Multiple-Conclusion Logics

PART 1: “Remarkable Phenomena”

João Marcos

http://geocities.com/jm_logica/

Uni-Log 2005

Montreux, CH

Introductory (and Motivational) Course

$\Gamma \Vdash \Delta$

An updated architecture of mathematics

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$\mathcal{S}$: a set of *sentences* $\Vdash$: a notion of *consequence*

A case for a new **mother-structure**?

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- distinguishing logics: a difficulty that lingers

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Disadvantage:

- difficult interpretation of what's going on?

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In case $\Lambda = \{\lambda\}$, the sentence λ is said to be *refuted* by $\mathcal{L}$.

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Glossary:

- *cut* translates Gentzen's 'Schnitt'
- *dilution* translates Gentzen's 'Verdünnung'

Closure Systems and single-conclusion CRs

Kuratowski (topological) closure:

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(C3) $\Gamma \subseteq \Lambda \;\Rightarrow\; \Gamma^{\Vdash} \subseteq \Lambda^{\Vdash}$ [derivable] — dilution

(CK1) $(\Gamma \cup \Sigma)^{\Vdash} = \Gamma^{\Vdash} \cup \Sigma^{\Vdash}$ — premise-apartness

(CK2) $\varnothing^{\Vdash} = \varnothing$ — no primitive theses

Closure Systems and single-conclusion CRs

Tarski 1930 closure:

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(CT1) $\Gamma^{\Vdash} = \cup\{(\Gamma_{\Phi})^{\Vdash} : \Gamma_{\Phi} \in \mathsf{Fin}(\Gamma)\}$ finitariness

(CT2) $|\mathcal{S}| \leq \aleph_0$ denumerable language

(CT3) $\bot^{\Vdash} = \mathcal{S}$, for some $\bot \in \mathcal{S}$ *ex falso*

where $\mathsf{Fin}(\Gamma) = \{\Gamma_{\Phi} : \Gamma_{\Phi}$ is a finite subset of $\Gamma\}$

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(C2) $\Lambda \Vdash \beta$ and $(\forall \lambda \in \Lambda)\Gamma \Vdash \lambda \;\Rightarrow\; \Gamma \Vdash \beta$ — full cut

(C3) $\Gamma \subseteq \Lambda \;\Rightarrow\; \Gamma^{\Vdash} \subseteq \Lambda^{\Vdash}$ — dilution

. . . .

Closure and single-conclusion consequence relation:

$$\Gamma \Vdash \alpha \;\Leftrightarrow\; \alpha \in \Gamma^{\Vdash}$$

Closure Systems and single-conclusion CRs

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Using classical rules for $\sim$ and $\supset$:

(ii) $\Vdash \sim(\gamma_1 \wedge \ldots \wedge \gamma_m)$,

(iii) $\Vdash (\gamma_1 \wedge \ldots \wedge \gamma_m) \supset (\delta_1 \vee \ldots \vee \delta_n)$.

Our Ancestors

$$\Gamma \Vdash \Delta$$

Some early exponents:

Our Ancestors

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Some early exponents:

Gerhard Gentzen 1934-35

Our Ancestors

$$\boxed{\Gamma \Vdash \Delta}$$

Some early exponents:

Gerhard Gentzen 1934-35: *Natural Deduction*

Our Ancestors

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But what is **multiple-conclusion** good for?

Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	T
F	T	F

Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
T	T	F
F	F	F

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T	T	T
F	T	F

$$\alpha \wedge \beta \Vdash^{\mathsf{s}} \alpha$$

Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
T	T	F
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T	T	T
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$$\alpha \wedge \beta \Vdash^{\mathsf{s}} \alpha$$

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Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
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F	T	F

$$\alpha \wedge \beta \Vdash^{\mathsf{s}} \alpha$$

$$\alpha \wedge \beta \Vdash^{\mathsf{s}} \beta$$

$$\alpha, \beta \Vdash^{\mathsf{s}} \alpha \wedge \beta$$

Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	T
F	T	F

$$\alpha \wedge \beta \Vdash^{\mathsf{s}} \alpha$$

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$$\alpha, \beta \Vdash^{\mathsf{s}} \alpha \wedge \beta$$

$$\alpha \Vdash^{\mathsf{s}} \alpha \vee \beta$$

Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
T	T	F
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$\vee$	T	F
T	T	T
F	T	F

$\alpha \wedge \beta \Vdash^{\mathsf{s}} \alpha$

$\alpha \wedge \beta \Vdash^{\mathsf{s}} \beta$

$\alpha, \beta \Vdash^{\mathsf{s}} \alpha \wedge \beta$

$\alpha \Vdash^{\mathsf{s}} \alpha \vee \beta$

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Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	T
F	T	F

$\alpha \wedge \beta \Vdash^{\mathsf{s}} \alpha$

$\alpha \wedge \beta \Vdash^{\mathsf{s}} \beta$

$\alpha, \beta \Vdash^{\mathsf{s}} \alpha \wedge \beta$

$\alpha \Vdash^{\mathsf{s}} \alpha \vee \beta$

$\beta \Vdash^{\mathsf{s}} \alpha \vee \beta$

$\alpha \vee \beta \Vdash^{\mathsf{s}}$ **??**

Reading rules from truth-tables: $\wedge$ and $\vee$

$\wedge$	T	F
T	T	F
F	F	F

$\vee$	T	F
T	T	T
F	T	F

$\alpha \wedge \beta \Vdash^{\mathsf{s}} \alpha$

$\alpha \wedge \beta \Vdash^{\mathsf{s}} \beta$

$\alpha, \beta \Vdash^{\mathsf{s}} \alpha \wedge \beta$

$\alpha \Vdash^{\mathsf{s}} \alpha \vee \beta$

$\beta \Vdash^{\mathsf{s}} \alpha \vee \beta$

$\alpha \vee \beta \Vdash^{\mathsf{s}}$ **??**

but

$\alpha \vee \beta \Vdash^{\mathsf{m}} \alpha, \beta$

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

$\alpha, \sim\alpha \Vdash^{\mathsf{s}} \beta$

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

$\alpha, \sim\alpha \Vdash^{\mathsf{s}} \beta$

$\alpha \Vdash^{\mathsf{s}} \sim\alpha \quad \Rightarrow \quad \Vdash^{\mathsf{s}} \sim\alpha$, or

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

$$\alpha, {\sim}\alpha \Vdash^{\mathsf{s}} \beta$$

$\alpha \Vdash^{\mathsf{s}} {\sim}\alpha \quad \Rightarrow \quad \Vdash^{\mathsf{s}} {\sim}\alpha$, or

$$\Vdash^{\mathsf{s}} \alpha \vee {\sim}\alpha$$

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

$\alpha, \sim\alpha \Vdash^{\mathsf{s}} \beta$

$\alpha \Vdash^{\mathsf{s}} \sim\alpha \quad \Rightarrow \quad \Vdash^{\mathsf{s}} \sim\alpha$, or

$\Vdash^{\mathsf{s}} \alpha \vee \sim\alpha$

while, with $\Vdash^{\mathsf{m}}$,

$\alpha, \sim\alpha \Vdash^{\mathsf{m}}$

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

$\alpha, \sim\alpha \Vdash^{\mathsf{s}} \beta$

while, with $\Vdash^{\mathsf{m}}$,

$\alpha \Vdash^{\mathsf{s}} \sim\alpha \quad \Rightarrow \quad \Vdash^{\mathsf{s}} \sim\alpha$, or

$\alpha, \sim\alpha \Vdash^{\mathsf{m}}$

$\Vdash^{\mathsf{s}} \alpha \vee \sim\alpha$

$\Vdash^{\mathsf{m}} \sim\alpha, \alpha$

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

$$\alpha, \sim\alpha \Vdash^{\mathsf{s}} \beta$$

$$\alpha \Vdash^{\mathsf{s}} \sim\alpha \quad \Rightarrow \quad \Vdash^{\mathsf{s}} \sim\alpha, \text{ or}$$

$$\Vdash^{\mathsf{s}} \alpha \vee \sim\alpha$$

while, with $\Vdash^{\mathsf{m}}$,

$$\alpha, \sim\alpha \Vdash^{\mathsf{m}}$$

$$\Vdash^{\mathsf{m}} \sim\alpha, \alpha$$

or, more simply,

$$\Vdash^{\mathsf{m}} \alpha \quad \Rightarrow \quad \sim\alpha \Vdash^{\mathsf{m}}$$

Reading rules from truth-tables: $\sim$

	$\sim$
T	F
F	T

$\alpha, \sim\alpha \Vdash^{\mathsf{s}} \beta$

$\alpha \Vdash^{\mathsf{s}} \sim\alpha \quad \Rightarrow \quad \Vdash^{\mathsf{s}} \sim\alpha$, or

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Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

$$\Gamma, \alpha \Vdash^{\mathsf{k}} \beta, \Delta \quad \Rightarrow \quad \Gamma \Vdash^{\mathsf{k}} \alpha \supset \beta, \Delta$$

Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

$$\Gamma, \alpha \Vdash^{\mathsf{k}} \beta, \Delta \quad \Rightarrow \quad \Gamma \Vdash^{\mathsf{k}} \alpha \supset \beta, \Delta$$

$\Gamma \Vdash^{\mathsf{k}} \alpha, \Delta$ and $\Gamma', \beta \Vdash^{\mathsf{k}} \Delta' \quad \Rightarrow \quad \Gamma', \Gamma, \alpha \supset \beta \Vdash^{\mathsf{k}} \Delta, \Delta'$

Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

$$\Gamma, \alpha \Vdash^{\mathsf{k}} \beta, \Delta \quad \Rightarrow \quad \Gamma \Vdash^{\mathsf{k}} \alpha \supset \beta, \Delta$$

$$\Gamma \Vdash^{\mathsf{k}} \alpha, \Delta \text{ and } \Gamma', \beta \Vdash^{\mathsf{k}} \Delta' \quad \Rightarrow \quad \Gamma', \Gamma, \alpha \supset \beta \Vdash^{\mathsf{k}} \Delta, \Delta'$$

If $\mathsf{k} = \mathsf{s}$, then $\Delta = \varnothing$ and Δ' is a singleton.

Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

$$\Gamma, \alpha \Vdash^{\mathsf{k}} \beta, \Delta \quad \Rightarrow \quad \Gamma \Vdash^{\mathsf{k}} \alpha \supset \beta, \Delta$$

$$\Gamma \Vdash^{\mathsf{k}} \alpha, \Delta \text{ and } \Gamma', \beta \Vdash^{\mathsf{k}} \Delta' \quad \Rightarrow \quad \Gamma', \Gamma, \alpha \supset \beta \Vdash^{\mathsf{k}} \Delta, \Delta'$$

If $\mathsf{k} = \mathsf{s}$, then $\Delta = \varnothing$ and Δ' is a singleton. But then the above rules characterize only intuitionistic implication!

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$\supset$	T	F
T	T	F
F	T	T

$$\Gamma, \alpha \Vdash^{\mathsf{k}} \beta, \Delta \quad \Rightarrow \quad \Gamma \Vdash^{\mathsf{k}} \alpha \supset \beta, \Delta$$

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If $\mathsf{k} = \mathsf{s}$, then $\Delta = \varnothing$ and Δ' is a singleton. But then the above rules characterize only intuitionistic implication!

Alternatively, for the classical implication, one could take:

Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

$$\Gamma, \alpha \Vdash^{\mathsf{k}} \beta, \Delta \quad \Rightarrow \quad \Gamma \Vdash^{\mathsf{k}} \alpha \supset \beta, \Delta$$

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$$\Gamma, \alpha, \alpha \supset \beta \Vdash^{\mathsf{m}} \beta, \Delta$$

Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

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Reading rules from truth-tables: $\supset$

$\supset$	T	F
T	T	F
F	T	T

$$\Gamma, \alpha \Vdash^{\mathsf{k}} \beta, \Delta \quad \Rightarrow \quad \Gamma \Vdash^{\mathsf{k}} \alpha \supset \beta, \Delta$$

$$\Gamma \Vdash^{\mathsf{k}} \alpha, \Delta \text{ and } \Gamma', \beta \Vdash^{\mathsf{k}} \Delta' \quad \Rightarrow \quad \Gamma', \Gamma, \alpha \supset \beta \Vdash^{\mathsf{k}} \Delta, \Delta'$$

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$$\Gamma \Vdash^{\mathsf{m}} \alpha, \alpha \supset \beta, \Delta$$

Interlude: Duality and multiple-conclusion

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For any given logic $\mathcal{L}_{\triangleright} = \langle \mathcal{S}, \triangleright \rangle$

Interlude: Duality and multiple-conclusion

For any given logic $\mathcal{L}_{\triangleright} = \langle \mathcal{S}, \triangleright \rangle$
let the dual logic $\mathcal{L}_{\blacktriangleright} = \langle \mathcal{S}, \blacktriangleright \rangle$
be such that:

$$(\Gamma \blacktriangleright \Delta) \text{ iff } (\Delta \triangleright \Gamma).$$

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be such that:

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Similarly,

given rules for some $\overline{\wedge}$:

$\alpha, \beta \triangleright \alpha \,\overline{\wedge}\, \beta$

$\alpha \,\overline{\wedge}\, \beta \triangleright \alpha$

$\alpha \,\overline{\wedge}\, \beta \triangleright \beta$

Interlude: Duality and multiple-conclusion

For any given logic $\mathcal{L}_{\rhd} = \langle \mathcal{S}, \rhd \rangle$

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be such that:

$$(\Gamma \blacktriangleright \Delta) \text{ iff } (\Delta \rhd \Gamma).$$

Similarly,

given rules for some $\bar{\wedge}$:	we can consider their duals for $\underline{\vee}$:
$\alpha, \beta \rhd \alpha \mathbin{\bar{\wedge}} \beta$	$\alpha \mathbin{\underline{\vee}} \beta \blacktriangleright \beta, \alpha$
$\alpha \mathbin{\bar{\wedge}} \beta \rhd \alpha$	$\alpha \blacktriangleright \alpha \mathbin{\underline{\vee}} \beta$
$\alpha \mathbin{\bar{\wedge}} \beta \rhd \beta$	$\beta \blacktriangleright \alpha \mathbin{\underline{\vee}} \beta$

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be such that:

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Similarly,

given rules for some $\overline{\wedge}$:	we can consider their duals for $\veebar$:
$\alpha, \beta \rhd \alpha \,\overline{\wedge}\, \beta$	$\alpha \veebar \beta \blacktriangleright \beta, \alpha$
$\alpha \,\overline{\wedge}\, \beta \rhd \alpha$	$\alpha \blacktriangleright \alpha \veebar \beta$
$\alpha \,\overline{\wedge}\, \beta \rhd \beta$	$\beta \blacktriangleright \alpha \veebar \beta$

given rules for some $\overline{\sim}$:

$\alpha, \overline{\sim}\alpha \rhd$

$\rhd \overline{\sim}\alpha, \alpha$

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Similarly,

given rules for some $\overline{\wedge}$:	we can consider their duals for $\underline{\vee}$:
$\alpha, \beta \triangleright \alpha \,\overline{\wedge}\, \beta$	$\alpha \,\underline{\vee}\, \beta \blacktriangleright \beta, \alpha$
$\alpha \,\overline{\wedge}\, \beta \triangleright \alpha$	$\alpha \blacktriangleright \alpha \,\underline{\vee}\, \beta$
$\alpha \,\overline{\wedge}\, \beta \triangleright \beta$	$\beta \blacktriangleright \alpha \,\underline{\vee}\, \beta$
given rules for some $\overline{\sim}$:	we can consider their duals for $\underline{\sim}$:
$\alpha, \overline{\sim}\alpha \triangleright$	$\blacktriangleright \alpha, \underline{\sim}\alpha$
$\triangleright \overline{\sim}\alpha, \alpha$	$\alpha, \underline{\sim}\alpha \blacktriangleright$

Interlude: Duality and multiple-conclusion

For any given logic $\mathcal{L}_{\triangleright} = \langle \mathcal{S}, \triangleright \rangle$

let the dual logic $\mathcal{L}_{\blacktriangleright} = \langle \mathcal{S}, \blacktriangleright \rangle$

be such that:

$$(\Gamma \blacktriangleright \Delta) \text{ iff } (\Delta \triangleright \Gamma).$$

Similarly,

given rules for some $\overline{\wedge}$:

$\alpha, \beta \triangleright \alpha \,\overline{\wedge}\, \beta$

$\alpha \,\overline{\wedge}\, \beta \triangleright \alpha$

$\alpha \,\overline{\wedge}\, \beta \triangleright \beta$

we can consider their duals for $\underline{\vee}$:

$\alpha \,\underline{\vee}\, \beta \blacktriangleright \beta, \alpha$

$\alpha \blacktriangleright \alpha \,\underline{\vee}\, \beta$

$\beta \blacktriangleright \alpha \,\underline{\vee}\, \beta$

given rules for some $\overline{\sim}$:

$\alpha, \overline{\sim}\alpha \triangleright$

$\triangleright \overline{\sim}\alpha, \alpha$

we can consider their duals for $\underline{\sim}$:

$\blacktriangleright \alpha, \underline{\sim}\alpha$

$\alpha, \underline{\sim}\alpha \blacktriangleright$

And so on. . .

Underdetermined models for classical logic

$\wedge$	T	F_2	F_1	F
T	T	F_2	F_1	F
F_2	F_2	F_2	F	F
F_1	F_1	F	F_1	F
F	F	F	F	F

$\vee$	T	F_2	F_1	F
T	T	T	T	T
F_2	T	F_2	T	F_2
F_1	T	T	F_1	F_1
F	T	F_2	F_1	F

	$\sim$
T	F
F_2	F_1
F_1	F_2
F	T

From Shoesmith & Smiley 1978

Underdetermined models for classical logic

$\wedge$	T	F_2	F_1	F
T	T	F_2	F_1	F
F_2	F_2	F_2	F	F
F_1	F_1	F	F_1	F
F	F	F	F	F

$\vee$	T	F_2	F_1	F
T	T	T	T	T
F_2	T	F_2	T	F_2
F_1	T	T	F_1	F_1
F	T	F_2	F_1	F

	$\sim$
T	F
F_2	F_1
F_1	F_2
F	T

(Notice that there is no *single* redundant value!)

From Shoesmith & Smiley 1978

Underdetermined models for classical logic

$\wedge$	T	F_2	F_1	F
T	T	F_2	F_1	F
F_2	F_2	F_2	F	F
F_1	F_1	F	F_1	F
F	F	F	F	F

$\vee$	T	F_2	F_1	F
T	T	T	T	T
F_2	T	F_2	T	F_2
F_1	T	T	F_1	F_1
F	T	F_2	F_1	F

	$\sim$
T	F
F_2	F_1
F_1	F_2
F	T

(Notice that there is no *single* redundant value!)

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From Shoesmith & Smiley 1978

Underdetermined models for classical logic

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T	T	F_2	F_1	F
F_2	F_2	F_2	F	F
F_1	F_1	F	F_1	F
F	F	F	F	F

$\vee$	T	F_2	F_1	F
T	T	T	T	T
F_2	T	F_2	T	F_2
F_1	T	T	F_1	F_1
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(though $\vDash^{\mathsf{m}} \sim\alpha \vee \alpha$)

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Given some $\mathcal{S}$, here are the elements of a **(tarskian) interpretation** over it:

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(antitheses of $\Vdash^{s}$??)

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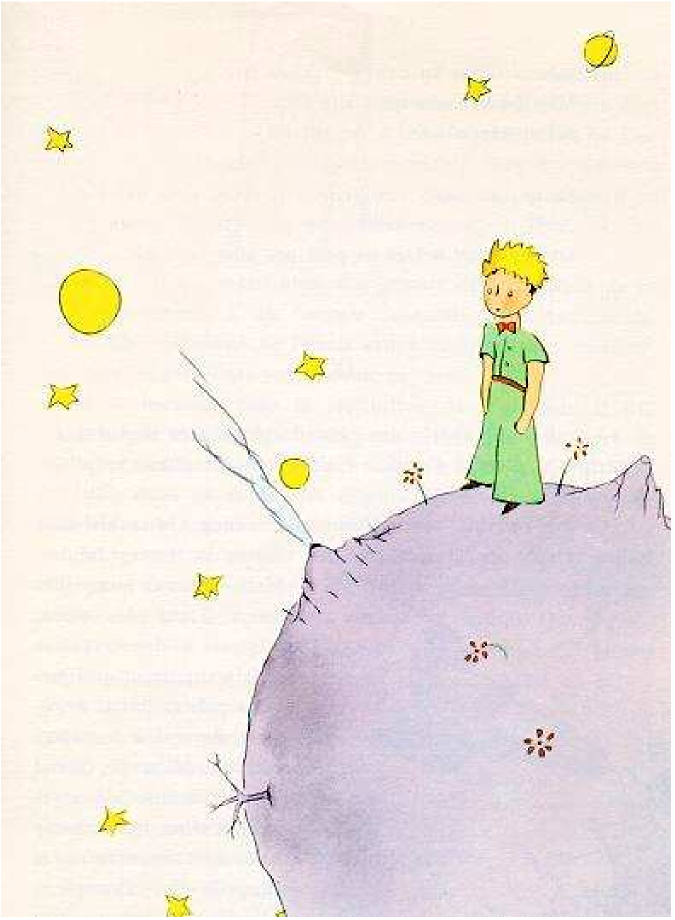
Why all the **bias towards truth**?



In 1869, Jules Verne published *Autour de la Lune*.
In 1959, *Luna* 3 photographed the far side of the Moon.

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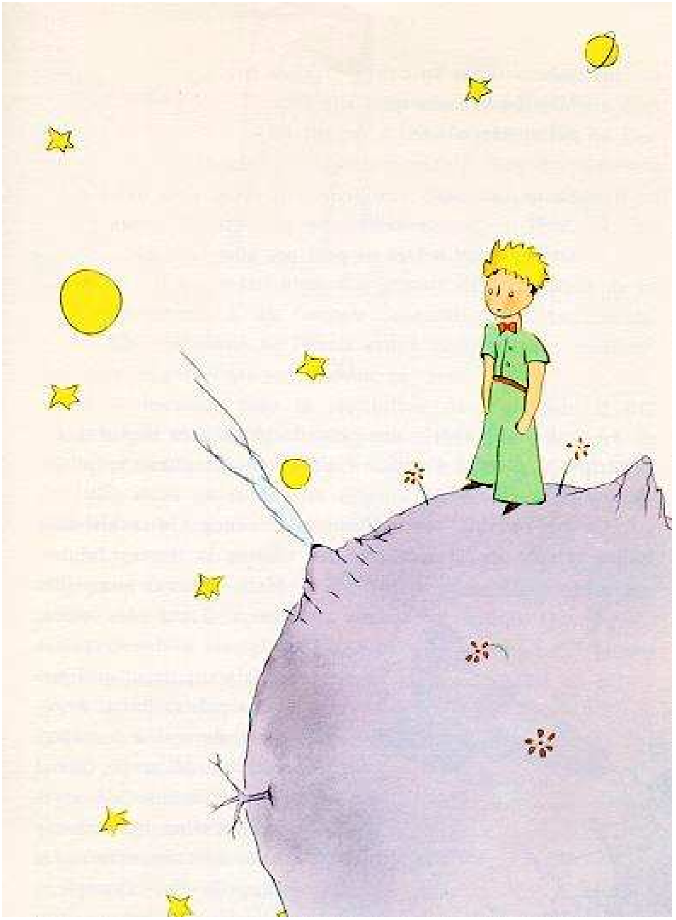

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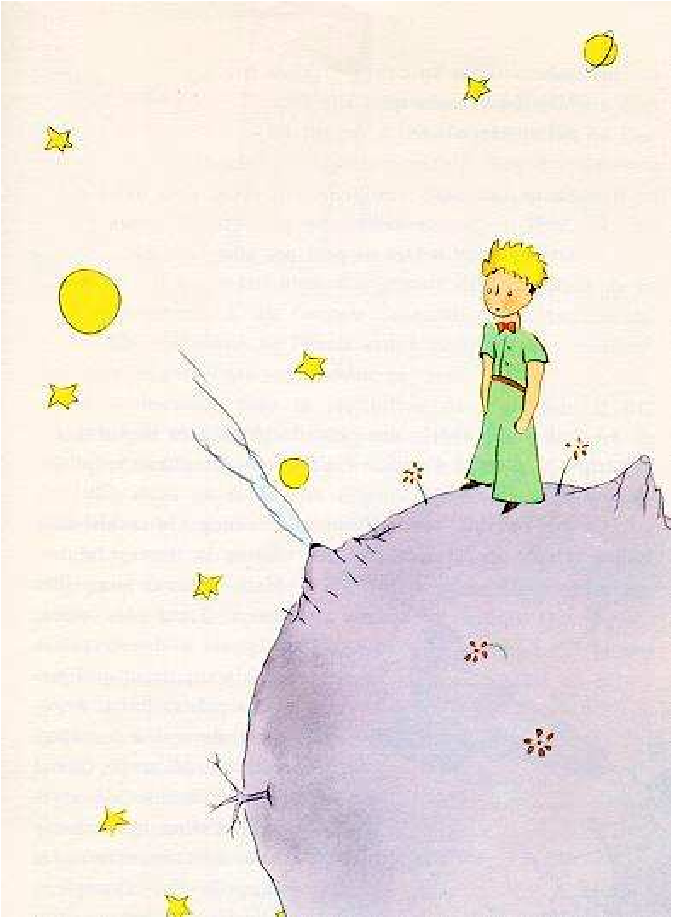

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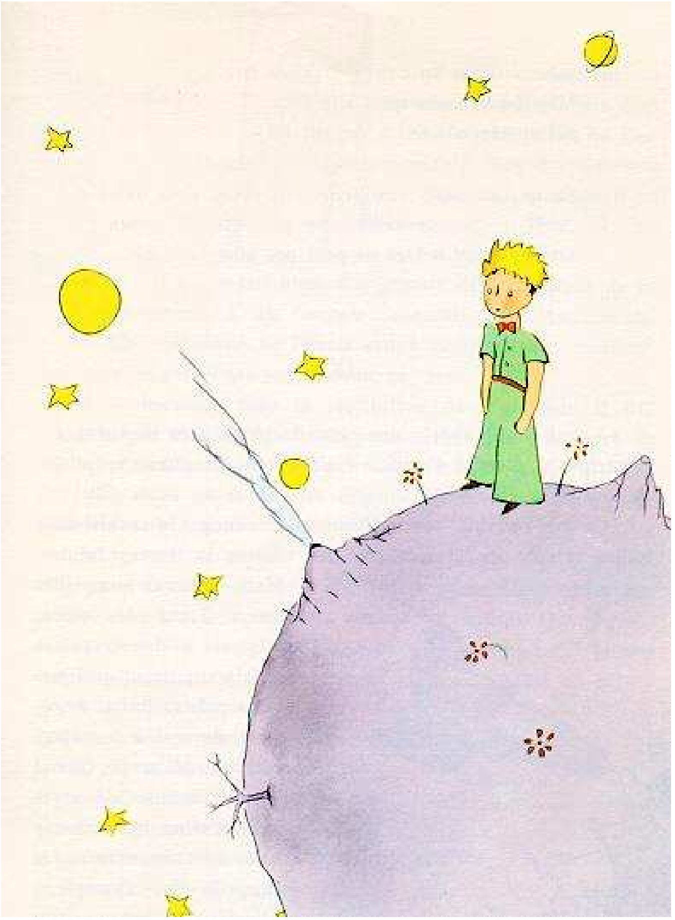

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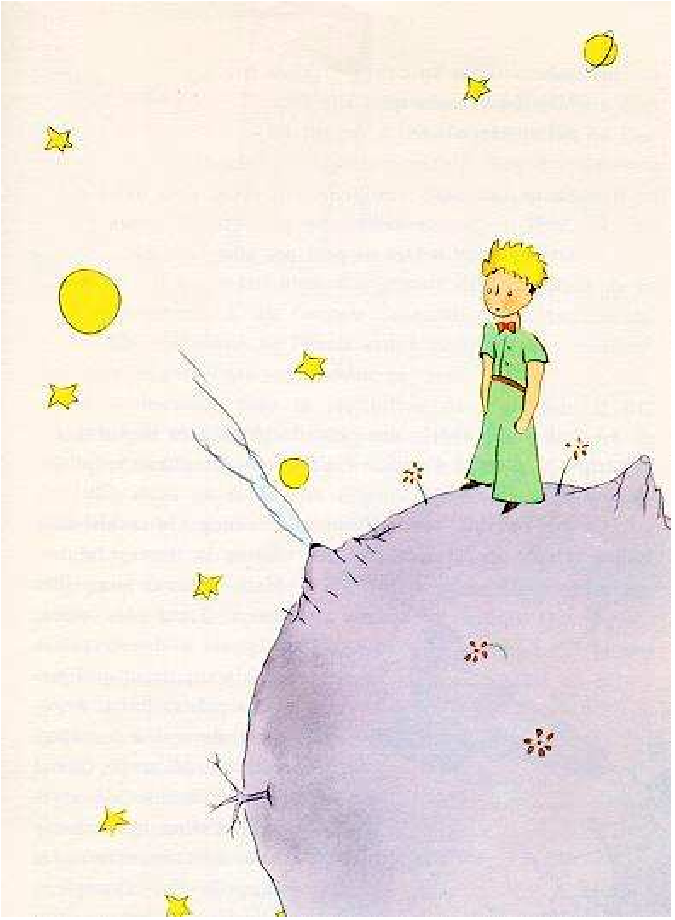

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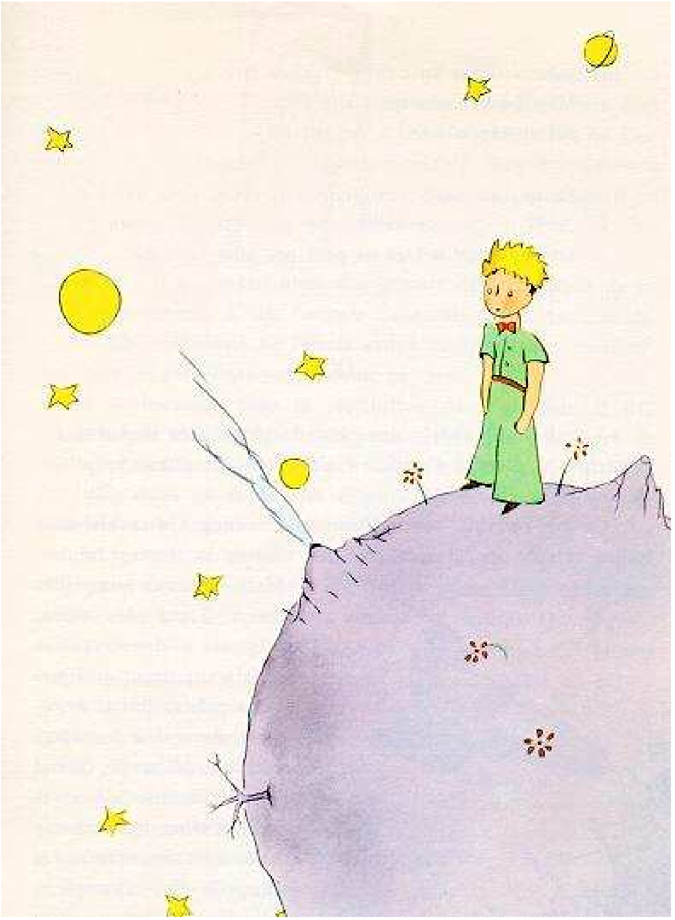

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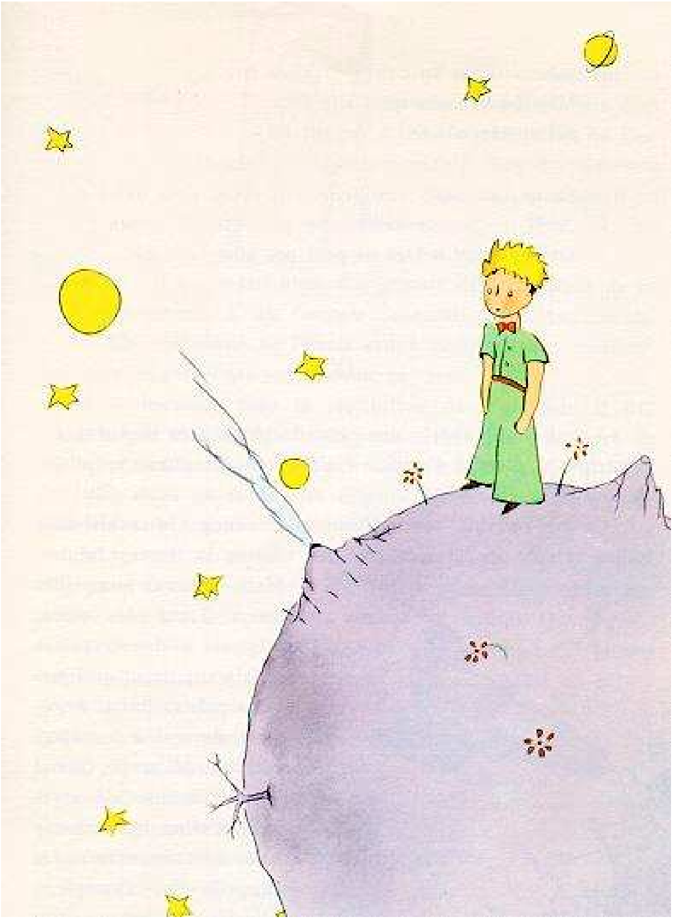

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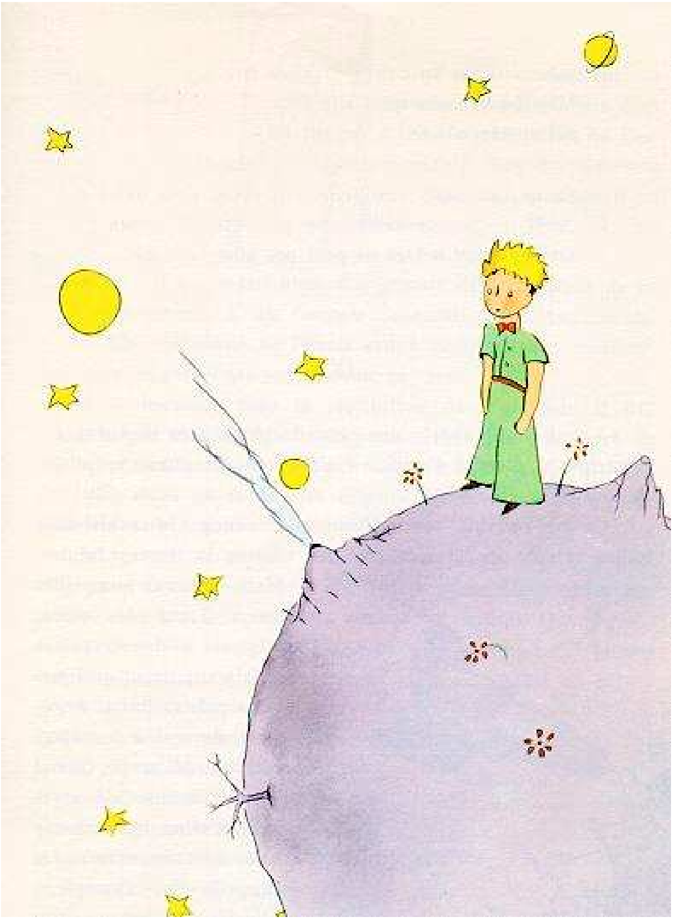

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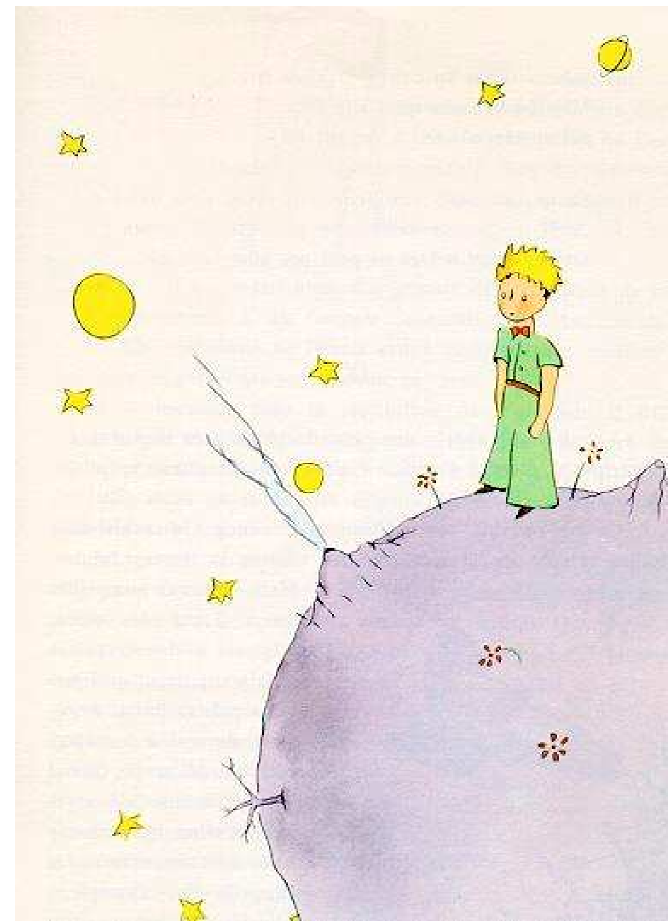

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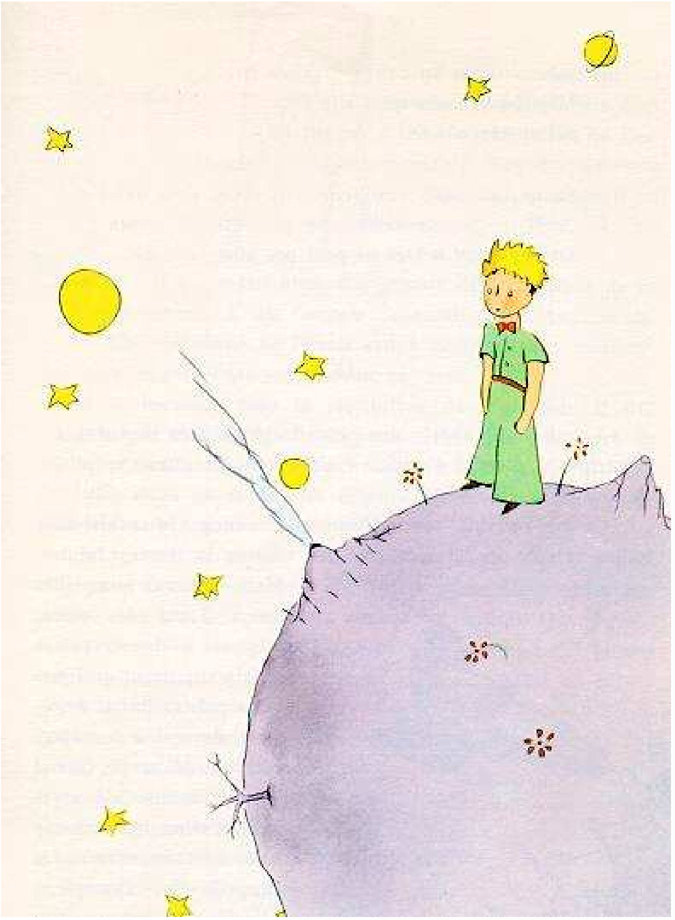

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$\mathcal{S} \not\models_\S$	iff	$\S \in$ Dada

Indecent Logics

$$\Gamma \Vdash \Delta$$

When the **nature** of inference does not really matter:

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Semantical conditions:

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Indecent Logics

$\Gamma \Vdash \Delta$

When the **nature** of inference does not really matter:

(i)	(ii)	(iii)
dadaistic	**nihilistic**	**semitrivial**

Semantical conditions:

$\mathcal{D}_{\mathsf{i}} \neq \varnothing$	$\mathcal{U}_{\mathsf{ii}} \neq \varnothing$	$\mathcal{D}_{\mathsf{iii}} \neq \varnothing$ and $\mathcal{U}_{\mathsf{iii}} \neq \varnothing$
$\mathsf{Sem}_{\mathsf{i}} \subseteq \mathsf{Dada}$	$\mathsf{Sem}_{\mathsf{ii}} \subseteq \mathsf{Nihil}$	$\mathsf{Sem}_{\mathsf{iii}} \subseteq \mathsf{Dada} \bigcup \mathsf{Nihil}$

Indecent Logics

$\Gamma \Vdash \Delta$

When the **nature** of inference does not really matter:

(i)	(ii)	(iii)	(iv)
dadaistic	**nihilistic**	**semitrivial**	**trivial**

Semantical conditions:

$\mathcal{D}_{\mathsf{i}} \neq \varnothing$	$\mathcal{U}_{\mathsf{ii}} \neq \varnothing$	$\mathcal{D}_{\mathsf{iii}} \neq \varnothing$ and $\mathcal{U}_{\mathsf{iii}} \neq \varnothing$
$\mathsf{Sem}_{\mathsf{i}} \subseteq \mathsf{Dada}$	$\mathsf{Sem}_{\mathsf{ii}} \subseteq \mathsf{Nihil}$	$\mathsf{Sem}_{\mathsf{iii}} \subseteq \mathsf{Dada} \bigcup \mathsf{Nihil}$

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Semantical conditions:

$\mathcal{D}_{\text{i}} \neq \varnothing$	$\mathcal{U}_{\text{ii}} \neq \varnothing$	$\mathcal{D}_{\text{iii}} \neq \varnothing$ and $\mathcal{U}_{\text{iii}} \neq \varnothing$	—
$\text{Sem}_{\text{i}} \subseteq \text{Dada}$	$\text{Sem}_{\text{ii}} \subseteq \text{Nihil}$	$\text{Sem}_{\text{iii}} \subseteq \text{Dada} \bigcup \text{Nihil}$	$\text{Sem}_{\text{iv}} \subseteq \text{Dada} \bigcap \text{Nihil} (= \varnothing)$

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Semantical conditions:

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Single-conclusion abstract characterizations:

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Single-conclusion abstract characterizations:

$(\forall \beta \Gamma)$

$\Gamma \Vdash^{\mathsf{s}}_{\mathsf{i}} \beta$

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(i)	(ii)	(iii)	(iv)
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Semantical conditions:

$\mathcal{D}_{\text{i}} \neq \varnothing$	$\mathcal{U}_{\text{ii}} \neq \varnothing$	$\mathcal{D}_{\text{iii}} \neq \varnothing$ and $\mathcal{U}_{\text{iii}} \neq \varnothing$	—
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Single-conclusion abstract characterizations:

$(\forall \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$
$\Gamma \Vdash^{\text{s}}_{\text{i}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{ii}} \beta$

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$\Gamma \Vdash^{\mathsf{s}}_{\mathsf{i}} \beta$	$\Gamma, \alpha \Vdash^{\mathsf{s}}_{\mathsf{ii}} \beta$	$\Gamma, \alpha \Vdash^{\mathsf{s}}_{\mathsf{iii}} \beta$	

Indecent Logics

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Semantical conditions:

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Single-conclusion abstract characterizations:

(i)	(ii)	(iii)	(iv)
$(\forall \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \beta \Gamma)$
$\Gamma \Vdash^{\text{s}}_{\text{i}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{ii}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{iii}} \beta$	$\Gamma \Vdash^{\text{s}}_{\text{iv}} \beta$

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Semantical conditions:

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Single-conclusion abstract characterizations:

$(\forall \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \beta \Gamma)$
$\Gamma \Vdash^{\mathsf{s}}_{\mathsf{i}} \beta$	$\Gamma, \alpha \Vdash^{\mathsf{s}}_{\mathsf{ii}} \beta$	$\Gamma, \alpha \Vdash^{\mathsf{s}}_{\mathsf{iii}} \beta$	$\Gamma \Vdash^{\mathsf{s}}_{\mathsf{iv}} \beta$

Multiple-conclusion abstract characterizations:

Indecent Logics

$\Gamma \Vdash \Delta$

When the **nature** of inference does not really matter:

(i)	(ii)	(iii)	(iv)
dadaistic	**nihilistic**	**semitrivial**	**trivial**

Semantical conditions:

$\mathcal{D}_{\mathsf{i}} \neq \varnothing$	$\mathcal{U}_{\mathsf{ii}} \neq \varnothing$	$\mathcal{D}_{\mathsf{iii}} \neq \varnothing$ and $\mathcal{U}_{\mathsf{iii}} \neq \varnothing$	—
$\mathsf{Sem}_{\mathsf{i}} \subseteq \mathsf{Dada}$	$\mathsf{Sem}_{\mathsf{ii}} \subseteq \mathsf{Nihil}$	$\mathsf{Sem}_{\mathsf{iii}} \subseteq \mathsf{Dada} \bigcup \mathsf{Nihil}$	$\mathsf{Sem}_{\mathsf{iv}} \subseteq \mathsf{Dada} \bigcap \mathsf{Nihil} (= \varnothing)$

Single-conclusion abstract characterizations:

$(\forall \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \beta \Gamma)$
$\Gamma \Vdash^{\mathsf{s}}_{\mathsf{i}} \beta$	$\Gamma, \alpha \Vdash^{\mathsf{s}}_{\mathsf{ii}} \beta$	$\Gamma, \alpha \Vdash^{\mathsf{s}}_{\mathsf{iii}} \beta$	$\Gamma \Vdash^{\mathsf{s}}_{\mathsf{iv}} \beta$

Multiple-conclusion abstract characterizations:

$(\forall \beta \Gamma \Delta)$

$\Gamma \Vdash^{\mathsf{m}}_{\mathsf{i}} \beta, \Delta$

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$\Gamma \Vdash^{\text{s}}_{\text{i}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{ii}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{iii}} \beta$	$\Gamma \Vdash^{\text{s}}_{\text{iv}} \beta$

Multiple-conclusion abstract characterizations:

$(\forall \beta \Gamma \Delta)$	$(\forall \alpha \Gamma \Delta)$
$\Gamma \Vdash^{\text{m}}_{\text{i}} \beta, \Delta$	$\Gamma, \alpha \Vdash^{\text{m}}_{\text{ii}} \Delta$

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Single-conclusion abstract characterizations:

$(\forall \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \beta \Gamma)$
$\Gamma \Vdash^{\text{s}}_{\text{i}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{ii}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{iii}} \beta$	$\Gamma \Vdash^{\text{s}}_{\text{iv}} \beta$

Multiple-conclusion abstract characterizations:

$(\forall \beta \Gamma \Delta)$	$(\forall \alpha \Gamma \Delta)$	$(\forall \alpha \beta \Gamma \Delta)$
$\Gamma \Vdash^{\text{m}}_{\text{i}} \beta, \Delta$	$\Gamma, \alpha \Vdash^{\text{m}}_{\text{ii}} \Delta$	$\Gamma, \alpha \Vdash^{\text{m}}_{\text{iii}} \beta, \Delta$

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$\Gamma \Vdash^{\text{s}}_{\text{i}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{ii}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{iii}} \beta$	$\Gamma \Vdash^{\text{s}}_{\text{iv}} \beta$

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$(\forall \beta \Gamma \Delta)$	$(\forall \alpha \Gamma \Delta)$	$(\forall \alpha \beta \Gamma \Delta)$	$(\forall \Gamma \Delta)$
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Now, compare:

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Semantical conditions:

(i)	(ii)	(iii)	(iv)
$\mathcal{D}_{\text{i}} \neq \varnothing$	$\mathcal{U}_{\text{ii}} \neq \varnothing$	$\mathcal{D}_{\text{iii}} \neq \varnothing$ and $\mathcal{U}_{\text{iii}} \neq \varnothing$	—
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(i)	(ii)	(iii)	(iv)
$(\forall \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \beta \Gamma)$
$\Gamma \Vdash^{\text{s}}_{\text{i}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{ii}} \beta$	$\Gamma, \alpha \Vdash^{\text{s}}_{\text{iii}} \beta$	$\Gamma \Vdash^{\text{s}}_{\text{iv}} \beta$

Multiple-conclusion abstract characterizations:

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$(\forall \beta \Gamma \Delta)$	$(\forall \alpha \Gamma \Delta)$	$(\forall \alpha \beta \Gamma \Delta)$	$(\forall \Gamma \Delta)$
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$(\forall \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \alpha \beta \Gamma)$	$(\forall \beta \Gamma)$
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Call a logic $\mathcal{L} = \langle \mathcal{S}, \vDash \rangle$ **consistent** in case:

(1) $\mathcal{L}$ is non-dadaistic (i.e., $\mathsf{Sem}_{\mathcal{L}} \not\subseteq \mathsf{Dada}$)

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yet:

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Ineffable Inconsistencies

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(2) $\mathcal{S}$ is $\mathcal{L}$-trivializing (i.e., $(\forall \Delta \subseteq \mathcal{S})\ \mathcal{S} \models^{\mathsf{m}} \Delta$)

Here is the **Paradox of Ineffable Inconsistencies**:

Given any consistent tarskian logic $\mathcal{L}$, one can always find an inconsistent logic $\mathcal{IL}$ such that:

$$\Gamma \models^{\mathsf{m}}_{\mathcal{IL}} \beta, \Delta \quad \text{iff} \quad \Gamma \models^{\mathsf{m}}_{\mathcal{L}} \beta, \Delta$$

yet: $\mathcal{S} \not\models^{\mathsf{m}}_{\mathcal{IL}}$.

HOW?

Ineffable Inconsistencies

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HOW? Just **add** to $\mathsf{Sem}_{\mathcal{L}}$ an arbitrary dadaistic valuation!