Exercises on Multiple-Conclusion Logics

ON DUALITY

Exploring symmetry:

1. What is the dual $\not\subset$ to the classical implication $\supset$?
   i) Define its rules and its matrices.
   ii) Define it in terms of other classical connectives.
   iii) Check that $\{\supset, \not\subset\}$ provide a functionally complete basis for classical logic.

2. Can you propose a general semantic method for dualization?

3. Check that NAND is dual to NOR.

4. What is the dual to the classical bi-implication?

5. Check that, in normal modal logics, $\Box$ is dual to $\Diamond$.

6. Check that, in first-order classical logic, $\forall$ is dual to $\exists$.

Extending the intuitions:

7. Consider Łukasiewicz's logic Ł$_3$, defined by the matrices:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

	$\neg$
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

where $\mathcal{D} = \{1\}$. What is the logic dual to Ł$_3$?

Hint: Find first an adequate bivaluation semantics for Ł$_3$.

ON TARSKIAN INTERPRETATIONS, AND BEYOND

Limit and degenerate cases:

8. What happens if one chooses $\mathcal{S} = \varnothing$? Which logics can be defined?

9. What if $\mathcal{D} \cup \mathcal{U} \neq \mathcal{V}$? (GAPS)

10. What if $\mathcal{D} \cap \mathcal{U} \neq \varnothing$? (GLUTS)

11. What if $\mathsf{Sem} = \varnothing$?

Local × Global:

12. Can you exemplify logics having local and global rules such that $\Vdash_{\mathrm{g}} \not\subseteq \Vdash_{\ell}$?

Indecency (with respect to the cardinalities of the sets of premisses and alternatives):

13. Why are there only FOUR indecent logics?

14. Show, for each indecent logic, the adequacy of each semantic characterization with respect to the corresponding abstract characterization.

15. Check the details of the Paradox of Ineffable Inconsistencies.

16. Which of the indecent logics have adequate matrix semantics?

Anything (quodcumque) × *whatever* (qualiscumque):

17. What is the semantic difference between *ex contradictione sequitur quodlibet* $(\Gamma, \alpha, \neg\alpha \Vdash \beta, \Delta)$ and *pseudo-scotus* $(\Gamma, \alpha, \neg\alpha \Vdash \Delta)$? Give an example of a $\mathbf{T}$-logic that respects the former principle but fails the latter.

Superlogics:

18. Given a family of logics $\mathcal{F} = \{\mathcal{L}_i\}_{i \in I} = \{\langle \mathcal{S}, \Vdash_i \rangle\}_{i \in I}$ and its superlogic given by $\mathcal{L}_{\mathcal{F}} = \langle \mathcal{S}, \cap_{i \in I} \Vdash_i \rangle$, check that:
    i) Properties (C1), (C2), (C2n), (C3) and (CLS) are all preserved from $\mathcal{F}$ into $\mathcal{L}_{\mathcal{F}}$.
    ii) Property (CC) is not preserved. Given an example of how it can fail.

19. Let $\mathcal{F} = \{\langle \mathcal{S}, \vDash_{\mathsf{Sem}[i]} \rangle\}_{i \in I}$ be a family of logics with tarskian interpretations. Then, check that:
    i) Each logic from the family respects properties (C1), (C2), (C2n), (C3).
    ii) In the superlogic of $\mathcal{F}$, we have $\vDash_{\mathcal{F}} \; = \; \vDash_{\bigcup_{i \in I} \mathsf{Sem}[i]}$.

ON ABSTRACT CONSEQUENCE RELATIONS

The single-conclusion environment:

20. A topological structure $\langle \mathcal{X}, \tau \rangle$ is a structure where τ is a collection of subsets of $\mathcal{X}$ with the restrictions that $\varnothing$ and $\mathcal{X}$ are in τ, and τ is closed under arbitrary unions and under finitely many intersections. The elements of τ are called *open sets* of the topology, and their complements are called *closed sets*.

    Recall now the axioms characterizing a Kuratowski topological closure.

    i) Check that dilution [(C3)] is indeed derivable from inclusion [(C1)] and premise-apartness [(CK1)].
    ii) Check that axioms (C1), (C2), (CK1) and (CK2) adequately characterize the 'semantics of closed sets'.
    iii) What is the 'logic of open sets' and interior operators?

21. Recall the definitions of naive cut [(C2n)] and of full cut [(C2)].

    i) Check that: (C1) + (C2) + (C3) $\Rightarrow$ (C2n).
    ii) Check that: (C1) + (C2n) + (C3) $\not\Rightarrow$ (C2).
       Hint: (Béziau 1995)
       Consider the logic $\mathcal{L}_{\mathbb{R}} = \langle \mathbb{R}, \Vdash \rangle$ such that $\mathbb{R}$ is the set of real numbers, and $\Vdash$ is defined as follows:

       $\Gamma \Vdash x$ iff $x \in \Gamma$, or
       $x = \frac{1}{n}$, for some $n \in \mathbb{N} \setminus \{0\}$, or
       there is a sequence $(x_n : n \geq 0)$ in Γ that converges to x.

       Have a look then at the sequence $(\frac{1}{n} : n \geq 0)$.
    iii) Check that compactness [(CC)] + (C1) + (C2n) + (C3) $\Rightarrow$ (C2).

Lindenbaum-Asser Extension Lemma:

22. Recall the definition of excessive, closed, and maximal theories. Check that:

    i) If Γ is excessive, then Γ is closed.
    ii) In classical logic, excessive $\Rightarrow$ maximal.

23. The 'constructive' extension. Check the details of the following alternative version of the Extension Lemma for compact $\mathbf{T}$-logics. Suppose that $\mathcal{S}$ is a denumerable set $\{\varphi_n\}_{n\in\mathbb{N}}$. Consider a theory Γ such that $\Gamma \not\Vdash \beta$, Let's extend this set into a β-excessive theory Γ_{exc}. Build the chain $\{\Gamma_n\}_{n\in\mathbb{N}}$ by defining:

$$\begin{array}{rcl} \Gamma_0 & = & \Gamma \\ \Gamma_{n+1} & = & \Gamma_n \cup \{\varphi_n\} \text{ if } \Gamma_n, \varphi_n \not\Vdash \beta \\ & & \Gamma_n, \text{ otherwise} \end{array}$$

Define $\Gamma_{\mathsf{exc}} = \bigcup_{n\in\mathbb{N}}(\Gamma_n)$. Check that Γ_{exc} is indeed a β-excessive theory. (For the case of a non-denumerable $\mathcal{S}$, use transfinite induction.)

The multiple-conclusion environment:

24. Recall the many guises of cut. Check their announced inter-relations:

(C2) $\Leftrightarrow$ (C2$\mathcal{S}$) $\{$(C3)$\}$	(C2$\mathbf{Lc}$) or (C2$\mathbf{Rc}$) $\Rightarrow$ (C2$\mathbf{for}$)
(C2$\mathbf{fin}$) $\Leftrightarrow$ (C2$\mathbf{for}$) $\{$(C3)$\}$	(C2$\mathbf{Lc}$) or (C2$\mathbf{Rc}$) $\boxed{\not\Leftarrow}$ (C2$\mathbf{for}$)
(C2$\mathbf{Lc}$) $\not\Leftrightarrow$ (C2$\mathbf{Rc}$) $\not\Leftrightarrow$ (C2$\mathbf{LR}$)	(C2) $\Rightarrow$ (C2$\mathbf{LR}$)
(C2$\mathbf{Lc}$) and (C2$\mathbf{Rc}$) $\Leftrightarrow$ (C2$\mathbf{LR}$) $\{$(C3)$\}$	(C2) $\boxed{\not\Leftarrow}$ (C2$\mathbf{LR}$)

(C2$\mathbf{for}$) $\Rightarrow$ (C2) $\{$(CC)$\}$

Hint: (Shoesmith & Smiley 1978)
To check the framed clauses consider the following logics, over an infinite $\mathcal{S}$:
   i) $\mathcal{L}_1$ is such that $\Gamma \Vdash \Delta$ iff $|\Gamma| \geq \aleph_0$ or $\Delta \neq \varnothing$
   ii) $\mathcal{L}_2$ is such that $\Gamma \Vdash \Delta$ iff $\Gamma \neq \varnothing$ or $|\Delta| \geq \aleph_0$
   iii) $\mathcal{L}_3$ is such that $\Gamma \Vdash \Delta$ iff either $|\Gamma| \geq \aleph_0$ or $|\Delta| \geq \aleph_0$ or $\Gamma \cap \Delta \neq \varnothing$

Completeness and categoricity:

25. Check the details of the 2-valued S-Reduction, that is, given a κ-valued semantics $\mathsf{Sem}(\kappa) = \{\S_j : \mathcal{S} \to \mathcal{D}_j \cup \mathcal{U}_j\}_{j \in J}$, define a set of bivaluations $\mathsf{Sem}(2) = \{b_j : \mathcal{S} \to \{T, F\}\}_{j \in J}$ by setting $b_j(\varphi) = T$ iff $\S_j(\varphi) \in \mathcal{D}_j$, and check that $\Sigma \vDash_{\mathsf{Sem}(2)} \Pi$ iff $\Sigma \vDash_{\mathsf{Sem}(\kappa)} \Pi$.

26. Fix a compact $\mathbf{T}$-logic $\mathcal{L}$ and a non-trivial theory Γ of this logic. Check that the semantics given by $\mathsf{Biv}(\mathcal{H})$ is complete for $\mathcal{L}$, given any collection of theories $\mathcal{H} \supseteq \mathsf{Exc}(\Gamma, \beta, \mathcal{L})$. *Hint:* Use the Extension Lemma.

27. Let $\mathcal{L}$ be a multiple-conclusion logic characterized by a bivaluation semantics.
   i) Check the Lemma on the 'Uniqueness of 2-valued counter-examples': $\Sigma \nvDash_b \Pi$ and $\Sigma \nvDash_c \Pi \Rightarrow b = c$, for any quasi-partition $\langle \Sigma, \Delta \rangle$ of $\mathcal{S}$ and any pair of bivaluations b and c.
   ii) Given a set of quasi-partitions $\mathcal{P}$, check that $\mathsf{Biv}(\mathcal{P})$ is adequate for $\mathcal{L}$ iff $\mathcal{P} = \mathsf{CQPart}(\mathcal{S}, \mathcal{L})$.

28. Fixed some $\mathcal{S}$, let $\mathcal{T}^{\mathcal{A}}$ be the collection of all abstract $\mathbf{T}$-logics over $\mathcal{S}$, and let $\mathcal{T}^{\mathcal{B}}$ be the collection of all tarskian bivaluation semantics over $\mathcal{S}$. Consider a mapping $\mathbf{BA}$ that to each bivaluation associates the consequence relation determined by it, and a mapping $\mathbf{AB}$ that to each abstract consequence relation associates the set of bivaluations that respect it. Check that:
   i) $\langle \mathbf{BA}, \mathbf{AB} \rangle$ is a Galois connection between $\langle \mathcal{T}^{\mathcal{A}}, \supseteq \rangle$ and $\langle \mathcal{T}^{\mathcal{B}}, \subseteq \rangle$, i.e.:
      1. (a) $\mathbf{BA}(\mathbf{AB}(\Vdash)) \supseteq \Vdash$ for every $\Vdash \in \mathcal{T}^{\mathcal{A}}$
         (b) $\mathsf{Biv} \subseteq \mathbf{AB}(\mathbf{BA}(\mathsf{Biv}))$ for every $\mathsf{Biv} \in \mathcal{T}^{\mathcal{B}}$
      2. both $\mathbf{BA}$ and $\mathbf{AB}$ are monotonic
   ii) On either single- or multiple-conclusion $\mathbf{T}$-logics, the converse of 1(a) amounts to completeness (as given by S-Reduction + W-Reduction).
   iii) On single-conclusion $\mathbf{T}$-logics, the converse of 1(b) cannot be obtained. On multiple-conclusion $\mathbf{T}$-logics it does obtain and it amounts to categoricity.