



Variable $L^{p(\cdot)}$
Spaces

David V.
Cruz-Uribe, SFO

Classical theory

Intuition on
 $L^{p(\cdot)}(\Omega)$

A Sufficient
Condition

(Almost)
Necessary
Conditions

Weaker Sufficient
Conditions

A Necessary &
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Variable Lebesgue Spaces

David V. Cruz-Uribe, SFO

Trinity College

Summer School and Workshop
Harmonic Analysis and Related Topics
Lisbon, June 21-25, 2010



Lecture 2

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The Maximal Operator on Variable Lebesgue Spaces



Outline

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The maximal operator

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For $f \in L^1_{\text{loc}}$, define

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy = \sup_{Q \ni x} \int_Q |f(y)| dy$$

Can replace cubes by centered cubes or by
(centered) balls



Motivation

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Calderón-Zygmund philosophy:

*To control a singular integral operator,
control the maximal operator*



L^p estimates

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- $p = \infty$: $\|Mf\|_\infty = \|f\|_\infty$

- $1 < p < \infty$:

$$\int_{\mathbb{R}^n} Mf(x)^p dx \leq C(p, n) \int_{\mathbb{R}^n} |f(x)|^p dx$$

- $p = 1$: any $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{C(n)}{\lambda} \int_{\mathbb{R}^n} |f(x)| dx$$



Calderón-Zygmund decomposition

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Given $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, and $\lambda > 0$, there exist disjoint dyadic cubes $\{Q_j^\lambda\}$ such that

$$\lambda < \int_{Q_j^\lambda} |f(y)| \, dy \leq 2^n \lambda$$

and $|f(x)| \leq \lambda$ a.e. on $\mathbb{R}^n \setminus \bigcup_j Q_j^\lambda$.

Further,

$$E_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\} \subset \bigcup_j 3Q_j^\lambda$$



Proof of L^p estimates

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Step 1: weak $(1, 1)$ via CZ-decomp:

$$|E_\lambda| \leq \sum_j |3Q_j^\lambda| \leq \frac{3^n}{\lambda} \sum_j \int_{Q_j^\lambda} |f| \, dy \leq \frac{3^n}{\lambda} \int_{\mathbb{R}^n} |f| \, dy.$$

Step 2: (p, p) inequalities follow by Marcinkiewicz interpolation

$$\int_{\mathbb{R}^n} Mf(x)^p \, dx = p \int_0^\infty \lambda^{p-1} |E_\lambda| \, d\lambda.$$



A simple example

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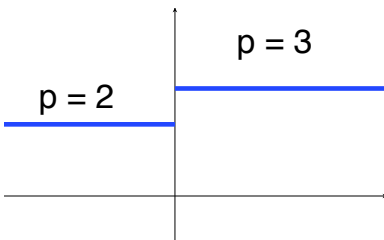
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$$p(x) = \begin{cases} 2 & -5 < x \leq 0 \\ 3 & 0 < x < 5. \end{cases}$$





A simple example (continued)

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Let $f(x) = |x|^{-1/3} \chi_{(-5,0)}(x): f \in L^{p(\cdot)}((-5, 5))$

For $0 < x < 5$,

$$Mf(x) \geq \int_{-x}^x |f(y)| dy = 3|x|^{-1/3} \notin L^{p(\cdot)}((-5, 5))$$



First question

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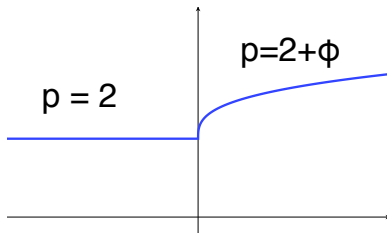
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Do we need $p(\cdot)$ continuous? How much regularity?





Behavior at infinity

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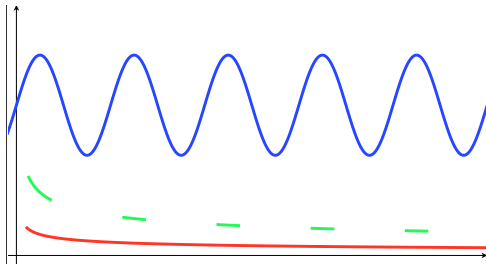
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Let $p(x) = 3 + \sin(\pi x)$

Define $f(x) = |x|^{-1/3} \sum_k \chi_{[1/4+2k, 3/4+2k]} \in L^{p(\cdot)}(\mathbb{R})$

$Mf(x) \approx c|x|^{-1/3} \notin L^{p(\cdot)}(\mathbb{R})$





Second question

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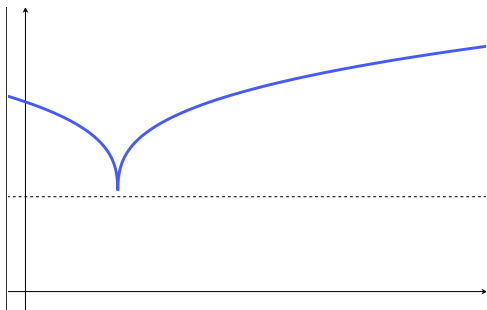
Do we need $p(\cdot)$ continuous at infinity? How much regularity?



Two more questions

Can $p(\cdot) = \infty$ on parts of the domain?

Can $p(\cdot)$ get arbitrarily close to 1?



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Local log-Hölder continuity

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Given $p(\cdot)$, we say $1/p(\cdot) \in LH_0$ if

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{C_0}{-\log(|x - y|)}, \quad |x - y| < \frac{1}{2}$$

If $p_+ < \infty$, $p(\cdot) \in LH_0$ iff $1/p(\cdot) \in LH_0$.

$1/p(\cdot) \in LH_0$ iff $1/p'(\cdot) \in LH_0$



Log-Hölder continuity at infinity

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Given $p(\cdot)$, we say $1/p(\cdot) \in LH_\infty$ if there exists p_∞ such that

$$\left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right| \leq \frac{C_\infty}{\log(e + |x|)}.$$

If $p_+ < \infty$, $p(\cdot) \in LH_\infty$ iff $1/p(\cdot) \in LH_\infty$.

Hereafter, let $LH = LH_0 \cap LH_\infty$.



$L^{p(\cdot)}$ estimates for the maximal operator

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Theorem

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $1/p(\cdot) \in LH$. Then for all $\lambda > 0$,

$$\|\lambda \chi_{\{x: Mf(x) > \lambda\}}\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}$$

If in addition $p_- > 1$, then

$$\|Mf\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}.$$



Outline of proof: $p_+ < \infty$

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- Assume f non-negative, bounded and compact support, $\|f\|_{p(\cdot)} = 1$
- Split f into $f \geq 1$ and $0 \leq f < 1$
- Apply Calderón-Zygmund decomposition
- Use LH_0 where f large; LH_∞ where f small
- Use classical result: M bounded on L^{p_-} / L^{p_∞}



First step: f small

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Let $f_1 = f \chi_{\{x: f(x) \geq 1\}}$. Suffices to prove

$$\int_{\mathbb{R}^n} Mf_1(x)^{p(x)} dx \leq C.$$

Apply Calderón-Zygmund decomposition:

$$\int_{\mathbb{R}^n} Mf_1(x)^{p(x)} dx \leq C \sum_{k,j} \int_{E_j^k} \left(\int_{3Q_j^k} f_1(y) dy \right)^{p(x)} dx$$

where $\{Q_j^k\}$ are CZ cubes at height A^k and $E_j^k \subset 3Q_j^k$ pairwise disjoint.



Intuition: Hölder's inequality

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$$\begin{aligned} & \sum_{k,j} \int_{E_j^k} \left(\int_{3Q_j^k} f_1(y) dy \right)^{p(x)} dx \\ & \leq C \sum_{k,j} \int_{E_j^k} \left(\int_{3Q_j^k} f_1(y)^{p(y)/p_-} dy \right)^{p_-} dx \\ & \leq C \sum_{k,j} \int_{E_j^k} M(f_1^{p(\cdot)/p_-})(x)^{p_-} dx \\ & \leq C \int_{\mathbb{R}^n} M(f_1^{p(\cdot)/p_-})(x)^{p_-} dx \\ & \leq C \int_{\mathbb{R}^n} f_1(x)^{p(x)} dx \leq C. \end{aligned}$$



A lemma to make this work

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Lemma

TFAE:

- $p(\cdot) \in LH_0$;
- *given any cube Q and $x \in Q$,*

$$|Q|^{p(x)-p_+(Q)} \leq C \quad \text{and} \quad |Q|^{p_-(Q)-p(x)} \leq C.$$



Second step: f large

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Let $f_2 = f\chi_{\{x:0 \leq f < 1\}}$. Again suffices to prove

$$\int_{\mathbb{R}^n} Mf_2(x)^{p(x)} dx \leq C.$$

Intuition: if function F “lives” far from origin, then

$$\int_{\mathbb{R}^n} F(x)^{p(x)} dx \approx \int_{\mathbb{R}^n} F(x)^{p_\infty} dx$$



Another key lemma

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Lemma

If $p(\cdot) \in LH_\infty$, and $0 \leq F(x) \leq 1$, then

$$\int_{\mathbb{R}^n} F(x)^{p(x)} dx \leq C \int_{\mathbb{R}^n} F(x)^{p_\infty} dx + C \int_{\mathbb{R}^n} R(x)^{p_\infty} dx$$

and

$$\int_{\mathbb{R}^n} F(x)^{p_\infty} dx \leq C \int_{\mathbb{R}^n} F(x)^{p(x)} dx + C \int_{\mathbb{R}^n} R(x)^{p_\infty} dx$$

where $R(x) = (e + |x|)^{-n}$.



The final step

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$$\begin{aligned} & \int_{\mathbb{R}^n} Mf_2(x)^{p(x)} dx \\ & \leq C \int_{\mathbb{R}^n} Mf_2(x)^{p_\infty} dx + C \int_{\mathbb{R}^n} R(x)^{p_\infty} dx \\ & \leq C \int_{\mathbb{R}^n} f_2(x)^{p_\infty} dx + C \\ & \leq C \int_{\mathbb{R}^n} f_2(x)^{p(x)} dx + C \int_{\mathbb{R}^n} R(x)^{p_\infty} dx + C \\ & \leq C. \end{aligned}$$



$p_- > 1$ necessary

If $p_- = 1$, then there exists a sequence of balls B_k and exponents $s_k \searrow 1$ such that

$$\frac{|E_k|}{|B_k|} = \frac{|\{x \in B_k : 1 < p(x) \leq s_k\}|}{|B_k|} \geq 1/2.$$

Let x_k be center of B_k . Define

$$f_k(x) = |x - x_k|^{-n + \frac{1}{k+1}} \chi_{E_k}(x).$$

Choose s_k so that $f_k \in L^{p(\cdot)}$ but

$$Mf_k(x) \geq c(n)(k+1)f_k(x).$$

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LH_0 and LH_∞ pointwise sharp

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Take any $\phi(\cdot) : [0, \infty) \rightarrow [0, 1]$, increasing, smooth
 $\phi(x) = 0$ and

$$\lim_{x \rightarrow 0^+} \phi(x) \log(x) = -\infty.$$

Define

$$p(x) = \begin{cases} 2 + \phi(x) & x \geq 0 \\ 2 & x < 0 \end{cases}$$

Then $p(\cdot) \notin LH_0$ and M not bounded on $L^{p(\cdot)}(\mathbb{R})$.

Similar construction holds for LH_∞ .



LH_0 and LH_∞ not necessary

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Define

$$\frac{1}{p(x)} = \begin{cases} \frac{1}{2} + \log(|x|)^{-1/2} & |x| < e^{-9} \\ 5/6 & |x| \geq e^{-9}. \end{cases}$$

Then M bounded on $L^{p(\cdot)}(\mathbb{R})$ but $p(\cdot) \notin LH_0$.

Similar example holds at infinity.



Continuity not necessary

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Define

$$p(x) = \begin{cases} 2 + \alpha \sin(\pi \log \log(1/|x|)) & |x| < e^{-e} \\ 2 & |x| \geq e^{-e} \end{cases}$$

Then M bounded on $L^{p(\cdot)}(\mathbb{R})$ but $p(\cdot)$ not continuous at 0.

Similar example holds at infinity.



Condition at infinity

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$p(\cdot) \in N_\infty$ if

$$\int_{\{p(x) \neq p_\infty\}} \exp\left(-k \left| \frac{1}{p(x)} - \frac{1}{p_\infty} \right| \right) dx < \infty.$$

If $p(\cdot) \in N_\infty$ and $1/p(\cdot) \in LH_0$, M bounded on $L^{p(\cdot)}$.

Use N_∞ condition to prove *Lemma at Infinity* above.



Local condition

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$p(\cdot) \in K_0$ if $\sup_Q |Q|^{-1} \|\chi_Q\|_{p(\cdot)} \|\chi_Q\|_{p'(\cdot)} < \infty$.

If $p(\cdot) \in K_0 \cap N_\infty$, M is bounded on $L^{p(\cdot)}$.

Use K_0 condition to prove the *Hölder inequality* argument above.

K_0 necessary and sufficient on bounded domains (only)!

$L^{p(\cdot)}$ version of Muckenhoupt A_p condition; proof uses theory of weights.



Averaging operators on cubes

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Define

$$\mathcal{Y} = \{ \mathcal{Q} = \{ Q \} : Q \text{ pairwise disjoint cubes} \}.$$

Given $\mathcal{Q} \in \mathcal{Y}$ define the averaging operator

$$A_{\mathcal{Q}}f(x) = \sum_{Q \in \mathcal{Q}} \int_Q |f(y)| dy \chi_Q(x).$$



Equivalent conditions

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Theorem

Given $p(\cdot)$ such that $1 < p_- \leq p_+ < \infty$, TFAE:

- M bounded on $L^{p(\cdot)}$
- A_Q uniformly bounded for all $Q \in \mathcal{Y}$
- M bounded on $L^{p'(\cdot)}$
- For some $s > 1$, $M(|\cdot|^s)^{1/s}$ bounded on $L^{p(\cdot)}$
- $\lim_{r \rightarrow 1^-} (1 - r) \|M(|\cdot|^r)^{1/r}\|_{p(\cdot)} = 0$.



Duality of M

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Find a direct proof of the equivalence of

- M bounded on $L^{p(\cdot)}$
- M bounded on $L^{p'(\cdot)}$

Special case: Prove this for classical Lebesgue spaces.



Tail condition

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Does there exist a “geometric” tail condition to replace LH_∞ that is necessary and sufficient?

Define

$$T_Q(x) = \frac{|Q|}{|Q| + |x - x_Q|^n} \approx M(\chi_Q)(x)$$

$$\sup_Q |Q|^{-1} \|T_Q\|_{p(\cdot)} \|T_Q\|_{p'(\cdot)} < \infty.$$



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D. Cruz-Uribe, L. Diening, and A. Fiorenza.
A new proof of the boundedness of maximal
operators on variable Lebesgue spaces.
Boll. Unione Mat. Ital. (9), 2(1):151–173, 2009.



References II

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L. Diening.

Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$.

Math. Inequal. Appl., 7(2):245–253, 2004.



D. Cruz-Uribe, A. Fiorenza, and C. J. Neugebauer.

The maximal function on variable L^p spaces.

Ann. Acad. Sci. Fenn. Math., 28(1):223–238, 2003.



References III

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References



A. Nekvinda.

Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$.
Math. Inequal. Appl., 7(2):255–265, 2004.



T. Kopaliani.

On the Muckenchaup condition in variable
Lebesgue spaces.
Proc. A. Razmadze Math. Inst., 148:29–33,
2008.



References IV

Variable $L^{p(\cdot)}$ Spaces

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Cruz-Uribe, SFO

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References



L. Diening.

Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces.

Bull. Sci. Math., 129(8):657–700, 2005.



A. K. Lerner and S. Ombrosi.

A boundedness criterion for general maximal operators.

Publ. Mat., 54(1):53–71, 2010.



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Cruz-Uribe, SFO

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Conditions

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Sufficient
Condition

Open Questions

References



B. Muckenhoupt and R. L. Wheeden.

Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform.

Studia Math., 55(3):279–294, 1976.