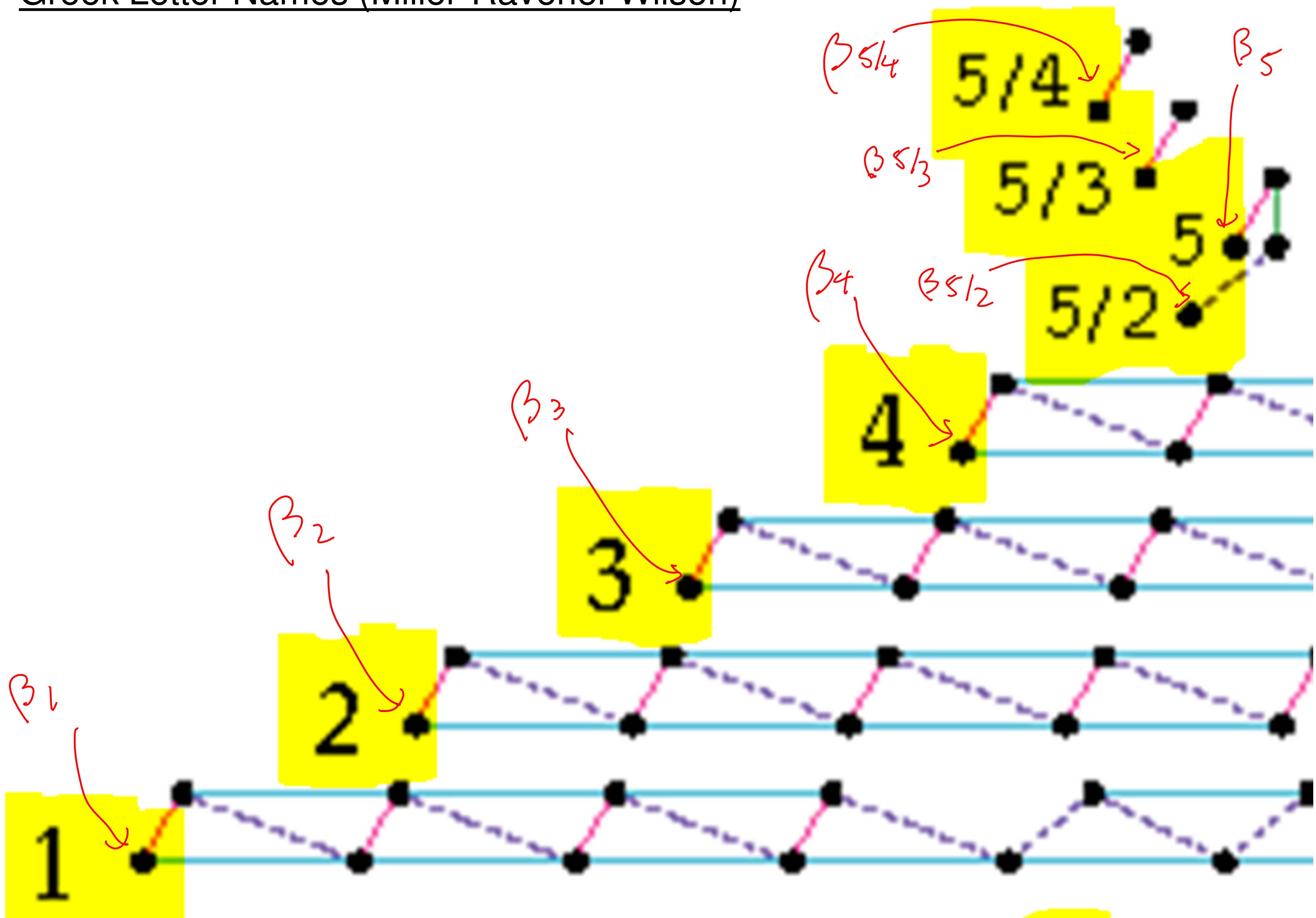
v_1-torsion in the v_2-family



"5/4" $\xrightarrow{v_1}$ "5/3" $\xrightarrow{v_1}$ "5/2" $\xrightarrow{v_1}$ "5" $\xrightarrow{v_1}$ 0

Greek Letter Names (Miller-Ravenel-Wilson)



β-family notation

$$\beta_{i/j,k} \in \left(\pi^S_{2(p^2-1)i-2(p-1)j-2}\right)_{(p)}$$

p^k-torsion

Convention

$$\beta_{i/j,1} =: \beta_{i/j}$$

$$\beta_{i/1} =: \beta_i$$

$$v_2 \beta_{i/j,k} = \beta_{i+1/j,k}$$

$$v_1 \beta_{i/j,k} = \beta_{i/j-1,k}$$

$$p \beta_{i/j,k} = \beta_{i/j,k-1}$$

β-elements and congruences of modular forms

Theorem (B)

Let $p \geq 5$. There is a bijective correspondence:

$$f \in M_n, \quad n = i(p^2-1)$$

$$\beta_{i/j,k} \in (\pi_*^s)_{(p)} \iff \begin{array}{l} (1)\ f(q) \not\equiv g(q) \mod p, \quad g \in M_{<n} \\ (2)\ f(q^\ell) - f(q) \equiv g(q) \mod p^k, \quad g \in M_{n-j(p-1)}(\Gamma_0(\ell)) \end{array}$$

1:18 β

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  - Greek letter elements
- Geometry
  - Witten genus
  - Derived algebraic geometry

Geometry of TMF: survey

- Question: What is the geometric nature of TMF?
- E.g. K-theory cocycles are given by vector bundles, what gives a TMF-cocycle?
- Beginning with Witten and Segal, and elaborated on by Stolz-Teichner, et. al., the belief is that a TMF-cocycle is given by a "conformal field theory". Much is conjectural.
- Lurie shows that TMF has an algebro-geometric significance, as the "derived" moduli space of elliptic curves.

Genera

Let G be a suitable group over O, and let

$$\Omega_d^G = \frac{d\text{−manifolds with } G\text{−stable normal structure}}{\text{cobordism}}$$

An *R_*-valued genus* is a graded ring homomorphism

$$\Phi: \Omega_*^G \to R_*$$

Examples of genera

These all arise from maps of commutative ring spectra

- Cardinality of 0-manifolds (mod 2)

$$\Omega_*^{O} \to \mathbb{Z}_2 \quad MO \to H\mathbb{Z}_2$$

- Signed cardinality of oriented 0-manifolds

$$\Omega_*^{SO} \to \mathbb{Z} \quad MSO \to H\mathbb{Z}$$

- The $\hat{A}$-genus

Atiyah-Bott-Shapiro

$$\Omega_*^{Spin} \to \pi_* KO \quad MSpin \to KO$$

The $\hat{A}$-genus of a spin manifold is the index of the Dirac operator acting on the sections of the associated spinor bundle

Witten Genus

Witten produced a genus

$$W: \Omega_*^{String} \rightarrow [M_*]_{\mathbb{Z}}$$

($String =$ 7-connected cover of O)

The idea: a string structure is a vanishing of the obstruction to quantizing a supersymmetric conformal field theory on a manifold. The partition function of the resulting QFT associates a number to every elliptic curve – a modular form!

Kevin Costello has a renormalization framework that actually makes some version of this statement mathematically precise

Witten Genus

Witten produced a genus

$$\text{W}: \Omega_*^{String} \to [M_*]_{\mathbb{Z}}$$

($String$ = 7-connected cover of O)

Theorem(Ando-Hopkins-Rezk)

The Witten genus refines to a map of ring spectra

$$W: MString \to TMF$$

"Hierarchy of genera"

String $\pi_0 = \pi_1 = \pi_3 = 0$

↓

Spin $\pi_0 = \pi_1 = 0$

↓

SO $\pi_0 = 0$

↓

O

$$
\begin{array}{ccc}
\Omega^{String} & \xrightarrow{W} & TMF^* \\
\downarrow & & \\
\Omega^{Spin} & \xrightarrow[\hat{A}]{} & K_{\mathbb{R}}^* \\
\downarrow & & \\
\Omega^{SO} & \xrightarrow[\widetilde{Card}]{} & H\mathbb{Z}^* \\
\downarrow & & \\
\Omega^{O}_* & \xrightarrow[Card]{} & H\mathbb{Z}/2^*
\end{array}
$$

Derived Algebraic Geometry (Lurie's approach)

A derived scheme consists of

- An ordinary scheme $(X, \mathcal{O}_X)$
- A sheaf $\underline{\mathcal{O}}_X$ of commutative ring spectra such that

$$\pi_0 \, \underline{\mathcal{O}}_X = \mathcal{O}_X$$

with a certain additional local condition…

A derived elliptic curve is a derived abelian group scheme whose underlying scheme is an elliptic curve.

Derived Algebraic Geometry (Lurie's approach)

Let E be a ring spectrum. An orientation of a derived elliptic curve C/E is an isomorphism

$$\mathrm{Spf}(E^{\mathbb{C}P^\infty}) \underset{\simeq}{\rightarrow} \hat{C}$$

Theorem(Lurie)

The moduli problem of oriented derived elliptic curves is representable. The representing Deligne-Mumford stack is

$$(\mathcal{M}_{ell}, \mathcal{O}_{ell})$$

Advantages to the DAG approach

- Gives a "pure thought" construction of TMF – Goerss-Hopkins-Miller rely on obstruction theory
- Gives a homotopically unique construction of TMF – the moduli space of solutions to the Goerss-Hopkins-Miller obstruction problem is not contractible (but does have one component).
- Generalizes to give equivariant TMF for compact Lie groups, a "genuine" equivariant theory in the sense of Lewis-May-Steinberger

Objectives for next 2 lectures:

Chromatic level	Group	Arithmetic Object	Cohomology theory	Cocycles represeted by:	Geometry
1	GL_1	multiplicative group	K-theory	Vector bundles	Spin
2	GL_2	elliptic curves	TMF	conformal field theories?	String
n	?	?	?	?	?

How does this generalize for arbitrary n?

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Chromatic level	Group	Arithmetic Object	Cohomology theory	Cocycles represeted by:	Geometry
1	GL_1	multiplicative group	K-theory	Vector bundles	Spin
2	GL_2	elliptic curves	TMF	conformal field theories?	String
n	U(1,n-1)	abelian varieties with complex multiplication	TAF	?	?