Existence results for optimal control problems with some special non-linear dependence on state and control

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***Abstract*— We present a general approach to prove existence of solutions for optimal control problems not based on typical convexity conditions which quite often are very hard, if not impossible, to check. By taking advantage of several relaxations of the problem, we isolate an assumption which guarantees the existence of solutions of the original optimal control problem. Showing the validity of this crucial hypothesis through various means and in various contexts is the main goal of this contribution. In each such situation, we end up with some existence result. In particular, we would like to stress a general result that takes advantage of the particular structure of both the cost functional and the state equation. One main motivation for our work here comes from a model for guidance and control of ocean vehicles. Some explicit existence results and comparison examples are given.**

I. Introduction

This paper focuses on the analysis of optimal control problems of the general form

$$(P_1) \text{ Minimize in } u: \quad \int_0^T [\sum_{i=1}^s c_i(x(t))\phi_i(u(t))]\, dt \qquad (1)$$

subject to

$$x'(t) = \sum_{i=1}^s Q_i(x(t))\phi_i(u(t)) \text{ in } (0,T), \qquad (2)$$

$$x(0) = x_0 \in \mathbf{R}^N, \qquad (3)$$

and

$$u \in L^\infty(0,T), \quad u(t) \in K, \qquad (4)$$

where $K \subset \mathbf{R}^m$ is compact. The state $x : (0,T) \to \mathbf{R}^N$ takes values in $\mathbf{R}^N$. The mappings

$$c_i : \mathbf{R}^N \to \mathbf{R}, \quad \phi_i : \mathbf{R}^m \to \mathbf{R}, \quad Q_i : \mathbf{R}^N \to \mathbf{R}^N$$

as well as the restriction set $K \subset \mathbf{R}^m$ will play a fundamental role. We assume, at this initial stage, that c_i are continuous, ϕ_i are of class $\mathcal{C}^1$, and each Q_i is Lipschitz so that the state system is well-posed.

In such a general form, we cannot apply results for not-necessarily convex problems like the ones in [1], [14] or [17]. Besides, techniques based on Bauer' Maximum Principle are quite difficult to extend to our general setting because it is hard to analyze the concavity of the cost functional when the dependence on both state and control comes in product form. Also the Rockafellar's variational reformulation introduced in [15], and well-described in [2], [5] or recently in [13], looks as if it cannot avoid assuming a separate dependence on the state and control variables, since this is the main structural restriction on the variational problem for which the existence of solution has been so far ensured ([3]).

Concerning the classical Filippov-Roxin theory introduced in [6] and [18], it is not easy at all to know if typical convexity assumptions hold, or when they may hold, as we can see from the examples and counter-examples in [2]. When analyzing explicit examples, one realizes such difficulties coming from the need of a deep understanding of typical orientor fields. The same troubles would arise when applying refinements of this result as the ones in [10] and [11].

Our aim is to provide hypotheses on the different ingredients of the problem so that existence of solutions can be achieved through an independent road. Actually, it is not easy to claim whether our results improve on classical or more recent general results. They provide an alternative tool which can be more easily used in practice than such results when one faces an optimal control problem under the special structure we consider here. As a matter of fact, convexity will also occur in our statements but in an unexpected and non-standard way.

Before stating our main general result, a bit of notation is convenient. We will write

$$c : \mathbf{R}^N \to \mathbf{R}^s, \quad \phi : \mathbf{R}^n \to \mathbf{R}^s, \quad Q : \mathbf{R}^N \to \mathbf{R}^{Ns}, \quad (5)$$

with components c_i, ϕ_i, and Q_i, respectively. Consider also a new ingredient of the problem related to ϕ. Suppose that there is a $\mathcal{C}^1$ mapping

$$\Psi : \mathbf{R}^s \to \mathbf{R}^{s-n}, \quad \Psi = (\psi_1, ..., \psi_{s-n}), \quad (s > n), \quad (6)$$

so that $\phi(K) \subset \{\Psi = 0\}$. This is simply saying, in a rough way, that the embedded (parametrized) manifold $\phi(K)$ of $\mathbf{R}^s$ is part of the manifold defined implicitly by $\Psi = 0$. In practical terms, it suffices to check that the composition $\Psi(\phi(u)) = 0$ for $u \in K$.

For a pair (c, Q), put

$$\mathcal{N}(c, Q) = \{v \in \mathbf{R}^s : Qv = 0, cv \le 0\}. \qquad (7)$$

Similarly, set

$$\mathcal{N}(K, \phi) = \qquad (8)$$

$$\{v \in \mathbf{R}^s : \nabla\Psi(\phi(u))v = 0 \vee \exists i, \nabla\psi_i(\phi(u))v > 0, \forall u \in K\}$$

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Our main general result is the following.

Theorem 1.1: Assume that the mapping Ψ as above is strictly convex (componentwise) and $\mathcal{C}^1$. If for each $x \in \mathbf{R}^N$, we have

$$\mathcal{N}(c(x), Q(x)) \subset \mathcal{N}(K, \phi), \tag{9}$$

then the corresponding optimal control problem (P_1) admits at least one optimal solution.

A particular, yet still under some generality, situation where this result can be implemented is the case of polynomial dependence where the ϕ_i's are polynomials of various degrees. The main structural assumption, in addition to the one coming from the set K, is concerned with the convexity of the corresponding mapping Ψ.

Suppose we take $\phi_i(u) = u_i$, for $i = 1, 2, \ldots, n$, and $\phi_{n+i}(u)$, $i = 1, 2, \ldots, s-n$, convex polynomials of whatever degree, or simply polynomials whose restriction to K is convex. In particular, K itself is supposed to be convex. Then we can take

$$\Psi_i(v) = \phi_{n+i}(\overline{v}) - v_{n+i}, \tag{10}$$

$$i = 1, 2, \ldots, s-n, \quad \overline{v} = (v_i)_{i=1,2,\ldots,n}.$$

In this case, it is clear that

$$\Psi(\phi(u)) = 0 \text{ for } u \in K,$$

by construction, and, in addition, Ψ is smooth and convex. The important constraint (9) can also be analyzed in more concrete terms, if we specify in a better way the structure of the problem.

As an illustration of the use of Theorem 1.1 above, though more general results are possible, we will concentrate on an optimal control problem of the type

$$(P) \text{ Minimize in } u:$$

$$\int_0^T \left[\sum_{i=1}^n c_i(x(t))u_i(t) + \sum_{i=1}^n c_{n+i}(x(t))u_i^2(t)\right] dt \tag{11}$$

subject to

$$x'(t) = Q_0(x(t)) + Q_1(x(t))u(t) + Q_2(x(t))u^2(t) \tag{12}$$

$$\text{in } (0, T),$$

$$x(0) = x_0 \in \mathbf{R}^n, \text{ and } u(t) \in K \subset \mathbf{R}^n. \tag{13}$$

We are taking here $N = n$. Q_1 and Q_2 are $n \times n$ matrices that, together with the vector Q_0, comply with appropriate technical hypotheses so that the state law is a well-posed problem. Set

$$Q = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & c_2 \end{pmatrix}, \tag{14}$$

where Q_1 is a non-singular $n \times n$ matrix, and $c_1 \in \mathbf{R}^n$. In addition, we put

$$D(x) = -(Q_1)^{-1}Q_2, \quad E(x) = c_1 D + c_2, \tag{15}$$

$$U(m, x) = 2\sum_i m_i e_i \otimes e_i D - \text{id}, \quad m = \phi(u), \tag{16}$$

where the e_i's stand for the vectors of the canonical basis of $\mathbf{R}^n$, and id is the identity matrix of size $n \times n$.

Theorem 1.2: Suppose that for the ingredients (c, Q, K) of (P), we have

1) the matrix U is always non-singular for $u \in K$, and $x \in \mathbf{R}^n$;
2) for such pairs (u, x), we always have $U^{-T}E < 0$, componentwise.

Then the corresponding optimal control problem admits optimal solutions.

Our strategy to prove these results is not new as it is based on the well-established philosophy of relying on relaxed versions of the original problem, and then, under suitable assumptions, prove that there are solutions of the relaxed problem which are indeed solutions of the original one ([4], [8], [19] and [20]). From this perspective, it is a very good example of the power of relaxed versions in optimization problems.

The relaxed version of the problem that we will be using is formulated in terms of Young measures associated with sequences of admissible controls. These so-called parametrized measures where introduced by L. C. Young ([20]), and have been extensively used in Calculus of Variations and Optimal Control Theory (see for example [11], [12] and [16]). Because of the special structure of the dependence on u, we will be concerned with (generalized) "moments" of such probability measures. Namely, the set

$$L = \{m \in \mathbf{R}^s : m_i = \phi_i(u), 1 \le i \le s, u \in K\}, \tag{17}$$

and the space of moments Λ given by

$$\left\{m \in \mathbf{R}^s : m_i = \int_K \phi_i(\lambda)\, d\mu(\lambda), 1 \le i \le s, \mu \in P(K)\right\} \tag{18}$$

will play a fundamental role. Here $P(K)$ is the convex set of all probability measures supported in K. Since the mapping

$$M : \mu \in P(K) \mapsto \Lambda, \quad M(\mu) = \int_K \phi(\lambda)\, d\mu(\lambda)$$

is linear, we easily conclude that Λ is a convex set of vectors, and, in addition, that the set of its extreme points is contained in L. In fact, for some particular ϕ_i's of polynomial type, the set of the extreme points of Λ is precisely L. This is closely related to the classical moment problem ([9]).

A crucial fact in our strategy is the following.

Assumption 1.1: For each fixed $x \in \mathbf{R}^N$, and $\xi \in Q(x)\Lambda \subset \mathbf{R}^N$, the minimum

$$\min_{m \in \Lambda} \{c(x) \cdot m : \xi = Q(x)m\}$$

is only attained in L.

Under this assumption, and the other technical requirements indicated at the beginning, one can show a general existence theorem of optimal solutions for our problem.

Theorem 1.3: Under Assumption 1.1 and the additional well-posedness hypotheses on (c, Q) indicated above, the initial optimal control problem (P_1) admits a solution.

Notice that we are not assuming any convexity on the set K in this statement. The proof of this theorem can be found in Section II. As remarked before, the proof is more-or-less standard, and it involves the use of an appropriate relaxed formulation of the problem in terms of moments of Young measures ([11], [16]).

Condition (9) in Theorem 1.1 is nothing but a sufficient condition to ensure Assumption 1.1 in a more explicit way. Ideally, one would like to provide explicit results saying that for a certain set $\mathcal{M}$, Assumption 1.1 holds if for each $x \in \mathbf{R}^N$, $(c(x), Q(x)) \in \mathcal{M}$. In fact, by looking at it from the point of view of duality, one can write a general statement whose proof is a standard exercise.

Proposition 1.1: If for every $x \in \mathbf{R}^N$, $(c, Q) = (c(x), Q(x))$ are such that for every $\eta \in \mathbf{R}^N$ there is a unique $m(\eta) \in L$ solution of the problem

$$\text{Minimize in } m \in L: \quad (c + \eta Q)m \tag{19}$$

then Assumption 1.1 holds.

One then says that $(c, Q) \in \mathcal{M}$ if this pair verifies the condition on this proposition. A full analysis of this set $\mathcal{M}$ turns out to depend dramatically on the ingredients of the problem. In particular, we will treat the cases $n = N = 1$, and the typical situation of algebraic moments of degree 2 and 3 in Section III, Section IV, and Section V, respectively.

Situations where either $N > 1$ or $n > 1$ are much harder to deal with, specially because existence results are more demanding on the structure of the underlying problem. In particular, we need a convexity assumption on how the non-linear dependence on controls occurs. We found that (9) turns out to be a general sufficient condition for the validity of Assumption 1.1, thus permitting to prove Theorem 1.1 based on Theorem 1.3. Theorem 1.2 follows then directly from Theorem 1.1 after some algebra. This can be found in Section VI.

Finally, we would like to point out that one particular interesting example, from the point of view of applications, that adapts to our results comes from the control of under-water vehicles (submarines). See [7]. This served as a clear motivation for our work. We plan to go back to this problem in the near future.

II. Proof of Theorem 1.3

Consider the following four formulations of the same underlying optimal control problem.

(P_1) The original optimal control problem described above in (1)-(4).

(P_2) The relaxed formulation in terms of Young measures ([11], [12] and [16]) associated with sequences of admissible controls:

$$\text{Minimize in } \mu = \{\mu_t\}_{t \in (0,T)}:$$

$$\tilde{I}(\mu) = \int_0^T [\int_K \sum_i c_i(x(t))\phi_i(\lambda)\, d\mu_t(\lambda)]\, dt$$

subject to

$$x'(t) = \int_K \sum_i Q_i(x(t))\phi_i(\lambda)\, d\mu_t(\lambda)$$

and $\text{supp}(\mu_t) \subset K$, $\quad x(0) = x_0 \in \mathbf{R}^N$. (P_3) The above relaxed formulation (P_2) rewritten by taking advantage of the moment structure of the cost density and the state equation. If we put $c = (c_1, ..., c_s) \in \mathbf{R}^s$, $Q \in M_{N \times s}$ and m such that

$$m_i = \int_K \phi_i(\lambda)\, d\mu_t(\lambda)\ \forall i \in \{1, ..., s\},$$

then we pretend to

$$\text{Minimize in } m \in \Lambda: \quad \int_0^T c(x(t)) \cdot m(t)\, dt$$

subject to

$$x'(t) = Q(x(t))m(t), \quad x(0) = x_0.$$

(P_4) Variational reformulation of formulation (P_3) ([2], [13], [15]). This amounts to defining an appropriate density by setting

$$\varphi(x, \xi) = \min_{m \in \Lambda}\{c(x) \cdot m\ : \xi = Q(x)m\}.$$

Then we would like to

$$\text{Minimize in } x(t): \quad \int_0^T \varphi(x(t), x'(t))\, dt$$

subject to $x(t)$ being Lipschitz in $(0, T)$ and $x(0) = x_0$.

We know that the three versions of the problem (P_2), (P_3), and (P_4) admit optimal solutions because they are relaxations of the original problem (P_1). In fact, since K is compact, (P_2) is a particular case of the relaxed problems studied in [11] and [16]. The existence of solution for the linear optimal control problem (P_3) is part of the classical theory ([2]). Indeed, (P_3) is nothing but (P_2) rewritten in terms of moments, so that the equivalence is immediate. (P_4) is the reformulated problem introduced in [15] whose equivalence to (P_3) was largely explored in [2] and [13].

Let $\tilde{x}$ be one such solution of (P_4). By Assumption 1.1 applied to a. e. $t \in (0, T)$, we have

$$\varphi(\tilde{x}(t), \tilde{x}'(t)) =$$

$$\min_{m \in \Lambda}\{c(\tilde{x}(t)) \cdot m(t) :\ \tilde{x}'(t) = Q(\tilde{x}(t))m(t)\} = c(\tilde{x}(t)) \cdot \tilde{m}(t)$$

for a measurable $\tilde{m}(t) \in L$, a solution of (P_3) (see [13]). The fundamental fact here (through Assumption 1.1) is that $\tilde{m}(t) \in L$ for a.e. $t \in (0, T)$, and this in turn implies that $\tilde{m}(t)$ is the vector of moments of an optimal Dirac-type Young measure $\mu = \{\mu_t\}_{t \in (0,T)} = \{\delta_{\tilde{u}(t)}\}_{t \in (0,T)}$ for an admissible $\tilde{u}$ for (P_1). This admissible control $\tilde{u}$ is optimal for (P_1). This finishes the proof.

From now on, we focus on furnishing conditions which in various contexts guarantee that Assumption 1.1 is fulfilled so that, whenever that is the case, through Theorem 1.3, we will have an existence result.

III. Polynomial Dependence. The case $N = n = 1, p = 2$

Until Section VII, we concentrate in the situation where

$$\phi : \mathbf{R}^n \to \mathbf{R}^s$$

is such that $\phi_i(u) = u_i$, for $i = 1, 2, \ldots, n$, and $\phi_{n+i}(u)$, $i = 1, 2, \ldots, s - n$ are convex polynomials of some degree

p, or simply polynomials whose restriction to K is convex. We will consider K itself to be convex.

Our goal is to explore different possibilities to apply directly Theorem 1.3 by ensuring Assumption 1.1. In other words, we will search for functions

$$c : \mathbf{R}^N \to \mathbf{R}^s, \quad Q : \mathbf{R}^N \to \mathbf{R}^{Ns},$$

such that for every $x \in \mathbf{R}^N$,

$$(c(x), Q(x)) \in \mathcal{M}$$

where $\mathcal{M}$ represents the set

$$\left\{(c, Q) : \forall \xi \in Q\Lambda, \ \arg\min_{m \in \Lambda}\{c \cdot m : \xi = Qm\} \in L\right\} \tag{20}$$

During the following three sections we will focus on the one dimensional case $N = n = 1$ and use some ideas based in duality (Proposition 1.1) and in geometric interpretations.

Next we explore various scenarios where Assumption 1.1 can be derived, and defer explicit examples until Section VI. In particular, we consider in this section the situation where ϕ is given by $\phi(a) = (a, a^2)$. We are talking about polynomial components of degree less than or equal to $p = 2$.

Let $K = [a_1, a_2]$, L, and Λ as in (17)-(18). Here, we have $s = 2$ and

$$c : \mathbf{R} \to \mathbf{R}^2, \quad Q : \mathbf{R} \to \mathbf{R}^2$$

can be identified with vectors in $\mathbf{R}^2$, or more precisely, with plane curves parametrized by x. To emphasize that function Q is not a matrix-valued but vector-valued, we will call it q. We illustrate this situation in Figure 1.

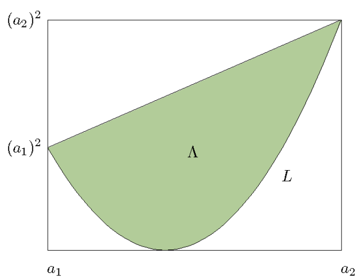

Fig. 1. $\Lambda = co(L)$ for $p = 2$

Next we describe sufficient conditions for $(c(x), q(x)) \in \mathcal{M}$.

Lemma 3.1: Let K, L and ϕ be as above. For every $x \in \mathbf{R}$, let $q = q(x)$ and $c = c(x)$ be vectors such that one of the following conditions is verified

1) $q_1 + q_2(a_1 + a_2) = 0$ and
$$\det\begin{pmatrix} c_1 & c_2 \\ q_1 & q_2 \end{pmatrix} \neq 0;$$
2) $q_1 + q_2(a_1 + a_2) \neq 0$ and
$$(q_1 + q_2(a_1 + a_2))\det\begin{pmatrix} c_1 & c_2 \\ q_1 & q_2 \end{pmatrix} < 0.$$

Then $(c, q) \in \mathcal{M}$, and consequently Assumption 1.1 is verified.

Proof: Suppose there is η such that the minimum of $(c + \eta q) \cdot m$ is attained in more than one point of $L = \phi(K)$. This means that the real function

$$g(t) = (c + \eta q) \cdot \phi(t) = (c_1 + \eta q_1)t + (c_2 + \eta q_2)t^2$$

has more than one minimum point over K. For that to happen, either g is constant on t, i. e.,

$$\begin{cases} c_1 + \eta q_1 = 0 \\ c_2 + \eta q_2 = 0 \end{cases} \Leftrightarrow \det\begin{pmatrix} c_1 & c_2 \\ q_1 & q_2 \end{pmatrix} = 0,$$

which contradicts our hypothesis; or else we must have

$$c_2 + \eta q_2 < 0, \quad g'\left(\frac{a_1 + a_2}{2}\right) = 0.$$

This condition can be written as

$$c_1 + (a_1 + a_2)c_2 + \eta[q_1 + (a_1 + a_2)q_2] = 0.$$

If $q_1 + q_2(a_1 + a_2) = 0$, but $c_1 + (a_1 + a_2)c_2 \neq 0$ (condition 1. in statement of lemma), then this equation can never be fulfilled. Otherwise, there is a unique value for η, by solving this equation, which should also verify the condition on the sign of $c_2 + \eta q_2$. It is elementary, after going through the algebra, that the condition on this sign cannot be true under the second condition on the statement of the lemma. ∎

IV. The case $N = n = 1, p = 3$

We study the case where $\phi(a) = (a, a^2, a^3)$, $s = 3$, and c and q can be identified as vectors in $\mathbf{R}^3$. This situation is represented in Figure 2. The understanding of the set Λ and its sections by planes in $\mathbf{R}^3$ is much more subtle however.

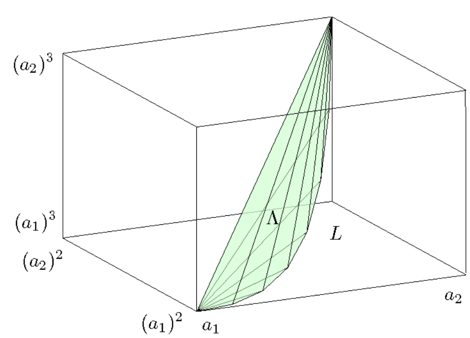

Fig. 2. $\Lambda = co(L)$ for $p = 3$

To repeat the procedure used for $p = 2$, and apply Proposition 1.1, we would like to give sufficient conditions for the function

$$g(t) = (c + \eta q) \cdot \phi(t) =$$
$$= (c_1 + \eta q_1)t + (c_2 + \eta q_2)t^2 + (c_3 + \eta q_3)t^3 \tag{21}$$

to have a single minimum over $K = [a_1, a_2]$ for every η. As indicated, and after some reflection, a complete analysis of the situation is rather confusing, and the conditions on the vectors c and q much more involved.

Searching some type of general condition would focus on considering the local maximizer and the local minimizer of $g(t)$, M_+ and M_-, respectively, and demanding that the interval $[a_1, a_2]$ have an empty intersection with the interval determined by M_+ and M_-. But this would lead to rather complicated expressions. Even so, some times under more specific hypotheses on the form of the vectors c and q, these conditions can be exploited.

Remark 4.1: Notice that the relation

$$ext(\Lambda) = L$$

is not true for a general K if it has positive and negative values. However, it is true if we consider $a_1 > 0$ or $a_2 < 0$.

Lemma 4.1: Let $K = [a_1, a_2]$ with $a_1 > 0$ and

$$(c, q) = ((0, c_2, c_3), (0, q_2, q_3))$$

such that

$$-\frac{q_2}{q_3} < 0, \quad (c_2, c_3) \cdot (1, -\frac{q_2}{q_3}) < 0.$$

Then the assumptions of Proposition 1.1 are valid, and consequently so is Assumption 1.1.

The proof is short and based in basic calculus ideas relative to the exact localization of M_+ and M_-. We should skip it for the sake of brevity.

V. A GEOMETRIC APPROACH TO THE CASE $N = n = 1, p = 3$.

Besides using Proposition 1.1, we can try to propose a general criteria for obtaining Assumption 1.1, based on a geometric approach.

We define the set $\mathcal{M}_1$ of pairs $(c, q) \in \mathbf{R}^3 \times \mathbf{R}^3$ through the following requirements:

- the quantity

$$(\phi'(t) \times (c \times q)) \cdot (\phi(s) - \phi(t))$$

  does not change sign over the pairs $t, s \in K,\ s \neq t$;
- whenever there is a unique $a \in K = [a_1, a_2]$ such that

$$(\phi(a_1) + \phi(a_2) - 2\phi(a)) \cdot q = 0, \tag{22}$$

  then

$$(\phi(a_1) + \phi(a_2) - 2\phi(a)) \cdot c > 0.$$

Hence we can establish the following result.

Proposition 5.1: Let $\mathcal{M}$ be as in (20).

If $a_1 > 0$, and $(c, q) \in \mathcal{M}_1$, then $(c, q) \in \mathcal{M}$ and Assumption 1.1 holds.

This type of result can be also deduced for the case $N = n = 1$, $p = 2$ where it can be seen to be equivalent to the conditions in Lemma 3.1. However, when the parameters N, n and p increase their values, it becomes very hard to give geometrically-based sufficient conditions in such an exhaustive manner as we have done here. Therefore, we skip the proof and we refer to Section VI where we show how to give more adequate sufficient conditions for interesting high dimensional particular situations, where some geometrical ideas can be used as a way to verify Assumption 1.1. Before going further to higher dimensional situations we illustrate how to apply these results for $N = n = 1$ with a simple example where Lemma 4.1 can be applied. Consider the problem of minimizing in u

$$\int_0^T [c(x(t))u^2(t) + u^3(t)]\, dt$$

under

$$x'(t) = [q(x(t))]u^2(t) + u^3(t), \quad x(0) = x_0$$

where $u(t) \in [a_0, a_1]$, $a_0 > 0$.

Lemma 5.1: If the functions $q(x)$ and $c(x)$ are Lipschitz,

$$c(x) < q(x)\ \forall x,$$

and $q(x)$ is always positive, then the optimal control problem admits solutions.

This result comes directly by applying Lemma 4.1 and Theorem 1.3. Nevertheless, note that checking the convexity of the set $\mathcal{A}_x$ defined by

$$\{(\xi, v) : v \geq c(x)u^2 + u^3, \xi = q(x)u^2 + u^3, u \in K = [a_0, a_1]\}$$

as asked in classical theory can be very difficult.

VI. THE CASE $N, n > 1$

The previous analysis makes it very clear that checking Assumption 1.1 may be a very hard task as soon as n and/or N become greater than 1. Yet in this section we would like to show that there are chances to prove some non-trivial results.

The three main ingredients in Assumption 1.1 are:

- the vector $c \in \mathbf{R}^s$ in the cost functional;
- the matrix $Q \in \mathbf{M}^{N \times s}$ occurring in the state equation;
- the convexification Λ of the set of moments L.

For (c, Q) given, consider the set $\mathcal{N}(c, Q)$ as it was defined in (7). Let Ψ be as in (6) and such that $\nabla\Psi(m)$ is a rank $s - n$ matrix and L can be seen as the embedded (parametrized) manifold of $\mathbf{R}^s$ in the manifold defined implicitly by $\Psi = 0$. This means that $\Psi(\phi(u)) = 0$ for all $u \in K$.

Consider also the set of vectors $\mathcal{N}(K, \phi)$ described in (8), that is, the set of "ascent" directions for Ψ at points of L.

We are now in a good condition to prove Theorem 1.1.

Proof: The proof is rather straightforward. Firstly, note that due to the convexity assumption on Ψ, and the fact that $L \subset \{\Psi = 0\}$, we have $\Lambda \subset \{\Psi \leq 0\}$.

Suppose that $m_0 \in L$ and $m_1 \in \Lambda$, so that $\Psi(m_0) = 0$,

$$\Psi(m_1) \leq 0,\ cm_1 \leq cm_0, \text{ and } Qm_1 = Qm_0\ (= \xi).$$

Then it is obvious that $m = m_1 - m_0 \in \mathcal{N}(c, Q)$. Because of our assumption, $m \in \mathcal{N}(K, \phi)$. We have two possibilities:

1) $\nabla\Psi(m_0)m = 0$. Because of the convexity of each component of Ψ, we have

$$\Psi(m_1) - \Psi(m_0) - \nabla\Psi(m_0)m \geq 0.$$

But then

$$0 = \Psi(m_0) \leq \Psi(m_1) \leq 0,$$

so that $m_1 \in L$. Because of the strict convexity of each component of Ψ, this means that $m_1 = m_0$, and Assumption 1.1 holds.

2) $\nabla\psi_i(m_0)m > 0$ for some i. Once again we have

$$\psi_i(m_1) - \psi_i(m_0) - \nabla\psi_i(m_0)m \geq 0.$$

But this is impossible because $\psi_i(m_1) > 0$ cannot happen for a vector in Λ.

■

Remark 6.1: Notice that if in the original problem (P_1) we would have considered the dynamics given by

$$Q(x)\phi(u) + Q_0(x)$$

instead of just $Q(x)$, Assumption 1.1 and Theorem 1.1 could be written exactly in the same way.

Though Theorem 1.1 can be applied to more general cases, we will focus on a particular situation motivated by the control of underwater vehicles ([7]). We will briefly describe the structure of the state equation. Indeed, it is just

$$x'(t) = Q_1(x)\phi(u) + Q_0(x)$$

where the state $x \in \mathbf{R}^{12}$ incorporates the position and orientation in body and world coordinates, and the control $u \in \mathbf{R}^{10}$ accounts for guidance and propulsion. Under suitable simplifying assumptions ([7]), the components of the control vector u only occur as either linear or pure squares, in such a way that $\phi(u) = (u, u^2) \in \mathbf{R}^{20}$, and $u^2 = (u_i^2)_i$, componentwise. Q_1 and Q_0 are matrices which may have essentially any kind of dependence on the state x.

To cover this sort of situations just described, we will concentrate on the optimal control problem (P) already stated in (11)-(13), and set D, E and U as in (14)-(15).

We can now prove Theorem 1.2.

Proof:

Notice that accordingly to (10), since $s = 2n$, we have, for $m \in \mathbf{R}^s$,

$$\psi_i(m) = m_i^2 - m_{n+i}, \quad i = 1, 2, \ldots, n,$$

which are certainly smooth and (strictly) convex. Moreover,

$$\nabla\Psi(m) = \begin{pmatrix} 2\tilde{m} & -\mathrm{id} \end{pmatrix}$$

where

$$\tilde{m} = 2\sum_i m_i e_i \otimes e_i,$$

and e_i is the canonical basis of $\mathbf{R}^n$.

Suppose we have, for a vector $v \in \mathbf{R}^{2n}$, $v = (v_1, v_2)$, that

$$Qv = 0, \quad cv \leq 0.$$

A more explicit way of writing this is

$$Q_1 v_1 + Q_2 v_2 = 0, \quad c_1 v_1 + c_2 v_2 \leq 0.$$

So

$$v_1 = Dv_2, \quad Ev_2 \leq 0.$$

We have to check that such a vector v is not a direction of descent for every function ψ_j, or it is an ascent direction for at least one of them. Note that

$$\nabla\Psi(m)v = Uv_2, \quad Ev_2 \leq 0.$$

It is an elementary Linear Algebra exercise to check that if $U^{-T}E < 0$, then condition (9) is verified so that Theorem 1.1 can be applied. ■

By using similar ideas, more general situations can be treated. In a forthcoming work, we will see how to apply Theorem 1.1 to an optimal control problem modeling an underwater vehicle maneuvering, in a situation where the state variable lies in $\mathbf{R}^{12}$ while the control belongs to $\mathbf{R}^3$.

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