On the Optimal Control of Non-Newtonian Fluids

TELMA GUERRA[1] JORGE TIAGO[2] ADÉLIA SEQUEIRA[3]

Abstract. We consider optimal control problems of systems governed by stationary, incompressible generalized Navier-Stokes equations with shear-dependent viscosity in a two-dimensional or three-dimensional domain. We study a general class of viscosity functions which switches between shear-thinning and shear-thickening behaviour. We prove an existence result for such class of optimal control problems.

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1 Introduction

This paper is devoted to the proof of existence of solution for a distributed optimal control problem of a viscous and incompressible fluid. The control and state variables are constrained to satisfy a system with shear dependent viscosity which switches between shear-thinning and shear thickening behavior.

More specifically, we deal with a quasi-linear generalization of the stationary Navier-Stokes system described as

$$\begin{cases} -div\ (S(D\mathbf{y})) + \mathbf{y}\cdot\nabla\mathbf{y} + \nabla p = \mathbf{u} & \text{in } \Omega \\ div\,\mathbf{y} = 0 & \text{in } \Omega \\ \mathbf{y} = 0 & \text{on } \partial\Omega\ , \end{cases} \tag{1}$$

[1]Escola Superior de Tecnologia do Barreiro, IPS. E-mail:telma.guerra@estbarreiro.ips.pt.
Centro de Matemática e Aplicações, FCT-UNL.
[2]DMAT - IST-UTL. E-mail:jorge.tiago@dmat.ist.utl.pt
Cemat.
[3]DMAT - IST-UTL. E-mail:adelia.sequeira@dmat.ist.utl.pt
Cemat.

where S is the extra stress tensor given by

$$S(\eta) = (1 + |\eta|)^{\alpha(x)-2}\eta$$

and $\alpha(x)$ is a positive bounded continuous function. The vector $\mathbf{y}$ denotes the velocity field, p denotes the pressure, $D\mathbf{y} = \frac{1}{2}\big(\nabla\mathbf{y} + (\nabla\mathbf{y})^T\big)$ is the symmetric part of the velocity gradient, $\mathbf{u}$ is the given body force and Ω ($n = 2$ or $n = 3$) is an open bounded subset of $I\!R^n$.

System (1) can be used to model steady incompressible electro-rheological fluids. It is based on the assumption that electro-rheological materials, composed by suspesions of particles in a fluid, can be considered as a homegenized single continuum media. The corresponding viscosity has the property of switching between shear-thinning and shear-thickening under the application of a magnetic field. This model is described and analyzed in [13, 14, 15] or [7]. More recently, in [5], the authors proved the existence and uniqueness of a $C^{1,\gamma}(\bar{\Omega}) \cap \mathbf{W}^{2,2}(\Omega)$ solution under smallness data conditions for the system (1). This regularity result motivate us to the analysis of the associated distributed optimal control problem that we describe in the following.

Let us look for the control $\mathbf{u}$ and the corresponding $\mathbf{y}_u$ solution of (1) such that the pair $(\mathbf{y}_u, \mathbf{u})$ solves

$$(P_\alpha)\begin{cases} \text{Minimize } J(\mathbf{u}, \mathbf{y}_u) \\ \text{subject to (1)} \end{cases} \tag{2}$$

where $J : \mathbf{L}^2(\Omega) \times \mathbf{W}_0^{1,2}(\Omega) \to I\!R$ is given by

$$J(\mathbf{u}, \mathbf{y}) = \frac{1}{2}\int_\Omega |\mathbf{y}_u - \mathbf{y}_d|^2\, dx + \frac{\nu}{2}\int_\Omega |\mathbf{u}|^2\, dx. \tag{3}$$

and $\mathbf{y}_d$ denotes a fixed element of $\mathbf{L}^2(\Omega)$.

Such type of optimal control problems has been subject of intensive research in the past decades. For systems governed by non-Newtonian fluids we mention the results in [1, 3, 4, 8, 9, 12, 16] where the authors used several techniques to deal properly with the shear-thickening and shear-thinning viscosity laws, both in the 2D and 3D cases. For the existence of solution, such techniques consist in exploring correctly the properties of S in order to establish compactness results

necessary for the application of the direct method of the calculus of variations. Our purpose here is to show that, based on the regularity results in [5], it is possible to easily extend these ideas to the case of electro-rheological fluids modeled by (1). Treating the optimality conditions associated to (P_α) is also an important, yet delicate, issue. We will therefore postpone this to be treated elsewhere.
In section 2 we introduce the notation that we are going to use, and we recall some useful results. In section 3 we characterize the tensor S including the continuity, coercivity and monotonicity properties. Finally, in section 4 we prove our main existence result.

2 Notation and classical results

We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω, $\mathcal{D}'(\Omega)$ denotes its dual (the space of distributions). The standard Sobolev spaces are represented by $\mathbf{W}^{k,\alpha}(\Omega)$ ($k \in I\!N$ and $1 < \alpha < \infty$), and their norms by $\|\cdot\|_{k,p}$. We set $\mathbf{W}^{0,\alpha}(\Omega) \equiv \mathbf{L}^\alpha(\Omega)$ and $\|\cdot\|_\alpha \equiv \|\cdot\|_{L^\alpha}$. The dual space of $\mathbf{W}_0^{1,\alpha}(\Omega)$ is denoted by $\mathbf{W}^{-1,\alpha'}(\Omega)$ and its norm by $\|\cdot\|_{-1,\alpha'}$. We consider the space of divergence free functions defined by

$$\mathcal{V} = \{\psi \in \mathcal{D}(\Omega) \mid \nabla \cdot \psi = 0\},$$

to eliminate the pressure in the weak formulation. The space $\mathbf{V}_\alpha$ is the closure of $\mathcal{V}$ with respect to the gradient norm , i.e.

$$\mathbf{V}_\alpha = \left\{\psi \in \mathbf{W}_0^{1,\alpha}(\Omega) \mid \nabla \cdot \psi = 0\right\}.$$

The space of Hölder contínuos functions is a Banach space defined as

$$C^{m,\gamma}(\bar{\Omega}) \equiv \{\mathbf{y} \in C^m(\bar{\Omega}) : \|\mathbf{y}\|_{C^{m,\gamma}(\bar{\Omega})} < \infty\}$$

where

$$\|\mathbf{y}\|_{C^{m,\gamma}(\bar{\Omega})} \equiv \sum_{|\alpha|=0}^{m} \|D^\alpha \mathbf{y}\|_\infty + [\mathbf{y}]_{C^{m,\gamma}(\bar{\Omega})}, \tag{4}$$

$$[\mathbf{y}]_{C^{m,\gamma}}(\bar{\Omega}) \equiv \sum_{|\alpha|=m} \sup_{\{x_1,x_2\in\bar{\Omega},x_1\neq x_2\}} \frac{|D^\alpha \mathbf{y}(x_1)-D^\alpha \mathbf{y}(x_2)|}{|x_1-x_2|^\gamma} < +\infty,$$

for m a nonnegative integer and $0 < \gamma < 1$ and

$$D^\alpha \mathbf{y} \equiv \frac{\partial^{|\alpha|}\mathbf{y}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$$

for $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \in I\!N_0$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

We recall now two important inequalities.

Lemma 2.1 (Poincaré's inequality)**.** *Let* $\mathbf{y} \in \mathbf{W}_0^{1,\alpha}(\Omega)$ *with* $1 \leq \alpha < +\infty$. *Then there exists a constant* C_1 *depending on* α *and* Ω *such that*

$$\|\mathbf{y}\|_\alpha \leq C_1(\alpha, \Omega)\|\nabla \mathbf{y}\|_\alpha.$$

Proof. [2].

Lemma 2.2 (Korn's inequality)**.** *Let* $\mathbf{y} \in \mathbf{W}_0^{1,\alpha}(\Omega)$ *with* $1 < \alpha < +\infty$. *Then there exists a constant* C_2 *depending on* Ω *such that*

$$C_2(\Omega)\|\mathbf{y}\|_{1,\alpha} \leq \|D\mathbf{y}_\alpha\|_\alpha.$$

Proof. [11].

Finally two simple, yet very useful, properties of the convective term.

Lemma 2.3. *Let us consider* $\mathbf{u}$ *in* $\mathbf{V}_2$ *and* $\mathbf{v}$, $\mathbf{w}$ *in* $\mathbf{W}_0^{1,2}(\Omega)$, *then*

$$(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) = -(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) \quad \textit{and} \quad (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{v}) = 0. \tag{5}$$

3 Extra tensor properties

We assume that the tensor $S : I\!R^{n\times n}_{sym} \longrightarrow I\!R^{n\times n}_{sym}$ has a potential, i.e. there exists a function $\Phi \in C^2(I\!R^+_0, I\!R^+_0)$ with $\Phi(0) = 0$ such that

$$S_{ij}(\eta) = \frac{\partial \Phi(|\eta|^2)}{\partial \eta_{ij}} = 2\Phi'(|\eta|^2)\,\eta_{ij}, \qquad S(0) = 0$$

for all $\eta \in I\!R^{n\times n}_{sym}$ $\big($here $I\!R^{n\times n}_{sym}$ consists of all symetric $(n \times n)$-matrices). An example of such tensor is the one we are going to work with:

$$S(\eta) = (1 + |\eta|)^{\alpha(x)-2}\eta$$

where $\alpha(x)$ is a continuous function in $\bar{\Omega}$ such that

$$\alpha(x) : \bar{\Omega} \to (1, +\infty)$$

and

$$1 < \alpha_\infty \leq \alpha(x) \leq \alpha_0 < +\infty \tag{6}$$

$$\min \alpha(x) = \alpha_\infty,$$

$$\max \alpha(x) \leq \alpha_0 \quad \text{for} \quad \alpha_0 > 2.$$

The case $1 < \alpha(x) < 2$ corresponds to shear-thinning behaviour fluids for viscosity, whereas $\alpha(x) > 2$ corresponds to shear-thickening. The case $\alpha(x) = 2$ corresponds Newtonian case. For such $\alpha(x)$, it can be proved that S satisfies the standard properties

Properties **3.1.** *Consider $\alpha \in (1, \infty)$, C_3 and C_4 positive constants we have*

A1 -

$$\left| \frac{\partial S_{k\ell}(\eta)}{\partial \eta_{ij}} \right| \leq C_3\,(1 + |\eta|)^{\alpha(x)-2}.$$

A2 -

$$S'(\eta) : \zeta : \zeta = \sum_{ijk\ell} \frac{\partial S_{k\ell}(\eta)}{\partial \eta_{ij}} \zeta_{k\ell}\zeta_{ij} \geq C_4\,(1 + |\eta|)^{\alpha(x)-2}\,|\zeta|^2$$

for all $\eta, \zeta \in I\!R^{n\times n}_{sym}$ and $i, j, k, \ell = 1, \cdots, d$.

Proof. In fact,

$$\begin{aligned} \left| \frac{\partial S_{kl}}{\partial \eta_{ij}} \right| &= \left| (\alpha(x) - 2)\,(1 + |\eta|)^{\alpha(x)-3}\,\delta_{ik}\delta_{jl} + (1 + |\eta|)^{\alpha(x)-2}\,\delta_{ik}\delta_{jl} \right| \\ &= \left| (\alpha(x) - 2)\,(1 + |\eta|)^{\alpha(x)-3}\,\delta_{ik}\delta_{jl} + (1 + |\eta|)^{\alpha(x)-2}\,\delta_{ik}\delta_{jl} \right| \\ &\leq |\alpha(x) - 2|\,(1 + |\eta|)^{\alpha(x)-3}\,|\delta_{ik}\delta_{jl}| + (1 + |\eta|)^{\alpha(x)-2}\,|\delta_{ik}\delta_{jl}|. \end{aligned} \tag{7}$$

Taking into account that

$$\delta_{ik}\delta_{jl} = \begin{cases} 1 & \text{if } i=k, j=l \\ 0 & \text{otherwise} \end{cases}$$

we can write

$$\begin{aligned}(7) &\leq |\alpha(x)-2|\,(1+|\eta|)^{\alpha(x)-3} + (1+|\eta|)^{\alpha(x)-2} \\ &= (1+|\eta|)^{\alpha(x)-3}\left(|\alpha(x)-2| + (1+|\eta|)\right) \end{aligned} \tag{8}$$

Considering that $\alpha(x) \leq \alpha_0$, we have

$$\begin{aligned}(8) &\leq (1+|\eta|)^{\alpha(x)-3}\left((\alpha_0-2) + (1+|\eta|)\right) \\ &\leq \left(1+|\eta|^2\right)^{\alpha(x)-3}\left((\alpha_0-2)\,(1+|\eta|) + (1+|\eta|)\right) \\ &= \left(1+|\eta|^2\right)^{\alpha(x)-3}(\alpha_0-1)(1+|\eta|) \\ &= (\alpha_0-1)\,(1+|\eta|)^{\alpha(x)-2},\end{aligned}$$

and therefore we have **A1** with $C_3 = \alpha_0 - 1$. To obtain **A2** we write

$$\begin{aligned}&S'(\eta):\zeta:\zeta = \sum_{ijk\ell}\frac{\partial S_{k\ell}(\eta)}{\partial \eta_{ij}}\zeta_{k\ell}\zeta_{ij} \\ &= \sum_{ijk\ell}\left[(\alpha(x)-2)\,(1+|\eta|)^{\alpha(x)-3}\,\delta_{ik}\delta_{jl} + (1+|\eta|)^{\alpha(x)-2}\,\delta_{ik}\delta_{jl}\right]\zeta_{ij}\zeta_{k\ell} \\ &= (\alpha(x)-2)\,(1+|\eta|)^{\alpha(x)-3}\sum_{ijk\ell}\delta_{ik}\delta_{jl}\zeta_{ij}\zeta_{k\ell} + (1+|\eta|)^{\alpha(x)-2}\sum_{ijk\ell}\delta_{ik}\delta_{jl}\zeta_{ij}\zeta_{k\ell}\end{aligned} \tag{9}$$

Considering that

$$\sum_{ijk\ell}\delta_{ik}\delta_{jl}\zeta_{ij}\zeta_{k\ell} = \sum_{ij}\zeta_{ij}\zeta_{ij} = |\zeta|^2$$

expression (9) is equal to

$$\begin{aligned}&(\alpha(x)-2)\,(1+|\eta|)^{\alpha(x)-3}\,|\zeta|^2 + (1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2 \\ &= (1+|\eta|)^{\alpha(x)-3}\left[(\alpha(x)-2)| + (1+|\eta|)\right]|\zeta|^2\end{aligned} \tag{10}$$

Taking account $\alpha(x)-2<0$ and $\alpha_\infty \leq \alpha(x)$ it follows that

$$
\begin{aligned}
(10) &\geq (1+|\eta|)^{\alpha(x)-3}\left[(\alpha(x)-2)(1+|\eta|)+(1+|\eta|)\right]|\zeta|^2 \\
&= (1+|\eta|)^{\alpha(x)-3}\left[(\alpha(x)-1)(1+|\eta|)\right]|\zeta|^2 \\
&= (\alpha(x)-1)\,(1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2 \\
&\geq (\alpha_\infty-1)\,(1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2.
\end{aligned}
$$

Instead, $\alpha(x)-2 \geq 0$ gives

$$
\begin{aligned}
(10) &\geq (1+|\eta|)^{\alpha(x)-3}\,(1+|\eta|)\,|\zeta|^2 \\
&= (1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2.
\end{aligned}
$$

Then we have

$$
S'(\eta):\zeta:\zeta \geq \begin{cases} (\alpha_\infty-1)\,(1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2 & \text{if} \quad \alpha(x)-2<0 \\ (1+|\eta|)^{\alpha(x)-2}\,|\zeta|^2 & \text{if} \quad \alpha(x)-2\geq 0. \end{cases}
$$

Assumptions **A1-A2** imply the following standard properties for S (see [11]):

- Continuity

$$
|S(\eta)| \leq (1+|\eta|)^{\alpha(x)-2}|\eta|, \tag{11}
$$

- Coercivity

$$
S(\eta):\eta \geq \begin{cases} \nu(1+|\eta|)^{\alpha(x)-2}|\eta|^2 & \text{if} \quad \alpha(x)-2<0 \\ |\eta|^2 & \text{if} \quad \alpha(x)-2\geq 0 \end{cases} \tag{12}
$$

- Monotonicity

$$
(S(\eta)-S(\zeta)):(\eta-\zeta) \geq \nu(1+|\eta|+|\zeta|)^{\alpha(x)-2}|\eta-\zeta|^2. \tag{13}
$$

For continuity we have

$$
|S(\eta)| = \left|(1+|\eta|)^{\alpha(x)-2}\eta\right| = \left|(1+|\eta|)^{\alpha(x)-2}\right|\,|\eta| = (1+|\eta|)^{\alpha(x)-2}|\eta|.
$$

Coercivity is equivalent to monotonocity taking $S(\zeta)=\zeta=0_M$ therefore is enough to prove monotonocity. Taking account that

$$S_{ij}(\eta) - S_{ij}(\zeta) = \int_0^1 \frac{\partial}{\partial t} S_{ij}(t\eta + (1-t)\zeta)\, dt$$
$$= \int_0^1 \sum_{kl} \frac{\partial S_{ij}(t\eta + (1-t)\zeta)}{\partial D_{kl}} (\eta - \zeta)_{kl}$$

we can write

$$(S(\eta) - S(\zeta) : (\eta - \zeta) = \int_0^1 \sum_{ij} \sum_{kl} \frac{\partial S_{ij}(t\eta + (1-t)\zeta)}{\partial D_{kl}} (\eta - \zeta)_{kl} : (\eta - \zeta)_{ij}\, dt$$
$$= \int_0^1 S'(t\eta + (1-t)\zeta)) : (\eta - \zeta) : (\eta - \zeta)\, dt \tag{14}$$

Using **A2** and considering $\alpha(x) - 2 < 0$ we have

$$(14) \geq \int_0^1 (\alpha_\infty - 1)\, (1 + |t\eta + (1-t)\zeta|)^{\alpha(x)-2}\, |\eta - \zeta|^2\, dt \tag{15}$$

Once we have ($t \in [0, 1]$)

$$1 + |t\eta + (1-t)\zeta| \leq 1 + |\eta + \zeta| \leq 1 + |\eta| + |\zeta| \tag{16}$$

we can write

$$(15) \geq \int_0^1 (\alpha_\infty - 1)\, ((1 + |\eta| + |\zeta|))^{\frac{\alpha(x)-2}{2}}\, |\eta - \zeta|^2\, dt$$

Using **A2** and considering $\alpha(x) - 2 \geq 0$ we have

$$(14) \geq \int_0^1 (1 + |t\eta + (1-t)\zeta|)^{\alpha(x)-2}\, |\eta - \zeta|^2\, dt \tag{17}$$
$$\geq \int_0^1 1^{\alpha(x)-2} |\eta - \zeta|^2\, dt \tag{18}$$
$$\geq |\eta - \zeta|^2 \tag{19}$$

Then we conclude that

$$(S(\eta) - S(\zeta)) : (\eta - \zeta) \geq \begin{cases} (\alpha_\infty - 1)(1 + |\eta| + |\zeta|)^{\alpha(x)-2} |\eta - \zeta|^2 & \text{if} \quad \alpha(x) - 2 < 0 \\ |\eta - \zeta|^2 & \text{if} \quad \alpha(x) - 2 \geq 0 \end{cases}$$

4 Main Result

Definition **4.1.** *Assume that* $\mathbf{u} \in \mathbf{L}^2(\Omega)$. *A function* $\mathbf{y}$ *is a* $C^{1,\gamma}$*-solution of* (1) *if* $\mathbf{y} \in C^{1,\gamma}(\bar{\Omega})$, *for* $\gamma \in (0,1)$, *div* $\mathbf{y} = 0$, $\mathbf{y}|_{\partial\Omega} = 0$ *and it satisfies the following integral equality*

$$(S(D\mathbf{y}), D\varphi) + (\mathbf{y} \cdot \nabla \mathbf{y}, \varphi) = (\mathbf{u}, \varphi), \quad \text{for all } \varphi \in \mathbf{V}_2. \tag{20}$$

Next proposition, due to [5], present us an existence and uniqueness result of a C^{1,γ_0} solution of the system (1) with certain conditions made on $\mathbf{u}$, but without adicional conditions on exponent α.

Proposition **4.2.** *We assume* $\mathbf{u} \in \mathbf{L}^q(\Omega)$ *for some* $q > n$. *Let us consider* Ω *a* C^{1,γ_0} *domain, and* $\alpha \in C^{0,\gamma_0}(\bar{\Omega})$, *with* $\gamma_0 = 1 - \frac{n}{q}$. *Then there exist positive constants* C_5 *and* C_6, *depending on* $\|\alpha\|_{C^{0,\gamma}(\bar{\Omega})}$, n, q *and* Ω *such that, if* $\|\mathbf{u}\|_q < C_5$, *for some* $\gamma < \gamma_0$ *there exists a* $C^{1,\gamma}$ *solution* $(\mathbf{y}, p)$ *of problem* (1) *with*

$$\|\mathbf{y}\|_{C^{1,\gamma}(\bar{\Omega})} + \|p\|_{C^{0,\gamma}(\bar{\Omega})} \leq C_6 \|\mathbf{u}\|_q. \tag{21}$$

Furthermore, there exists a constant C_7 *depending on* α_∞, $\|\alpha\|_{C^{0,\gamma}(\bar{\Omega})}$, n, q *and* Ω *such that if* $\|\mathbf{u}\|_q \leq C_7$ *the solution is unique.*

Proposition **4.3.** *Assume that* **A1** *and* **A2** *are fullfilled. Considering* $\mathbf{y} \in C^{1,\gamma}(\bar{\Omega})$ *we have*

$$\|D\mathbf{y}\|_2 \leq C_8 \|\mathbf{u}\|_2 \tag{22}$$

where $\mathbf{y}$ *is the associated state to* $\mathbf{u}$.

Proof. Taking $\varphi = \mathbf{y}$ in (20) and recalling the convective term properties we have

$$(S(D\mathbf{y}), D\mathbf{y}) = (\mathbf{u}, \mathbf{y}). \tag{23}$$

The fact $\mathbf{y} \in C^{1,\gamma}(\bar{\Omega})$ implies that $\mathbf{y}$ belongs to $C^1(\bar{\Omega})$ which means that $\mathbf{y}$ and $D\mathbf{y}$ are limited functions in $\bar{\Omega}$ and consequently belong to $\mathbf{L}^\alpha(\Omega)$ for any $\alpha > 1$, in particular we consider $\mathbf{y} \in \mathbf{L}^2(\Omega)$. Then, on one hand, by using Holder´s inequality and the Poincaré and Korn inequalities there existes a constant C_8 such that

$$|(\mathbf{u}, \mathbf{y})| \leq \|\mathbf{u}\|_2 \|\mathbf{y}\|_2 \leq C_8 \|\mathbf{u}\|_2 \|D\mathbf{y}\|_2.$$

On the other hand, by coercivity we write

$$\|D\mathbf{y}\|_2^2 \leq (S(D\mathbf{y}), D\mathbf{y})$$

Putting together both inequalities with (23) we have the pretended result.

Once we have the guaratee of existence of a solution for (1) provided by Proposition 4.2 and a estimative of energy for $D\mathbf{y}$ given by proposition 4.3, we can now formulate and prove the following existence result for the control problem (P_α).

THEOREM **4.4** (Main Result)**.** *Assume that* **A1**-**A2** *are fulfilled, with* $1 < \alpha \leq 2$. *Then* (P_α) *admits at least a solution.*

To prove this theorem we have to establish some important results:

PROPOSITION **4.5.** *Assume that* $(\mathbf{u}_k)_{k>0}$ *converges to* $\mathbf{u}$ *weakly in* $\mathbf{L}^2(\Omega)$. *Then there exists* $\mathbf{y} \in \mathbf{W}_0^{1,2}(\Omega)$ *and* $\tilde{S} \in \mathbf{L}^2(\Omega)$ *such that*

$$(\mathbf{y}_k)_k \rightharpoonup \mathbf{y} \quad \textit{in} \quad \mathbf{W}_0^{1,2}(\Omega) \tag{24}$$

$$(D\mathbf{y}_k)_k \rightharpoonup D\mathbf{y} \quad \textit{in} \quad \mathbf{L}^2(\Omega) \tag{25}$$

$$(S(D\mathbf{y}_k))_k \rightharpoonup \tilde{S} \quad \textit{in} \quad \mathbf{L}^2(\Omega). \tag{26}$$

Proof. The convergence of $(\mathbf{u}_k)_{k>0}$ to $\mathbf{u}$ in the weak topology of $\mathbf{L}^2(\Omega)$ implies that $(\mathbf{u}_k)_{k>0}$ is bounded, ie, there exists a positive constant M such that

$$\|\mathbf{u}_k\|_2 \leq M, \quad \text{for } k > k_0. \tag{27}$$

Due to (22) and (27), it follows that

$$\|D\mathbf{y}_k\|_2 \le C_8 M$$

By Korn´s inequality $\mathbf{y}_k$ is then bounded in $\mathbf{W}_0^{1,2}(\Omega)$ and thus there is a subsequence still indexed in k that weakly converges to a certain $\mathbf{y}_k$ in $\mathbf{W}_0^{1,2}(\Omega)$. Also, by using a Sobolev's compact injection, $\mathbf{y}_k$ converges strongly (then weakly) to $\mathbf{y}$ in $\mathbf{L}^2(\Omega)$. It is straightforward to conclude (25).
Finally, the previous estimate, together with (11) implies

$$\begin{aligned}
\|S(D\mathbf{y}_k)\|_2^2 &\le \int_\Omega (1+|D\mathbf{y}_k|)^{(\alpha(x)-2)2}|D\mathbf{y}_k|^2\,dx \\
&\le \int_\Omega (1+|D\mathbf{y}_k|)^{(\alpha(x)-2)2}(1+|D\mathbf{y}_k|)^2\,dx \\
&= \int_\Omega (1+|D\mathbf{y}_k|)^{2(\alpha(x)-1)}\,dx \\
&\le C_8 \int_\Omega (1+|D\mathbf{y}_k|^{2(\alpha(x)-1)})\,dx \\
&\le C_8 \left(|\Omega| + \int_\Omega |D\mathbf{y}_k|^{2(\alpha_0-1)}\,dx\right) \\
&= C_8 \left(|\Omega| + \|D\mathbf{y}_k\|_{2(\alpha_0-1)}^{2(\alpha_0-1)}\right).
\end{aligned}$$

The last expression is then bounded once $D\mathbf{y}_k \in C(\bar{\Omega})$ and consequently the sequence $(S(D\mathbf{y}_k))_k$ is bounded in $\mathbf{L}^2(\Omega)$ and we finish the proof by establishing the existence of a subsequence, still indexed by k, and $\tilde{S} \in \mathbf{L}^2(\Omega)$ such that $(S(D\mathbf{y}_k))_{k>0}$ weakly converges to $\tilde{S} \in \mathbf{L}^2(\Omega)$.

Proposition **4.6.** *Assume that* (25), (24) *and* (26) *are verified. Then the weak limit of* $(\mathbf{y}_k)_k$, $\mathbf{y}$, *is the solution of* (20) *corresponding to* $\mathbf{u} \in \mathbf{L}^2(\Omega)$.

Proof. Let us consider

$$(S(D\mathbf{y}_k) - S(\mathbf{y}), D\varphi) + (\mathbf{y}_k \cdot \nabla\mathbf{y}_k - \mathbf{y}\cdot\nabla\mathbf{y}, \varphi) = (\mathbf{u}_k - \mathbf{u}, \varphi), \qquad (28)$$

for all $\varphi \in \mathbf{V}_2$. Taking account the convective term properties and the regularity results assumed on $\mathbf{y}$, we have

$$
\begin{aligned}
&|(\mathbf{y}_k \cdot \nabla \mathbf{y}_k - \mathbf{y} \cdot \nabla \mathbf{y}, \varphi)| \\
&= |((\mathbf{y}_k - \mathbf{y}) \cdot \nabla \mathbf{y}_k, \varphi) + (\mathbf{y} \cdot \nabla (\mathbf{y}_k - \mathbf{y}), \varphi)| \\
&= |((\mathbf{y}_k - \mathbf{y}) \cdot \nabla \mathbf{y}_k, \varphi) - (\mathbf{y} \cdot \nabla \varphi, (\mathbf{y}_k - \mathbf{y}))| \\
&\leq |((\mathbf{y}_k - \mathbf{y}) \cdot \nabla \mathbf{y}_k, \varphi)| + |(\mathbf{y} \cdot \nabla \varphi, (\mathbf{y}_k - \mathbf{y}))| \\
&\leq C_E^2 \left(\|\nabla \mathbf{y}_k\|_2 \|\varphi\|_4 + \|\mathbf{y}\|_4 \|\nabla \varphi\|_2 \right) \|\mathbf{y}_k - \mathbf{y}\|_4 \\
&\to 0 \quad \text{when} \quad k \to +\infty.
\end{aligned}
$$

This result is a consequence of the compact injection of $\mathbf{W}_0^{1,2}(\Omega)$ into $\mathbf{L}^4(\Omega)$ which provide a strong convergence in $\mathbf{L}^4(\Omega)$ once we have (24). Note that C_E corresponds to the embedding constant.
Hence, passing to the limit in

$$(S(D\mathbf{y}_k), D\varphi) + (\mathbf{y}_k \cdot \nabla \mathbf{y}_k, \varphi) = (\mathbf{u}_k, \varphi), \quad \text{for all} \quad \varphi \in \mathbf{V}_2,$$

we obtain

$$(\tilde{S}, D\varphi) + (\mathbf{y} \cdot \nabla \mathbf{y}, \varphi) = (\mathbf{u}, \varphi) \quad \text{for all} \quad \varphi \in \mathbf{V}_2, \tag{29}$$

In particular, taking $\varphi = \mathbf{y}$ and considering (5) we may write

$$(\tilde{S}, D\mathbf{y}) = (\tilde{S}, D\mathbf{y}) + (\mathbf{y} \cdot \nabla \mathbf{y}, \mathbf{y}) = (\mathbf{u}, \mathbf{y}). \tag{30}$$

On the other hand, the monotonocity assumption (13) gives

$$(S(D\mathbf{y}_k) - S(D\varphi), D(\mathbf{y}_k) - D\varphi) \geq 0 \quad \text{for all} \quad \varphi \in \mathbf{V}_2. \tag{31}$$

Since,

$$(S(D\mathbf{y}_k), D\mathbf{y}_k) = (\mathbf{u}_k, \mathbf{y}_k),$$

replacing the first member in (31), we obtain

$$(\mathbf{u}_k, \mathbf{y}_k) - (S(D\mathbf{y}_k), D\varphi) - (S(D\varphi), D\mathbf{y}_k - D\varphi) \geq 0 \quad \text{for all} \quad \varphi \in \mathbf{V}_2.$$

Passing to the limit it follows

$$(\mathbf{u}, \mathbf{y}) - (\tilde{S}, D\varphi) - (\tau(D\varphi), D\mathbf{y} - D\varphi) \geq 0 \quad \text{for all} \quad \varphi \in \mathbf{V}_2.$$

This inequality together with (30), implies that

$$(\tilde{S} - S(D\varphi), D\mathbf{y} - D\varphi) \geq 0 \quad \text{for all} \quad \varphi \in \mathbf{V}_2$$

Taking $\varphi = \mathbf{y} - \lambda\mathbf{v}$ (see [10]), which is possible considering any $\mathbf{v} \in \mathbf{V}_2$ and $\lambda > 0$, we have

$$(\tilde{S} - S(D(\mathbf{y} - \lambda\mathbf{v})), D\mathbf{y} - (D\mathbf{y} - \lambda\mathbf{v})) \geq 0 \quad \text{for all} \quad \varphi \in \mathbf{V}_2 \tag{32}$$

which is equivalent to

$$\lambda(\tilde{S} - S(D(\mathbf{y} - \lambda\mathbf{v})), D\mathbf{v})) \geq 0 \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_2 \tag{33}$$

once $\lambda > 0$,

$$(\tilde{S} - S(D(\mathbf{y} - \lambda\mathbf{v})), D\mathbf{v})) \geq 0 \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_2 \tag{34}$$

Passing to the limit when $\lambda \to 0$ and considering the continuity of S we obtain

$$(\tilde{S} - S(D(\mathbf{y})), D\mathbf{v})) \geq 0 \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}_2, \tag{35}$$

and this implies that

$$\tilde{S} = S(D(\mathbf{y}))$$

and then

$$(S(D\mathbf{y}), D\varphi) + (\mathbf{y} \cdot \nabla\mathbf{y}, \varphi) = (\mathbf{u}, \varphi) \quad \text{for all} \quad \varphi \in \mathbf{V}_2.$$

Hence, $\mathbf{y} \equiv \mathbf{y}_u$, i.e, $\mathbf{y}$ is the solution associated to $\mathbf{u}$.

PROPOSITION **4.7.** *Assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ are satisfied. Then $(\mathbf{y}_k)_k$ strongly converges to $\mathbf{y}_u$ in $\mathbf{W}_0^{1,2}(\Omega)$.*

Proof. Setting $\varphi = y_k - y_u$ in (20) and taking (13) we obtain

$$(S(D\mathbf{y}_k) - S(D\mathbf{y}_u), D(\mathbf{y}_k - \mathbf{y}_u)) \geq \|D(\mathbf{y}_k - \mathbf{y}_u)\|_2^2 \tag{36}$$

Therefore, using (5) and classical embedding results, we obtain

$$\begin{aligned}
&\lim_{k\to+\infty} \|D(\mathbf{y}_k - \mathbf{y}_u)\|_2^2 \\
&\leq \lim_{k\to+\infty} (S(D\mathbf{y}_k) - S(D\mathbf{y}_u), D(\mathbf{y}_k - \mathbf{y}_u)) \\
&= \lim_{k\to+\infty} ((\mathbf{u}_k - \mathbf{u}, \mathbf{y}_k - \mathbf{y}_u) - (\mathbf{y}_k \cdot \nabla \mathbf{y}_k - \mathbf{y}_u \cdot \nabla \mathbf{y}_u, \mathbf{y}_k - \mathbf{y}_u)) \\
&= \lim_{k\to+\infty} ((\mathbf{u}_k - \mathbf{u}, \mathbf{y}_k - \mathbf{y}_u) - ((\mathbf{y}_k - \mathbf{y}_u) \cdot \nabla \mathbf{y}_u, \mathbf{y}_k - \mathbf{y}_u)) \\
&= \lim_{k\to+\infty} \big((\mathbf{u}_k - \mathbf{u}, \mathbf{y}_k - \mathbf{y}_u) - \|\mathbf{y}_k - \mathbf{y}_u\|_4^2 \|\nabla \mathbf{y}_u\|_2\big) = 0.
\end{aligned}$$

and therefore, by Korn's inequality,

$$\|\mathbf{y}_k - \mathbf{y}\|_{1,2} \to 0.$$

We can now prove our main result.

Proof of Theorem 4.4. Let $(\mathbf{u}_k)_k$ be a minimizing sequence in $\mathbf{L}^2(\Omega)$ and $(\mathbf{y}_k)_k$ the sequence of associated states. Considering the properties of J, we obtain

$$\tfrac{\nu}{2}\|\mathbf{u}_k\|_2^2 \leq J(\mathbf{u}_k, \mathbf{y}_k) \leq J(0, \mathbf{y}_0) \quad \text{for} \quad k > k_0$$

implying that $(\mathbf{u}_k)_k$ is bounded in $\mathbf{L}^2(\Omega)$. From Proposition 4.7, we deduce that (y_k) converges strongly to y_u. On one hand J is convex once it is a sum of quadratic terms, on the other hand, if

$$(\mathbf{v}_k, \mathbf{z}_k) \to (\mathbf{v}, \mathbf{z}) \quad \text{in} \quad \mathbf{L}^2(\Omega) \times \mathbf{W}_0^{1,2}(\Omega)$$

implies

$$J(\mathbf{v}_k, \mathbf{z}_k) \to J(\mathbf{v}, \mathbf{z}) \quad \text{in} \quad I\!R$$

then the funcional J is also a continuous function. In fact, once we have

$$\begin{aligned}
&|J(\mathbf{v}_k, \mathbf{z}_k) - J(\mathbf{v}, \mathbf{z})| = \big|\|\mathbf{z}_k - \mathbf{y}_d\|_2^2 + \|\mathbf{v}_k\|_2^2 - \|\mathbf{z} - \mathbf{y}_d\|_2^2 - \|\mathbf{v}\|_2^2\big| \\
&\leq \left|(\|(\mathbf{z}_k - \mathbf{y}_d) - (\mathbf{z} - \mathbf{y}_d)\|_2 + \|(\mathbf{z} - \mathbf{y}_d)\|_2)^2 + (\|\mathbf{v}_k - \mathbf{v}\|_2 + \|\mathbf{v}\|_2)^2 - \|\mathbf{z} - \mathbf{y}_d\|_2^2 - \|\mathbf{v}\|_2^2\right| \\
&\leq \left|(\|(\mathbf{z}_k - \mathbf{z}\|_2 + \|(\mathbf{z} - \mathbf{y}_d)\|_2)^2 + (\|\mathbf{v}_k - \mathbf{v}\|_2 + \|\mathbf{v}\|_2)^2 - \|\mathbf{z} - \mathbf{y}_d\|_2^2 - \|\mathbf{v}\|_2^2\right|.
\end{aligned}$$

Since $\mathbf{z}_k \to \mathbf{z}$ strongly also in $\mathbf{L}^2$, last expression converges to zero when $k \to \infty$ and therefore, J is a semicontinuos function (see result in [2]). We may now apply the direct method of the Calculus os Variations (see for instance [6])

$$\inf_k J \leq J(\mathbf{u}, \mathbf{y}_u) \leq \liminf_k J(\mathbf{v}_k, \mathbf{z}_k) \leq \inf_k J$$

to conclude that $(\mathbf{u}, \mathbf{y}_u)$ is in fact a minimizer and therefore a solution for (P_α).

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