Variable Lebesgue Spaces

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Rubio de Francia Extrapolation on Variable Lebesgue Spaces

Outline

Operators on $L^{p(\cdot)}$

Our goal: prove a variety of operators—singular integrals, Riesz potentials, square functions—are bounded on $L^{p(\cdot)}$.

Our philosophy: follow Calderón-Zygmund program and use our control of the maximal operator.

Our approach: use theory of weights and Rubio de Francia extrapolation.

Weighted norm inequalities

A weight: $w \in L^1_{\text{loc}}$, $w \geq 0$

A weighted norm inequality for an operator T:

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \leq C(w,n,T) \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx,$$

where $1 \leq p < \infty$.

Equivalently, $\|Tf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$.

A_p weights

For $1 < p < \infty$, $w \in A_p$ if

$$[w]_{A_p} = \sup_Q \fint_Q w(x)\,dx \left(\fint_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty.$$

Equivalently,

$$\sup_Q |Q|^{-1} \|w\chi_Q\|_p \|w\chi_Q\|_{p'} < \infty.$$

A_p weights (cont.)

For $p = 1$, $w \in A_1$ if $Mw(x) \leq Cw(x)$ a.e.

$$[w]_{A_1} = \operatorname*{ess\,sup}_x \frac{Mw(x)}{w(x)} < \infty.$$

For all $p > 1$, $A_1 \subset A_p$.

$w \in A_p$, $1 < p < \infty$, iff $\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$

Rubio de Francia extrapolation

Theorem

Given an operator T, suppose that for some p_0, $1 \leq p_0 < \infty$, and every $w \in A_{p_0}$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)\, dx \leq C(n,T) \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\, dx.$$

Then for every p, $1 < p < \infty$, and every $w \in A_p$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p} w(x)\, dx \leq C(n,T) \int_{\mathbb{R}^n} |f(x)|^{p} w(x)\, dx.$$

A pithy summary

There are no L^p spaces,

only weighted L^2.

— A. Cordoba

Generalized extrapolation theory

Work over the past decade has shown that Rubio de Francia extrapolation can be generalized in many directions.

There are no Banach function spaces,

only weighted L^2.

Convolution

Given locally integrable functions f, g, define

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy.$$

Approximate identities

Given $\phi \in L^1$, $\int \phi(x)\,dx = 1$, and $t > 0$, define $\phi_t(x) = t^{-n}\phi(x/t)$.

The family of operators $\phi_t * f$ is called an approximate identity.

$\{\phi_t\}$ is a potential-type approximate identity if

$$\Phi(x) = \sup_{|y|\geq|x|} |\phi(y)| \in L^1.$$

Classical result

Given an approximate identity $\{\phi_t\}$:

- For $1 \leq p < \infty$, $\|\phi_t * f - f\|_p \to 0$ as $t \to 0$.
- If ϕ_t is a potential-type approximate identity, for $1 \leq p \leq \infty$, $\phi_t * f \to f$ pointwise a.e.

Pointwise convergence on $L^{p(\cdot)}$

Theorem

If $\{\phi_t\}$ is an approximate identity, then given $p(\cdot)$, for all $f \in L^{p(\cdot)}$, $\phi_t * f \to f$ pointwise a.e.

Proof:

- $f = f_1 + f_2$, $f_1 \in L^{p_-}$, $f_2 \in L^{p_+}$
- $\phi_t * f = \phi_t * f_1 + \phi_t * f_2$
- Apply classical result.

Norm convergence on $L^{p(\cdot)}$

Theorem

Given $p(\cdot)$, suppose M is bounded on $L^{p'(\cdot)}$. Then given any potential-type approximate identity $\{\phi_t\}$,

- $\sup_{t>0} \|\phi_t * f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$
- $\|\phi_t * f - f\|_{p(\cdot)} \to 0$ *as* $\to \infty$.

The proof : set up

WLOG: replace ϕ by Φ since $|\phi_t * f| \leq \Phi_t * |f|$.

Claim: $\sup_{t>0} |\Phi_t * f(x)| \leq CMf(x)$ a.e.

- Approximate Φ by $\sum_k a_k \chi_{B_k}(x)$, where B_k are nested balls centered at origin.
- Apply definition of convolution and monotone convergence theorem

The proof: duality

$$\begin{aligned}
\|\Phi_t * f\|_{p(\cdot)} &\leq C \int_{\mathbb{R}^n} (\Phi_t * f)(x) g(x)\, dx \qquad \|g\|_{p'(\cdot)} = 1 \\
&= C \int_{\mathbb{R}^n} f(x)(\Phi_t * g)(x)\, dx \\
&\leq C \int_{\mathbb{R}^n} f(x) Mg(x)\, dx \\
&\leq C \|f\|_{p(\cdot)} \|Mg\|_{p'(\cdot)} \\
&\leq C \|f\|_{p(\cdot)}
\end{aligned}$$

Extrapolation in $L^{p(\cdot)}$

Theorem

Given an operator T, suppose that for some p_0, $1 \leq p_0 < \infty$, and all $w \in A_1$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)\, dx \leq C([w]_{A_1}) \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\, dx.$$

Then given $p(\cdot)$ such that $p_0 \leq p_- \leq p_+ < \infty$ and M is bounded on $L^{(p(\cdot)/p_0)'}$,

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Proof: iteration algorithm

Define operator

$$\mathcal{R}h = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|^k_{L^{(p(\cdot)/p_0)'}}}.$$

Properties:

- If h non-negative, $h \leq \mathcal{R}h$.
- $\|\mathcal{R}h\|_{L^{(p(\cdot)/p_0)'}} \leq 2\|h\|_{L^{(p(\cdot)/p_0)'}}$.
- $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^{(p(\cdot)/p_0)'}}$

Proof: duality

$$
\begin{aligned}
\|Tf\|_{p(\cdot)}^{p_0} &= \||Tf|^{p_0}\|_{p(\cdot)/p_0} \\
&\leq C\int_{\mathbb{R}^n} |Tf(x)|^{p_0} g(x)\,dx \qquad \|g\|_{(p(\cdot)/p_0)'} = 1 \\
&\leq C\int_{\mathbb{R}^n} |Tf(x)|^{p_0} \mathcal{R}g(x)\,dx \\
&\leq C\int_{\mathbb{R}^n} |f(x)|^{p_0} \mathcal{R}g(x)\,dx \\
&\leq C\||f|^{p_0}\|_{p(\cdot)/p_0}\|\mathcal{R}g\|_{(p(\cdot)/p_0)'} \\
&\leq C\|f\|_{p(\cdot)}^{p_0}.
\end{aligned}
$$

Generalizations

- Arbitrary domains Ω
- Weak type inequalities:

$$\|\lambda\chi_{\{x:|Tf(x)|>\lambda\}}\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$$

- Vector-valued inequalities: $1 < q < \infty$,

$$\|\|Tf_i(\cdot)\|_{\ell^q}\|_{p(\cdot)} \leq C\|\|f_i(\cdot)\|_{\ell^q}\|_{p(\cdot)}$$

- Coifman-Fefferman type inequalities:

$$\|Tf\|_{p(\cdot)} \leq C\|Sf\|_{p(\cdot)}$$

- Off-diagnoal estimates:

$$\|Tf\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}, \qquad p(x)^{-1} - q(x)^{-1} = \alpha/n.$$

Singular integrals

Let $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ be such that:

- $\widehat{K} \in L^\infty$
- $|K(x)| \leq C|x|^{-n}$, $x \neq 0$
- $|\nabla K(x)| \leq |x|^{-(n+1)}$, $x \neq 0$.

Define the singular integral operator $Tf(x) = K * f(x)$.

Examples:

- Hilbert transform: $K(x) = \dfrac{1}{x}$
- Riesz transforms: $K(x) = \dfrac{x_j}{|x|^{n+1}}$, $1 \leq j \leq n$

$L^{p(\cdot)}$ estimates

Theorem

Given $p(\cdot)$, $1 < p_- \leq p_+ < \infty$, if M is bounded on $L^{p(\cdot)}$, then

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$$

where T is a singular integral.

Proof of $L^{p(\cdot)}$ estimates

If $1 < p_- \leq p_+ < \infty$ and M is bounded on $L^{p(\cdot)}$, $\exists s > 1$ such that M bounded on $L^{p(\cdot)/s}$, and so bounded on $L^{(p(\cdot)/s)'}$.

Let $p_0 = p_-/s > 1$. For $w \in A_{p_0}$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\, dx.$$

Hence, by extrapolation,

$$\|Tf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Variable Sobolev spaces

Given Ω and $p(\cdot)$ define

$$W^{k,p(\cdot)}(\Omega) = \{f : f, \partial^\alpha f \in L^{p(\cdot)}(\Omega), |\alpha| \leq k\}.$$

$$\|f\|_{W^{k,p(\cdot)}(\Omega)} = \|f\|_{p(\cdot)} + \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{p(\cdot)}.$$

$W^{k,p(\cdot)}(\Omega)$ is a Banach function space.

$W_0^{k,p(\cdot)}(\Omega)$: closure of $C_c(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$

Density of smooth functions

Question: When are smooth functions dense in $W^{1,p(\cdot)}(\Omega)$?

Let B be the unit disk in $\mathbb{R}^2$ and define for $x = (r, \theta)$

$$p(x) = \begin{cases} 3/2 & 0 < \theta < \frac{\pi}{2}, \pi < \theta < \frac{3\pi}{2} \\ 4 & \frac{\pi}{2} < \theta < \pi, \frac{3\pi}{2} < \theta < 2\pi. \end{cases}$$

Then continuous functions are not dense in $W^{1,p(\cdot)}(B)$.

Density of $C_c^\infty(\mathbb{R}^n)$

Theorem

Let $p(\cdot) \in LH_0$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^n)$.

- Use smooth cut-off function to show functions of compact support are dense.
- Fix ball B; M bounded on $L^{p(\cdot)}(2B)$
- Fix approximate identity $\{\phi_t\}$, $\phi \in C_c^\infty$
- If $\mathrm{supp}(f) \subset B$, $\phi_t * f \in C_c^\infty(2B)$
- $\|\phi_t * f - f\|_{L^{p(\cdot)}(2B)} \to 0$

Sobolev embedding theorem

Theorem

Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < n$, define $p^*(\cdot)$ by $p(x)^{-1} - p^*(x)^{-1} = 1/n$. If M is bounded on $L^{(p^*(\cdot)/(p^*)_-)'}(\Omega)$, then $W_0^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$ and

$$\|f\|_{p^*(\cdot)} \leq \|\nabla f\|_{p(\cdot)}.$$

Proof of Sobolev embedding

- Weak (p, p^*) inequality for I_1: $w \in A_1$, $1 \leq p < n$,

$$\|I_1 f\|_{L^{p^*,\infty}(w)} \leq C\|f\|_{L^p(w^{p^*/p})}$$

- Long-Nie decomposition to get weighted Sobolev embedding: $w \in A_1$, $1 \leq p < n$,

$$\|f\|_{L^{p^*}(w)} \leq C\|\nabla f\|_{L^p(w^{p^*/p})}$$

- Off-diagonal extrapolation.

Calderón-Zygmund theorem

Theorem

Let $\Omega \subset \mathbb{R}^n$ be bounded, and $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $p(\cdot) \in LH_0$ and $p_+ < n/2$. Given $f \in L^{p(\cdot)}(\Omega)$, if u is a solution to

$$\Delta u = f,$$

then $u \in W^{2,p(\cdot)}(\Omega)$.

Norm Inequalities $p_+ = \infty$

Does there exist an extrapolation theory into $L^{p(\cdot)}$ if $p_+ = \infty$?

Does there exist $p(\cdot)$, $p_+ = \infty$ but $|\Omega_\infty| = 0$, such that singular integrals are bounded on $L^{p(\cdot)}$?

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