A local existence result for an optimal control problem modeling the manoeuvring of an underwater vehicle

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Abstract

We prove a local existence result for the manoeuvrability control of a submarine. The problem is formulated as an optimal control problem with a nonlinear and highly coupled system of ODEs for the state law, a Lagrange type cost function, and nonlinear controls which take values on a convex and compact subset of $\mathbb{R}^3$. Finally, the existence of solution for this problem is obtained by applying a recent general existence result [11] which, however, requires some modifications to be used in our specific case.

keywords Submarine, manoeuvrability control, optimal control problem, relaxation, Young measures, existence of solution.

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1 Introduction

In this paper we turn over the existence of solution for the model of manoeuvrability control of a submarine which has been recently proposed in [4]. It corresponds to a real-life engineering problem so that all the hypotheses and ingredients that we will consider in the sequel are motivated by real (non-academic) requirements. To describe such model a state vector is defined

$$\mathbf{x} = (x, y, z, \phi, \theta, \psi, u, v, w, p, q, r) \in \Omega \subset \mathbb{R}^{12}, \tag{1}$$

where $X_{world} = (x, y, z; \phi, \theta, \psi)$ indicates the position and orientation of the submarine in the world fixed coordinate system, and $V_{body} = (u, v, w; p, q, r)$ is the vector of linear and angular velocities measured in the body coordinate system. Throughout this paper we follow the usual SNAME notation [3]. Permitted ranges of Euler angles are

$$-\pi < \phi < \pi, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad 0 < \psi < 2\pi, \tag{2}$$

so that

$$\Omega = \mathbb{R}^3 \times]-\pi, \pi[\times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\times]-0, 2\pi[\times \mathbb{R}^6.$$

The control vector is

$$\mathbf{u} = (\delta_b, \delta_s, \delta_r), \tag{3}$$

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where δ_b and δ_s represent, respectively, the angle of the bow and stern coupled planes, and δ_r is deflection of rudder. These controls act on the system in linear and quadratic form. Therefore, it is convenient to consider the mapping

$$\Phi(\mathbf{u}) = \left(\mathbf{u}, \mathbf{u}^2\right) \equiv \left(\delta_b, \delta_s, \delta_r, \delta_b^2, \delta_s^2, \delta_r^2\right) \in \mathbb{R}^6.$$

Admissible controls $\mathbf{u}$ are measurable functions that should lie in a certain set $K \subset \mathbb{R}^3$, which, in our case, is given by

$$K = [-a_1, a_1] \times [-a_2, a_2] \times [-a_3, a_3],$$

with $0 < a_1, a_2, a_3 < \pi/2$. Finally, the state law is described by a system of twelve ordinary differential equations

$$\mathbf{x}'(t) = Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)) \tag{4}$$

where

$$Q : \mathbb{R}^{12} \to \mathcal{M}^{12\times 6} \quad \text{and} \quad Q_0 : \mathbb{R}^{12} \to \mathbb{R}^6$$

will be described in Section 3. At this point, we just indicate that the right-hand side of (4) includes both kinematic and dynamic equations of motion (see [2, 3, 4, 5] for more details).

The manoeuvrability control problem for an underwater vehicle describes a situation where we want to reach (or to be very close to) a final state $\mathbf{x}^T$ in time T, while minimizing the use of control during the time interval $[0, T]$. The latter can be understood as minimizing the typical cost

$$\int_0^T \|\mathbf{u}(t)\|^2 \, dt$$

while the first aspect can be seen as minimizing

$$\begin{aligned}
\frac{1}{2}\left\|\mathbf{x}(T) - \mathbf{x}^T\right\|^2 &= \frac{1}{2}\int_0^T \frac{d}{dt}\left\|\mathbf{x}(t) - \mathbf{x}^T\right\|^2 dt + \frac{1}{2}\left\|\mathbf{x}(0) - \mathbf{x}^T\right\|^2 \\
&= \int_0^T < \mathbf{x}(t) - \mathbf{x}^T,\ Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)) > dt + \frac{1}{2}\left\|\mathbf{x}(0) - \mathbf{x}^T\right\|^2.
\end{aligned}$$

Hence, we consider the cost

$$\begin{aligned}
&\int_0^T \left[< \mathbf{x}(t) - \mathbf{x}^T,\ Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)) > + \|\mathbf{u}(t)\|^2\right] dt \\
= &\int_0^T \left[c(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + c_0(\mathbf{x}(t))\right] dt
\end{aligned}$$

where the vector c is given by

$$\begin{cases}
c_i(\mathbf{x}) = \sum_{j=1}^{12}\left(\mathbf{x} - \mathbf{x}^T\right)_j Q_{ji}, & i = 1, 2, 3, \\
c_i(\mathbf{x}) = \sum_{j=1}^{12}\left(\mathbf{x} - \mathbf{x}^T\right)_j Q_{ji} + 1, & i = 4, 5, 6,
\end{cases}$$

and

$$c_0(\mathbf{x}) =< \mathbf{x} - \mathbf{x}^T,\ Q_0(\mathbf{x}) > .$$

Typically, some penalty parameters are introduced to weigh at convenience the above two goals, but for simplicity and since it does not change mathematically the problem we have not considered such weights.

To sum up, we can write the manoeuvrability control problem as

$$(\mathrm{P}) \quad \begin{cases}
\text{Minimize in } \mathbf{u} : & \int_0^T \left[c(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + c_0(\mathbf{x}(t))\right] dt \\
\text{subject to} & \\
& \mathbf{x}'(t) = Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)), \quad 0 < t < T \\
& \mathbf{x}(0) = \mathbf{x}^0 \in \Omega \\
& \mathbf{x}(t) \in \Omega \quad \text{and} \quad \mathbf{u}(t) \in K,\ 0 \le t \le T.
\end{cases}$$

The main goal of this paper is to prove the following local existence result.

Theorem 1.1. *For $T > 0$, small enough, there exists an optimal solution of (P).*

We notice that the constraint on T is imposed to be able to guarantee that the state law is well-posed. The existence of T will be established during the proof of Theorem 1.1. As we will see later on, the fundamental question for this existence result is the relation between the vector c, the matrix Q, the mapping Φ and the set K. The role played by Q_0 is related to the existence and uniqueness of solution for the state law, and c_0 does not influence at all. To prove Theorem 1.1 we will apply a very recent general existence result [11] which requires some modifications to adapt the specific structure of our model. Section 2 is devoted to present this general result (Theorem 2.1) with its corresponding changes. In Section 3 we will check that our model satisfies the hypotheses required by this last theorem.

2 A general existence and uniqueness result for some specific optimal control problems

Throughout this section we basically follow the same ideas as in [11], but since our problem is slightly different from the one considered there and to make the paper easier for readers we include detailed statements and proofs.

To study the existence of solution for (P) we will turn ourself over the general optimal control problem of the type

$$(CP) \qquad \text{Minimize in } \mathbf{u} : \quad \int_0^T c(\mathbf{x}) \cdot \Phi(\mathbf{u}) + c_0(\mathbf{x}) dt \tag{5}$$

subject to

$$\mathbf{x}' = Q(\mathbf{x})\Phi(\mathbf{u}) + Q_0(\mathbf{x}) \tag{6}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^N,$$

and

$$\mathbf{u}(t) \in K, \tag{7}$$

where $K \subset \mathbb{R}^m$. We search a control $\mathbf{u}$ in $L^\infty((0,T), K)$ corresponding to an absolutely continuous state function $\mathbf{x} : (0,T) \to \mathbb{R}^N$.

The mappings

$$\Phi(\mathbf{u}) \in \mathbb{R}^s,$$

$$Q : \mathbb{R}^N \to \mathcal{M}^{N\times s},$$

$$Q_0, c : \mathbb{R}^N \to \mathbb{R}^s$$

should be such that the cost function is defined and takes finite values for admissible pairs $(\mathbf{x}, \mathbf{u})$ and the state system is well-posed.

As we will see, the fundamental question for the existence result is the relation between the vector c, the matrix Q, the application Φ and set K. For a better understanding of such relations we consider additionally a C^1 mapping

$$\Psi : R^s \to R^{s-m}, \quad \Psi = (\psi_1, ..., \psi_{s-m}), \quad (s > m), \tag{8}$$

so that $\Phi(K) \subset \{\Psi = 0\}$. This means that we are embedding the image space $\Phi(K)$ into a level surface (submanifold) defined by Ψ. Notice for example that for problem (P) where

$$\Phi(\mathbf{u}) = (u_1, u_2, u_3, (u_1)^2, (u_2)^2, (u_3)^2) \in \mathbb{R}^6$$

we have

$$\Psi(v) = ((v_1)^2 - v_4, (v_2)^2 - v_5, (v_3)^2 - v_6) \in \mathbb{R}^3.$$

Also we define for every pair (c, Q) the set

$$\mathcal{N}(c, Q) = \{v \in \mathbb{R}^s : Qv = 0, cv \leq 0\}. \tag{9}$$

Similarly, we consider

$$\begin{gathered}\mathcal{N}(K, \Phi) = \\ \{v \in \mathbb{R}^s : \text{ for each } \mathbf{u} \in K, \text{ either } \nabla\Psi(\Phi(\mathbf{u}))v = 0 \text{ or } \exists i \text{ s. t. } \nabla\psi_i(\Phi(\mathbf{u}))v > 0\},\end{gathered} \tag{10}$$

the set of "growth directions" of Ψ over $\Phi(K)$. We are now in conditions to state the existence result proved in [11] adapted to our frame.

Theorem 2.1. *Assume that the mapping Ψ as above is component-wise convex and C^1. If for each $\mathbf{x} \in \mathbb{R}^n$, we have*

$$\mathcal{N}(c(\mathbf{x}), Q(\mathbf{x})) \subset \mathcal{N}(K, \Phi), \tag{11}$$

then the corresponding optimal control problem (CP) has at least one solution. If, in addition, Φ is component-wise one to one, convex and strictly convex for at least one component over K, then the solution of (CP) is unique.

Notice that in the statement of Theorem 2.1, we have dropped the strictly convexity of Ψ as it was asked in [11]. Also we have included a sufficient condition which ensures the uniqueness of such a solution.

An essential tool to the proof of this result is the verification of the assumption

Assumption 2.1. *For each fixed $\mathbf{x} \in \mathbb{R}^N$, and $\xi \in Q(\mathbf{x})\Lambda + Q_0(\mathbf{x}) \subset \mathbb{R}^N$, the minimum*

$$\min_{m \in \Lambda} \{c(\mathbf{x}) \cdot m + c_0(\mathbf{x}) : \xi = Q(\mathbf{x})m + Q_0(\mathbf{x})\}$$

is only attained in L, where $L = \Phi(K)$ and $\Lambda = co(L)$.

In fact this hypothesis has a very simple geometrical meaning, as we show in Figure **??** for the simple case were $N = n = 1$, $K = [a_1, a_2]$ and $\Phi(\mathbf{u}) = (\mathbf{u}, \mathbf{u}^2)$. The set L is part of onedi curve parameterized by Φ and Λ is its convex hull. As the figure shows, for fixed ξ and Q, c must be oriented from the convex curve L towards the interior of its convex hull, in such a way that the minimum of $c \cdot m$ over

$$\{\xi = Qm\} \bigcap co(L)$$

must be attained exclusively over L.

This assumption allows us to proceed through a relaxation process using Young measures (as in [7], [9], [11], [13] and [14]) and conclude that there is a Dirac-type solution of the relaxed problem which corresponds to a solution of the original problem.

Before starting the proof of the existence result, let us first consider the following Lemma.

Lemma 2.1. *Let Ψ be as in Theorem 2.1. If c, Q, Φ and K in (CP) are such that condition (11) is satisfied, then Assumption 2.1 holds.*

Proof. We want to see that for every fixed $\mathbf{x} \in \mathbb{R}^N$ and $\xi \in Q(\mathbf{x})\Lambda + Q_0(\mathbf{x})$ the minimizer of $c(\mathbf{x}) \cdot v + c_0(\mathbf{x})$ over the set of vectors in Λ verifying the restriction $\xi = Q(\mathbf{x})v + Q_0(\mathbf{x})$ can only be in L, where both L and Λ are as in Assumption 2.1.

Suppose that $v_0 \in L$ and $v_1 \in \Lambda$ both belong to the manifold

$$\{\xi = Q(\mathbf{x})v + Q_0(\mathbf{x})\}$$

but they verify

$$c(\mathbf{x})v_1 + c_0(\mathbf{x}) \leq c(\mathbf{x})v_0 + c_0(\mathbf{x}).$$

As Ψ is component-wise convex and $L \subset \{\Psi = 0\}$, we have $\Lambda = co(L) \subset \{\Psi \leq 0\}$. Hence,

$$\Psi(v_0) = 0,\ \Psi(v_1) \leq 0,\ c \cdot v_1 \leq c \cdot v_0, \text{ and } Qv_1 = Qv_0\ (= \xi - Q_0).$$

Therefore it is obvious that $v = v_1 - v_0 \in \mathcal{N}(c(\mathbf{x}), Q(\mathbf{x}))$. Due to condition (11), $v \in \mathcal{N}(K, \Phi)$. Accordingly to the definition of $\mathcal{N}(K, \Phi)$ either $\nabla\psi_i(v_0)v > 0$ for some i or $\nabla\Psi(v_0)v = 0$. Suppose we are in the first situation. Because of the convexity of Ψ,

$$\psi_i(v_1) - \psi_i(v_0) - \nabla\psi_i(v_0)v \geq 0 \Leftrightarrow$$

$$\psi_i(v_1) \geq \nabla\psi_i(v_0)v > 0.$$

But this is impossible because $\psi_i(v_1) > 0$ cannot happen for a vector in Λ.

Suppose now that $\nabla\Psi(v_0)v = 0$. Again by convexity of each component of Ψ, we have

$$\Psi(v_1) - \Psi(v_0) - \nabla\Psi(v_0)v \geq 0,$$

that is,

$$0 = \Psi(v_0) \leq \Psi(v_1) \leq 0.$$

Hence, as $v_1 \in \Lambda = (\Lambda \setminus L) \cup L$ and

$$\Lambda \setminus L \subset \{\Psi(v) \leq 0, \quad \exists i \text{ s.t. },\ \psi_i(v) < 0\}$$

we conclude that $v_1 \in L$ and Assumption 2.1 holds. □

We can now prove Theorem 2.1.

Proof. We begin by the relaxation of (CP) using Young measures associated with sequences of admissible controls. Consider the problem

$$(RP) \quad \text{Minimize in } \mu = \{\mu_t\}_{t\in(0,T)}: \quad \tilde{I}(\mu) = \int_0^T [\int_K c(\mathbf{x}(t)) \cdot \Phi(\lambda) d\mu_t(\lambda)] + c_0(\mathbf{x}(t)) dt$$

subject to

$$\mathbf{x}'(t) = \int_K Q(\mathbf{x}(t))\Phi(\lambda)d\mu_t(\lambda) + Q_0(\mathbf{x}(t))$$

and

$$\text{supp}(\mu_t) \subset K, \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^N.$$

Notice that the theory of Young measures ([7], [9], [13], [14]) allows us to conclude that this formulation is, in particular, well posed, as having $\mathbf{u} \in L^\infty([0, T], K)$ for K bounded implies (see [8]) that the associated Young measures $\{\mu_t\}_t$ belongs to

$$\mathcal{Y}^p((0, T), P(K)) =$$

$$\left\{\mu = \{\mu_t\}_{t\in(0,T)} : \int_0^T \int_K ||\lambda||^p d\mu_t(\lambda)dt < \infty,\ \mu_t \in P(K)\right\} \quad \text{for every } p > 1,$$

where $P(K)$ is the space of probability measures supported in K. The existence of an optimal measure for this problem is immediately established by applying the existence result in [7] for the particular case where K is bounded.

In addition, (RP) can be rewritten by taking advantage of the moment structure of the cost density and the state equation. If we consider the set

$$\Lambda = \{m \in \mathbb{R}^s : m = \int_K \Phi(\lambda)d\nu(\lambda), \nu \in P(K)\},$$

then for each Young measure $\mu = \{\mu_t\}_t$ we can associate a function in $L^\infty([0,T],\Lambda)$ given by

$$m(t) = \int_K \Phi(\lambda) d\mu_t(\lambda).$$

This relation is not one-to-one but we can also associate at least one Young measure to each function in $L^\infty([0,T],\Lambda)$. The set Λ is very especial. Indeed, notice that L defined above as $L = \Phi(K)$ is part of Λ as it corresponds to generalized moments associated to Dirac-type Young measures. Moreover, in [6] it was shown that when K is a compact and convex set we have

$$\Lambda = \overline{co(L)} = co(L)$$

so that Λ is a convex, compact set, defined as

$$\Lambda = co(\Phi(K)).$$

This considerations allow us to conclude that the relaxed problem (RP) is equivalent to the linear optimal control problem

$$(LP) \quad \text{Minimize in } m \in \Lambda : \quad \int_0^T c(\mathbf{x}(t)) \cdot m(t) + c_0(\mathbf{x}(t)) dt$$

subject to

$$\mathbf{x}'(t) = Q(\mathbf{x}(t))m(t) + Q_0(\mathbf{x}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

whose optimal solution (for the existence of such a solution see [1]) corresponds to a Young measure which is an optimal solution (not necessarily unique) of (RP). Next, we will characterize this optimal solution, say $\tilde{m}(.)$ of (LP). To that purpose consider the function

$$\varphi(\mathbf{x},\xi) =$$

$$\begin{cases} \min_{m\in\Lambda}\{c(\mathbf{x}) \cdot m + c_0(\mathbf{x}) \ : \xi = Q(\mathbf{x})m + Q_0(\mathbf{x})\} & \text{if } \xi \in Q(\mathbf{x})\Lambda + Q_0(\mathbf{x}) \\ +\infty & \text{else.} \end{cases}$$

This density function is the typical integrand of the cost which defines the equivalent variational problem (VP)

$$\text{Minimize in } \mathbf{x}(t) : \quad \int_0^T \varphi(\mathbf{x}(t), \mathbf{x}'(t)) dt$$

subject to $\mathbf{x}(0) = \mathbf{x}_0$, $\mathbf{x}(t) \in AC([0,T],\mathbb{R}^N)$. The equivalence between problems (VP) and (LP) is well known and can be found in [12] , [1] and in more recent works under a similar framework [9], [10]. Accordingly, there is a solution for (VP), let us say $\tilde{\mathbf{x}}(.)$, whose connection to $\tilde{m}(.)$ is established through the relation

$$\varphi(\tilde{\mathbf{x}}(t), \tilde{\mathbf{x}}'(t)) = \min_{m\in\Lambda}\{c(\tilde{\mathbf{x}}(t)) \cdot m(t) + c_0(\tilde{\mathbf{x}}(t) : \ \tilde{\mathbf{x}}'(t) = Q(\tilde{\mathbf{x}}(t))m(t) + Q_o(\tilde{\mathbf{x}}(t))\}$$

$$= c(\tilde{\mathbf{x}}(t)) \cdot \tilde{m}(t) + c_0(\tilde{\mathbf{x}}(t)) \quad a.e. \quad t \in (0,T).$$

By Lemma 2.1,

$$\tilde{m}(t) \in L = \Phi(K)$$

so that there is a Dirac-type Young measure μ solution of (RP), associated to $\tilde{m}$. As a consequence, (CP) has an optimal solution $\mathbf{u} \in L^\infty([0,T],K)$ such that $\mu = \{\delta_{\mathbf{u}(t)}\}_{t\in(0,T)}$.

Let us now prove the second part of the theorem. Suppose that $\mathbf{u}_1(.)$ and $\mathbf{u}_2(.)$ are different optimal solutions of (CP). Then $\mu_1 = \{\delta_{\mathbf{u}_1(t)}\}_t$ and $\mu_2 = \{\delta_{\mathbf{u}_2(t)}\}_t$ are optimal solutions of (RP). As Φ is component-wise one to one, the corresponding generalized moments defined by $m_1(t) = \Phi(\mathbf{u}_1(t))$ and $m_2(t) = \Phi(\mathbf{u}_2(t))$ are different optimal solutions of (LP). Hence for $\lambda \in]0,1[$, we have that $m = \lambda m_1 + (1-\lambda)m_2$ is also an optimal solution of the linear problem (LP) and therefore $m \in L$. But since $L = \Phi(K)$ and Φ is strictly convex for some component i, m does not belong to L. A contradiction. Therefore we must have $\mathbf{u}_1 = \mathbf{u}_2$. □

3 Proof of Theorem 1.1

In this section we will apply the first part of Theorem 2.1 to the optimal control problem (P). In our case, Φ is not injective so that we cannot conclude about uniqueness. In fact, some numerical simulations (see [4]) suggest that the solution of (P) is not unique. We proceed in several steps:

3.1 Step 1: the matrices Q and Q_0

We start by paying some attention to the matrices Q and Q_0 of the control system, as it is fundamental to verify the well-posedness character of the state law and condition (11) of Theorem 2.1. We recall the notation introduced in Section 1 where we have set

$$\mathbf{x} = (x, y, z, \phi, \theta, \psi, u, v, w, p, q, r) \in \Omega \subset \mathbb{R}^{12},$$

with $X_{world} = (x, y, z; \phi, \theta, \psi)$ and $V_{body} = (u, v, w; p, q, r)$. Using this notation, accordingly to what we have seen also in Section 1 and using the data in [4] we know that Q is given by

$$Q = \begin{pmatrix} 0_{6\times 6} \\ M^{-1}F(V_{body}) \end{pmatrix}$$

where the matrix M is given by

$$M = \begin{pmatrix} m - \frac{\rho}{2}L^3X'_{\dot u} & 0 & 0 & 0 & mZ_G & -mY_G \\ 0 & m - \frac{\rho}{2}L^3Y'_{\dot v} & 0 & -mZ_G - \frac{\rho}{2}L^4Y'_{\dot p} & 0 & mX_G - \frac{\rho}{2}L^4Y'_{\dot r} \\ 0 & 0 & 0 & m - \frac{\rho}{2}L^3Z'_{\dot w} & -mX_G - \frac{\rho}{2}L^4Z'_{\dot q} & mY_G \\ 0 & -mZ_G - \frac{\rho}{2}L^4K'_{\dot v} & mY_G & I_x - \frac{\rho}{2}L^5K'_{\dot p} & -I_{xy} & -I_{xz} - \frac{\rho}{2}L^5K'_{\dot r} \\ mZ_G & 0 & -mX_G - \frac{\rho}{2}L^4M'_{\dot w} & -I_{xy} & I_y - \frac{\rho}{2}L^5M'_{\dot q} & -I_{yz} \\ -mY_G & mX_G - \frac{\rho}{2}L^4N_{\dot v} & 0 & -I_{xz} - \frac{\rho}{2}L^5N'_{\dot p} & -I_{yz} & I_z - \frac{\rho}{2}L^5N'_{\dot r} \end{pmatrix}$$

and $F = (G,\ H)$, $G, H \in \mathcal{M}^{6\times 3}$, with

$$G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\rho}{2}l^2(Y'_{\delta_r} + Y'_{\delta_r\eta}(\eta - \frac{1}{C})C)u^2 \\ \frac{\rho}{2}l^2(Z'_{\delta_b})u^2 & \frac{\rho}{2}l^2(Z'_{\delta_s} + Z'_{\delta_s\eta}(\eta - \frac{1}{C})C)u^2 & 0 \\ 0 & 0 & \frac{\rho}{2}l^3(K'_{\delta_r})u^2 \\ \frac{\rho}{2}l^3(M'_{\delta_b})u^2 & \frac{\rho}{2}l^3(M'_{\delta_s} + M'_{\delta_s\eta}(\eta - \frac{1}{C})C)u^2 & 0 \\ 0 & 0 & \frac{\rho}{2}l^3(N'_{\delta_r} + N'_{\delta_r\eta}(\eta - \frac{1}{C})C)u^2 \end{pmatrix}$$

and

$$H = \begin{pmatrix} \frac{\rho}{2}l^2(X'_{\delta_b\delta_b})u^2 & \frac{\rho}{2}l^2(X'_{\delta_s\delta_s})u^2 & \frac{\rho}{2}l^2(X'_{\delta_r\delta_r})u^2 \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \end{pmatrix}.$$

Finally, considering the dimensionless hydrodynamic coefficients in [4, Appendix] gives

$$Q(\mathbf{x}) = u^2 \begin{pmatrix} & 0_{6\times 6} & & & & \\ Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ 0 & 0 & Q_{23} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ 0 & 0 & Q_{43} & 0 & 0 & 0 \\ Q_{51} & Q_{52} & Q_{53} & 0 & 0 & 0 \\ 0 & 0 & Q_{63} & 0 & 0 & 0 \end{pmatrix}$$

$$= (x_7)^2 \begin{pmatrix} & & 0_{6\times 6} & & & \\ -0.0056307 & -0.0056219 & 0.0002292 & -0.0028418 & -0.0011310 & -0.0037067 \\ 0 & 0 & -0.0001291 & 0 & 0 & 0 \\ 1.527832 & 1.4903911 & -0.0617573 & -0.0001656 & -0.0000659 & -0.0002160 \\ 0 & 0 & 0.0001049 & 0 & 0 & 0 \\ -0.0162938 & -0.0162684 & 0.0006631 & 0 & 0 & 0 \\ 0 & 0 & -0.0002773 & 0 & 0 & 0 \end{pmatrix}.$$

We remark that Q, the 12×6 matrix of the coefficients interacting with the control, only depends on the surge velocity. Such particularity allows us to verify condition (11) quite easily, as we will see after.

As for Q_0, it is given by

$$Q_0 = \begin{pmatrix} \mathcal{T}(\phi, \theta, \psi) V_{body} \\ M^{-1} F_0(V_{body}, \phi, \theta, \psi) \end{pmatrix} \in \mathbb{R}^{12}.$$

where $\mathcal{T}$ is the transformation matrix in the kinematic equations

$$(X_{world})' = \mathcal{T}(\phi, \theta, \psi) V_{body}$$

defined by

$$\mathcal{T} = \begin{pmatrix} J_1(\phi, \theta, \psi) & 0_{3\times 3} \\ & \\ 0_{3\times 3} & J_2(\phi, \theta, \psi) \end{pmatrix}$$

with

$$J_1(\phi, \theta, \psi) = \begin{pmatrix} \cos\psi\cos\theta & -\sin\psi\cos\theta + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\phi\sin\theta\sin\psi & -\cos\psi\sin\phi + \sin\theta\sin\psi\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

and

$$J_2(\phi, \theta, \psi) = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{pmatrix}.$$

Concerning F_0, it is defined in [4] through the ordinary differential system of six equations

$$MV'_{body} = F_0(V_{body}, \phi, \theta, \psi) + F(V_{body})\Phi(\mathbf{u})$$

so that it corresponds to the terms independent of the controls. To obtain Q_0 we write F_0 with the data given in [4] and multiply it by M^{-1}, just as we have done for Q. Using the state notation

$$\mathbf{x} = (x_j), \quad \bar{F}_0(\mathbf{x}) = ((\bar{F}_0)_j) = M^{-1} F_0(\mathbf{x}), \quad 1 \le j \le 6,$$

we obtain

$$\begin{gathered}(\bar{F}_0)_1 = 0.21\sin x_4\cos x_5 + 5.593 x_{12}|x_{12}| - 10.68 x_{12}^2 - 7.234 x_{11} x_{12} + 2.905 x_{10} x_{12} - 0.93 x_8 x_{12} - 0.11 x_7 x_{12} \\ -19.65 x_{11}|x_{11}| + 5.658 x_{11}^2 + 0.015 x_{10} x_{11} - 1.809 x_9 x_{11} + 0.61 x_7 x_{11} + 7.252 x_{10}|x_{10}| - 0.4 x_{10}^2 + 0.14 x_9 x_{10} \\ -2.477 x_8 x_{10} + 0.21 x_7 x_{10} - 0.0085\sqrt{x_9^2 + x_8^2}\,|x_9| - 0.0022 x_7|x_9| - 0.0056 x_8\sqrt{x_9^2 + x_8^2} + 0.0074 x_9^2 \\ -0.015 x_8 x_9 - 0.022 x_7 x_9 + 0.012 x_8|x_8| + 0.22 x_8^2 + 0.013 x_7 x_8 - 0.0012 x_7^2 - 0.014 x_7 + 0.2\end{gathered}$$

$$
\begin{aligned}
(\bar{F}_0)_2 = {} & 0.032 \sin x_4 \cos x_5 + 4.918 x_{12} |x_{12}| - 1.028 x_{11} x_{12} - 0.21 x_7 x_{12} + 0.064 x_{10} x_{11} + 1.101 x_{10} |x_{10}| \\
& +0.41 x_9 x_{10} - 0.0073 x_7 x_{10} - 0.023 x_8 \sqrt{x_9^2 + x_8^2} - 0.061 x_8 x_9 + 0.0017 x_8 |x_8| \\
& -0.01 x_7 x_8 + 2.4985 \times 10^{-7} x_7^2 - 5.6213 \times 10^{-5} x_7 + 0.0012
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_3 = {} & -0.43 \sin x_5 - 57.56 \sin x_4 \cos x_5 - 1508. x_{12} |x_{12}| + 5212. x_{12}^2 + 1951. x_{11} x_{12} + 98.94 x_{10} x_{12} + 884.9 x_8 x_{12} \\
& +30.0 x_7 x_{12} + 5149. x_{11} |x_{11}| + 108.1 x_{11}^2 - 4.058 x_{10} x_{11} - 0.047 x_9 x_{11} - 166.7 x_7 x_{11} - 1956. x_{10} |x_{10}| \\
& +107.8 x_{10}^2 - 38.2 x_9 x_{10} + 667.5 x_8 x_{10} - 57.7 x_7 x_{10} + 2.215 \sqrt{x_9^2 + x_8^2} |x_9| + 0.59 x_7 |x_9| + 1.501 x_8 \sqrt{x_9^2 + x_8^2} \\
& +4.2833 \times 10^{-4} x_9^2 + 3.913 x_8 x_9 + 6.062 x_7 x_9 - 3.109 x_8 |x_8| - 54.46 x_8^2 - 3.376 x_7 x_8 + 0.088 x_7^2 + 0.099 x_7 - 2.205
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_4 = {} & -0.098 \sin x_4 \cos x_5 - 2.562 x_{12} |x_{12}| + 3.317 x_{11} x_{12} + 0.051 x_7 x_{12} - 0.0069 x_{10} x_{11} - 3.325 x_{10} |x_{10}| \\
& -0.065 x_9 x_{10} - 0.098 x_7 x_{10} + 0.0025 x_8 \sqrt{x_9^2 + x_8^2} + 0.0066 x_8 x_9 - 0.0053 x_8 |x_8| - 0.0057 x_7 x_8 \\
& -7.5427 \times 10^{-7} x_7^2 + 1.697 \times 10^{-4} x_7 - 0.0038
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_5 = {} & 0.62 \sin x_4 \cos x_5 + 16.2 x_{12} |x_{12}| - 56.57 x_{12}^2 - 20.96 x_{11} x_{12} - 9.622 x_8 x_{12} - 0.32 x_7 x_{12} - 56.86 x_{11} |x_{11}| \\
& -1.157 x_{11}^2 + 0.044 x_{10} x_{11} + 1.76 x_7 x_{11} + 21.01 x_{10} |x_{10}| - 1.157 x_{10}^2 + 0.41 x_9 x_{10} - 7.167 x_8 x_{10} + 0.62 x_7 x_{10} \\
& -0.025 \sqrt{x_9^2 + x_8^2} |x_9| - 0.0064 x_7 |x_9| - 0.016 x_8 \sqrt{x_9^2 + x_8^2} - 0.042 x_8 x_9 - 0.065 x_7 x_9 + 0.033 x_8 |x_8| \\
& +0.59 x_8^2 + 0.036 x_7 x_8 - 9.4993 \times 10^{-4} x_7^2 - 0.0011 x_7 + 0.024
\end{aligned}
$$

$$
\begin{aligned}
(\bar{F}_0)_6 = {} & 0.0037 \sin x_4 \cos x_5 + 2.308 x_{12} |x_{12}| - 0.12 x_{11} x_{12} - 0.079 x_7 x_{12} - 1.91 x_{10} x_{11} + 0.12 x_{10} |x_{10}| + 0.0063 x_9 x_{10} \\
& -0.0073 x_7 x_{10} - 0.0043 |x_8| \sqrt{x_9^2 + x_8^2} - 3.2111 \times 10^{-4} x_8 \sqrt{x_9^2 + x_8^2} - 0.071 x_8 x_9 + 1.9811 \times 10^{-4} x_8 |x_8| \\
& -0.0042 x_7 x_8 + 2.8285 \times 10^{-8} x_7^2 - 6.3637 \times 10^{-6} x_7 + 1.412 \times 10^{-4}
\end{aligned}
$$

Notice that in fact $\bar{F}_0$ does not depend on (x_1, x_2, x_3), but for simplicity we will still consider Q_0 as a vector function from $\mathbb{R}^{12}$ to $\mathbb{R}^{12}$ which is described by

$$
Q_0(\mathbf{x}) = \begin{pmatrix} \begin{pmatrix} J_1(x_4, x_5, x_6) & 0_{3\times 3} \\ 0_{3\times 3} & J_2(x_4, x_5, x_6) \end{pmatrix} \begin{pmatrix} x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{pmatrix} \\ \\ \bar{F}_0(\mathbf{x}) \end{pmatrix}
$$

where J_1, J_2 and $\bar{F}_0$ are as above.

3.2 Step 2: local existence and uniqueness of solutions for the state law

Let us now show that it is possible to find a time interval $I = [0, T]$ for which the initial value problem

$$\text{(IVP)} \begin{cases} \mathbf{x}'(t) = Q(\mathbf{x}(t))\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}(t)), & 0 < t < T \\ \mathbf{x}(0) = \mathbf{x}^0 \in \Omega \end{cases}$$

is well posed in the sense that for every control function $\mathbf{u} \in L^\infty(0, T; K)$ there is a unique solution. We start by recalling the classical theory on this subject and therefore we rewrite (IVP) in the standard way

$$\begin{cases} \mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t)), & 0 < t < T \\ \mathbf{x}(0) = \mathbf{x}^0 \in \Omega \subset \mathbb{R}^N, \end{cases} \tag{12}$$

with $\mathbf{f} : I \times \Omega \to \mathbb{R}^N$, $N = 12$ in our case. A (Carathéodory) solution of (12) is an absolutely continuous function

$$\mathbf{x} : (0, T_1) \to \Omega, \text{ with } T_1 \leq T,$$

such that for all $t \in (0, T_1)$

$$\mathbf{x}(t) = \mathbf{x}^0 + \int_0^t \mathbf{f}(s, \mathbf{x}(s))\,ds.$$

The solution $\mathbf{x} : (0, T_1) \to \Omega$ is said to be maximal if for another solution $\overline{\mathbf{x}} : (0, T_2) \to \Omega$ of (12) the two following conditions hold:

(i) $T_2 \leq T_1$, and

(ii) $\overline{\mathbf{x}}(t) = \mathbf{x}(t)$ for all $0 \leq t \leq T_2$.

As is well-known (see for instance [15, Appendix C]), if $\mathbf{f}$ satisfies conditions (H1)-(H4) below, then we can ensure the existence and uniqueness of a maximal solution for (12).

(H1) For each $\mathbf{x} \in \Omega$, the function $\mathbf{f}(\cdot, \mathbf{x}) : I \to \mathbb{R}^N$ is measurable,

(H2) for each $t \in I$, the function $\mathbf{f}(t, \cdot) : \Omega \to \mathbb{R}^N$ is continuous,

(H3) $\mathbf{f}$ is locally Lipschitz on $\mathbf{x}$, that is, for each $\mathbf{x}^0 \in \Omega$ there are a real number $\rho > 0$ and a locally integrable function

$$\alpha : I \to \mathbb{R}^+$$

such that the ball $B_\rho\left(\mathbf{x}^0\right)$ of radius ρ centered at $\mathbf{x}^0$ is contained in Ω and

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\| \leq \alpha(t)\,\|\mathbf{x} - \mathbf{y}\|$$

for each $t \in I$ and $\mathbf{x}, \mathbf{y} \in B_\rho\left(\mathbf{x}^0\right)$, and

(H4) $\mathbf{f}$ is locally integrable on t, that is, for each $\mathbf{x}^0 \in \Omega$ there exists a locally integrable function $\beta : I \to \mathbb{R}^+$ such that

$$\left\|\mathbf{f}\left(t, \mathbf{x}^0\right)\right\| \leq \beta(t) \quad \text{a. e. } t \in I.$$

Our next task is to check that (H1)-(H4) hold in our particular case. For any $\mathbf{u} \in L^\infty(\mathbb{R}; K)$, since the control variable $\mathbf{u}$ appears in linear and quadratic form, it is clear that the function

$$\mathbf{f}(t, \mathbf{x}) = Q(\mathbf{x})\,\Phi(\mathbf{u}(t)) + Q_0(\mathbf{x}) \tag{13}$$

is measurable with respect to t for each fixed $\mathbf{x} \in \Omega$. In addition, looking at the particular form of (13), it is clear that for each t, the function $\mathbf{x} \to \mathbf{f}(t, \mathbf{x})$ is continuous. With respect to conditions (H3) and (H4), again the form in which the controls appear let us conclude that (H4) is satisfied. As for the local Lipschitz condition (H3), since $\mathbf{f} = (f_1, \cdots, f_{12})$ is a vector function, we should check that condition for each component. Due to the constraints (2) and taking into account that the first six components of $\mathbf{f}$ only include the transformation matrix between body and world references frames, we have that $f_1, \cdots, f_6 \in C^\infty(\Omega)$ and therefore they are locally Lipschitz with respect to $\mathbf{x}$. As for the remaining $f_7, \cdots, f_{11}$, we notice that these components include by one side, polynomial terms, terms in the form of absolute value, terms with the structure of $\sqrt{x_j^2 + x_k^2}$, where $\mathbf{x} = (x_1, \cdots, x_{12})$, all of them locally Lipschitz, and products of locally Lipschitz functions, also locally Lipschitz, by the other.

Therefore we may state that for each $\mathbf{x}^0 \in \Omega$ and $\mathbf{u} \in L^\infty(\mathbb{R}; K)$ there exists a maximal time $T\left(\mathbf{x}^0, \mathbf{u}\right)$ and a unique maximal solution of (IVP) defined on $\left[0,\ T\left(\mathbf{x}^0, \mathbf{u}\right)\right]$. In fact, looking at the proof of the mentioned existence result (see [15]), we can see that $T\left(\mathbf{x}^0, \mathbf{u}\right)$ depends on both $\alpha(t) = \alpha(u(t))$ and $\beta(t) = \beta(u(t))$ in the sense that

$$\int_0^t \alpha(\tau)\, d\tau < 1 \quad \forall t \in [0, T(\mathbf{x}^0, \mathbf{u})]$$

and

$$\int_0^t \rho\alpha(\tau) + \beta(\tau)\, d\tau < \rho \quad \forall t \in [0, T(\mathbf{x}^0, \mathbf{u})].$$

Since Φ is continuous on the compact set K and taking into account the particular structure of matrices Q and Q_0, we can choose $\alpha(t)$ and $\beta(t)$ such that (H1)-(H4) are satisfied simultaneously to all $\mathbf{u} \in L^\infty(\mathbb{R}_+; K)$ and consequently we can choose T (uniformly in $\mathbf{u}$) such that problem (IVP) has a unique solution in $I = [0, T]$, with $T = T(\mathbf{x}^0)$, for every $\mathbf{u} \in L^\infty(I; K)$.

Remark 3.1. *It is not difficult to convince ourselves that for some suitable inputs* $\mathbf{u}$*, the corresponding solution* $\mathbf{x}$ *of the state law is not defined for all* $t > 0$ *because of the constraints (2). That is, we can not expect to have a global solution for all admissible* $\mathbf{u}$*. Moreover, in a real situation we also must impose some constraints on the state variables* (x, y, z) *due to the finite dimension of ocean. These restrictions, which are specially important in a situation in which the submarine is moving in littoral waters, may let the solution* $\mathbf{x}$ *blow-up in finite time.*

3.3 Step 3: checking condition (11) in Theorem 2.1

We need to describe for every $\mathbf{x} \in \mathbb{R}^{12}$ (and corresponding pair $(c(\mathbf{x}), Q(\mathbf{x}))$) the set

$$\mathcal{N}(c(\mathbf{x}), Q(\mathbf{x})) = \left\{v \in \mathbb{R}^6 : Q(\mathbf{x})v = 0, c(\mathbf{x}) \cdot v \leq 0\right\},$$

and check that such set is contained in

$$\mathcal{N}(K, \Phi) =$$
$$\left\{v = (v_1, \cdots, v_6) \in \mathbb{R}^6 : \text{ for each } \mathbf{u} \in K, \text{ either } \nabla\Psi(\Phi(\mathbf{u}))v = 0 \text{ or there is } i \text{ with } \nabla\Psi_i(\Phi(\mathbf{u}))v > 0\right\},$$

where Q is like described in the beginning of this section, where the data from [4] were used.

Let us first find the solution of $Qv = 0$. If the surge velocity $u = x_7$ is zero, then the solution is $\mathbb{R}^6$. Assuming that $x_7 \neq 0$ we have

$$\begin{cases} v_3 = 0 \\ v_6 = -\frac{1}{Q_{16}}(Q_{11}v_1 + Q_{12}v_2 + Q_{14}v_4 + Q_{15}v_5) \\ v_6 = -\frac{1}{Q_{36}}(Q_{31}v_1 + Q_{32}v_2 + Q_{34}v_4 + Q_{35}v_5) \\ v_2 = -\frac{Q_{51}}{Q_{52}}v_1. \end{cases}$$

Thus

$$\begin{cases} \cdots \\ \frac{1}{Q_{16}}(Q_{11} - \frac{Q_{51}}{Q_{52}}Q_{12})v_1 + \frac{Q_{14}}{Q_{16}}v_4 + \frac{Q_{15}}{Q_{16}}v_5 = \\ \frac{1}{Q_{36}}(Q_{31} - \frac{Q_{51}}{Q_{52}}Q_{32})v_1 + \frac{Q_{34}}{Q_{36}}v_4 + \frac{Q_{35}}{Q_{36}}v_5 \\ \cdots \end{cases}$$

but

$$\frac{Q_{14}}{Q_{16}} = 0.7666667 = \frac{Q_{34}}{Q_{36}}$$

and

$$\frac{Q_{15}}{Q_{16}} = 0.3051282 = \frac{Q_{35}}{Q_{36}}$$

so that

$$\begin{cases} \cdots \\ (Q_{11} - \frac{Q_{51}}{Q_{52}} Q_{12}) v_1 = (Q_{31} - \frac{Q_{51}}{Q_{52}} Q_{32}) v_1 \\ \cdots \end{cases} .$$

Since

$$Q_{11} - \frac{Q_{51}}{Q_{52}} Q_{12} = 0 \neq 0.0348637 = Q_{31} - \frac{Q_{51}}{Q_{52}} Q_{32},$$

we have

$$Qv = 0 \Leftrightarrow \begin{cases} v_1 = 0 \\ v_2 = 0 \\ v_3 = 0 \\ v_6 = -\frac{1}{Q_{16}}(Q_{14} v_4 + Q_{15} v_5) = \\ -\frac{1}{Q_{36}}(Q_{34} v_4 + Q_{35} v_5). \end{cases}$$

Before completing the characterization of $\mathcal{N}(c, Q)$ notice that the function Ψ used in describing $\mathcal{N}(K, \Phi)$ is given by

$$\Psi(m) = (m_1^2 - m_4, m_2^2 - m_5, m_3^2 - m_6), \quad m = (m_1, \cdots, m_6),$$

so that Ψ is obviously C^1 and convex. Moreover,

$$\nabla\Psi(m) = [2diag(m_1, m_2, m_3), -I_3].$$

Hence, for v such that $Qv = 0$ the vector $\nabla\Psi(m) \cdot v$ is in fact

$$2\,\mathtt{diag}[m_1, m_2, m_3] \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - I_3 \begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix}$$

$$= -\begin{pmatrix} v_4 \\ v_5 \\ v_6 \end{pmatrix}.$$

This means that for a vector v (in the manifold $Qv = 0$) to belong to $\mathcal{N}(K, \Phi)$, it must satisfy

$$v_4 = v_5 = v_6 = 0$$

or else one of those three components must be negative.

As a consequence, condition (11) can only hold if the vectors in $\mathcal{N}(c, Q)$ have one of the last three components strictly negative or either all null. But as we have seen, for the case where the surge velocity $u = x_7 \neq 0$ we have

$$v_6 = -\frac{1}{Q_{16}}(Q_{14} v_4 + Q_{15} v_5) = -\frac{1}{Q_{36}}(Q_{34} v_4 + Q_{35} v_5).$$

Hence, if both v_4 and v_5 are positive or null, we have v_6 necessarily negative or also null. If the surge velocity x_7 is zero then the matrix $Q(\mathbf{x})$ is null and therefore

$$c(\mathbf{x}) = (0, 0, 0, 1, 1, 1)$$

so that

$$c \cdot v \leq 0 \Leftrightarrow$$

$$v_4 + v_5 + v_6 \leq 0$$

which implies that either $v_4 = v_5 = v_6 = 0$ or at least one of them must be negative. Consequently

$$\mathcal{N}(c, Q) \subset \mathcal{N}(K, \Phi),$$

and applying Theorem 2.1 the proof is complete.

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