Multiple-Conclusion Logics

PART 2: “General Abstract Nonsense”

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Introductory (and Motivational) Course

$\Gamma \Vdash \Delta$

General Abstract Nonsense

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Consider logics $\mathcal{L}_{\Vdash} = \langle \mathcal{S}, \Vdash \rangle$ and $\mathcal{L}_{\vDash} = \langle \mathcal{S}, \vDash \rangle$
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Idea: To provide abstract axiomatizations for interesting semantical ideas, and vice-versa.

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(C1) $\Gamma \subseteq \Gamma^{\Vdash}$ — overlap

(C2) $(\Gamma^{\Vdash})^{\Vdash} \subseteq \Gamma$ — full cut

(C3) $\Gamma \subseteq \Lambda \ \Rightarrow\ \Gamma^{\Vdash} \subseteq \Lambda^{\Vdash}$ — dilution

(CK1) $(\Gamma \cup \Sigma)^{\Vdash} = \Gamma^{\Vdash} \cup \Sigma^{\Vdash}$ — premise-apartness

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(CK1) $\Sigma, \Gamma \Vdash \alpha \ \Leftrightarrow\ \Sigma \Vdash \alpha$ or $\Gamma \Vdash \alpha$ — premise-apartness

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…providing a Representation Theorem for the 'semantics of closed sets'.

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Here is a preliminary question:
Can (C2) be substituted by

(C2n) $\Sigma, \lambda \Vdash \beta$ and $\Gamma \Vdash \lambda \;\Rightarrow\; \Sigma, \Gamma \Vdash \beta$ naive cut

???

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Suppose we now define $\stackrel{\frown}{\smile} (\subseteq \text{Pow}(\mathcal{S}) \times \text{Pow}(\mathcal{S}))$ by setting $\Gamma \stackrel{\frown}{\smile} \Delta$ iff $((\forall\delta \in \Delta)\Gamma \Vdash \delta$ and $(\forall\gamma \in \Gamma)\Delta \Vdash \gamma)$.

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However:

- E1: with (C2) in the place of (C2n), $\stackrel{\frown}{\smile}$ *does* define an equivalence
- E2: (C1) + (C2) + (C3) $\Rightarrow$ (C2n)
- E3: (C1) + (C2n) + (C3) $\not\Rightarrow$ (C2)

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Axiom of Choice!

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Note that:

E4: (CC) + (C1) + (C2n) + (C3) $\Rightarrow$ (C2)

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substitutionality — *notion of 'logical form'!*

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(CLS) *is* preserved

(CC) is *not* preserved (ω-rules. . .)

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Say that $\langle \mathcal{S}, \vDash_{\mathsf{Sem}} \rangle$ is a κ-valued logic if $\kappa = \mathsf{Max}_{\S \in \mathsf{Sem}}(|\mathcal{V}_\S|)$.

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- Superlogics:

$$\bigcap_{i \in I} \models_{\mathsf{Sem}(i)} \;=\; \models_{\bigcup_{i \in I} \mathsf{Sem}[i]}$$

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Glossary:

- J.-Y. Béziau's *β-excessive* translates Günter Asser's 'vollständig in Bezug auf β'

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- In classical logic, excessive $\Rightarrow$ maximal.

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By *dilution* [(C3)], $\Phi \not\Vdash \beta$. By *compactness* [(CC)], $\bigcup \mathcal{C} \not\Vdash \beta$.
By *Zorn's Lemma*, Exc$(\Gamma, \beta, \mathcal{L})$ has a maximal element Γ_{exc}.

Q.E.D.

Automatic soundness

Fix some logic $\mathcal{L} = \langle \mathcal{S}, \Vdash \rangle$ and some theory Γ in what follows.

Call $\Gamma^{\Vdash} = \{\alpha : \Gamma \Vdash \alpha\}$ the right-closure of Γ.

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By (C1), $(\forall \delta \in \Delta)\, \Gamma^{\Vdash} \Vdash \delta$.

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Any single-conclusion $\mathbb{T}$-logic is many-valued

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Given some $\mathbf{T}$-logic $\mathcal{L}$, consider the superlogic $\mathcal{L}_{\mathcal{F}}$ of its Lindenbaum Bundle $\mathcal{F} = \{\mathcal{L}_\Gamma : \Gamma \subseteq \mathcal{S}\}$. Soundness is obvious.

[Now, for completeness: $\Delta \vDash_{\mathsf{Sem}(\bigcap \mathcal{F})} \beta \;\Rightarrow\; \Delta \Vdash \beta$.]

Suppose $\Delta \vDash_{\mathsf{Sem}(\bigcap \mathcal{F})} \beta$.

Thus, $\Delta \vDash_\Gamma \beta$, for every $\Gamma \subseteq \mathcal{S}$.

By the definition of $\vDash_\Gamma$, and the fact that $\mathcal{L}$ is a $\mathbf{T}$-logic, this means that $(\forall \Gamma \subseteq \mathcal{S})\ \Gamma, \Delta \Vdash \beta$.

In particular, for $\Gamma = \varnothing$, we have that $\Delta \Vdash \beta$. **Q.E.D.**

So:

Every single-conclusion $\mathbf{T}$-logic is κ-valued, for $\kappa = |\mathcal{S}|$.

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[Suszko's Reduction] ‘logical’ $\times$ ‘algebraic’ truth-values

For any many-valued valuation $\S : \mathcal{S} \longrightarrow \mathcal{V}_{\S}$ for a $\mathbf{T}$-logic $\mathcal{L}$,
with semantics $\text{Sem}(\kappa)$, consider its ‘binary print’:

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Collect such $b^\S$'s into Sem(2). Note that:

$$\Delta \vDash_{\text{Sem}(2)} \beta \text{ iff } \Delta \vDash_{\text{Sem}(\kappa)} \beta.$$

Q.E.D.

On the theory of (bi)valuations

Any theory $\Gamma \subseteq \mathcal{S}$ determines a characteristic bivaluation:

$$b_\Gamma(\varphi) = T \text{ iff } \varphi \in \Gamma.$$

Recall Exc$(\Gamma, \beta, \mathcal{L})$, the collection of all β-excessive theories extending Γ in $\mathcal{L}$.

Let Max$(\Gamma, \mathcal{L})$ be the collection of all maximal theories extending Γ in $\mathcal{L}$.

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- ⋆ If $\mathcal{H} \not\supseteq$ Exc$(\Gamma, \beta, \mathcal{L})$, completeness fails for Biv$(\mathcal{H})$ [Béziau 1999]
- ⋆ If Exc$(\Gamma, \beta, \mathcal{L}) \boxed{\subseteq \mathcal{H} \subseteq}$ Clo$(\Gamma, \mathcal{L})$, then Biv$(\mathcal{H})$ is an adequate semantics for $\mathcal{L}$. [da Costa & Béziau 1994ff]

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Then both $\{b_1\}$ and $\{b_1, b_2\}$ are adequate for $\mathcal{L}$.

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- CL with ineffable inconsistencies

Multiple-Conclusion $\mathbb{T}$-logics

Recall the abstract axioms of single-conclusion $\mathbf{T}$-logics:

(C1) $\Gamma, \beta \Vdash \beta$ — overlap

(C2) $\Lambda \Vdash \beta$ and $(\forall\lambda \in \Lambda)\Gamma \Vdash \lambda \;\Rightarrow\; \Gamma \Vdash \beta$ — full cut

(C3) $\Gamma \Vdash \beta \;\Rightarrow\; \Sigma, \Gamma \Vdash \beta$ — dilution

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And now consider multiple-conclusion approaches of them:

(C1) $\Gamma, \beta \Vdash \beta, \Delta$ — overlap

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¿(C2L)? $\Gamma, \Lambda \Vdash \Delta$ and $(\forall \lambda \in \Lambda)\Sigma \Vdash \lambda, \Pi \;\Rightarrow\; \Sigma, \Gamma \Vdash \Delta, \Pi$ left-cut

¿(C2R)? $\Gamma \Vdash \Lambda, \Delta$ and $(\forall \lambda \in \Lambda)\Sigma, \lambda \Vdash \Pi \;\Rightarrow\; \Sigma, \Gamma \Vdash \Delta, \Pi$ right-cut

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Call $\langle \Sigma, \Pi \rangle$ a *quasi-partition* of the set $\Theta \subseteq \mathcal{S}$ in case $\Sigma \cup \Pi = \Theta$ and $\Sigma \cap \Pi = \varnothing$.
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Note that:

- (C3**L**) + (C3**R**) $\Rightarrow$ (C3)
- (C2**L**) + (C2**R**) $\boxed{\not\Rightarrow}$ (C2)

The many ways of cutting

Recall the multiple-conclusion version of (C2):

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(C2**Lc**) $\not\Leftrightarrow$ (C2**Rc**) $\not\Leftrightarrow$ (C2**LR**)

(C2**Lc**) and (C2**Rc**) $\Leftrightarrow$ (C2**LR**) [(C3)]

(C2**Lc**) or (C2**Rc**) $\Rightarrow$ (C2**for**)

(C2**Lc**) or (C2**Rc**) $\not\Leftarrow$ (C2**for**)

Check Shoesmith & Smiley 1978

The many ways of cutting

Recall the multiple-conclusion version of (C2):

(C2) $(\exists\Theta \subseteq \mathcal{S})(\forall\langle\Sigma,\Pi\rangle \in \mathsf{QPart}(\Theta))\ \Sigma,\Gamma \Vdash \Delta,\Pi \ \Rightarrow\ \Gamma \Vdash \Delta$ full cut

Now, besides (C2**L**) and (C2**R**), one might also consider:

(C2$\mathcal{S}$) Fix $\Theta = \mathcal{S}$ in (C2)

(C2**fin**) Restrict (C2) to finite Θ

(C2**for**) Restrict (C2) by assuming Θ to be a singleton

(C2**Lc**) $\Gamma,\Lambda \Vdash \Delta$ and $(\forall\lambda \in \Lambda)\Gamma \Vdash \lambda,\Delta \ \Rightarrow\ \Gamma \Vdash \Delta$ [Fix $\Gamma = \Sigma$ and

(C2**Rc**) $\Gamma \Vdash \Lambda,\Delta$ and $(\forall\lambda \in \Lambda)\Gamma,\lambda \Vdash \Delta \ \Rightarrow\ \Gamma \Vdash \Delta$ $\Delta = \Pi$ in (C2**X**)]

(C2**LR**) $(\forall\pi \in \Pi)\Gamma \Vdash \pi,\Delta$ and $(\forall\sigma \in \Sigma)\Gamma \Vdash \sigma,\Delta$ and $\Sigma,\Gamma \Vdash \Delta,\Pi \Rightarrow \Gamma \Vdash \Delta$

Then, one can prove:

(C2) $\Leftrightarrow$ (C2$\mathcal{S}$) {(C3)}
(C2**fin**) $\Leftrightarrow$ (C2**for**) {(C3)}
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(C2$\mathbf{for}$) $\Rightarrow$ (C2) {(CC)}

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Lindenbaum Bundle, upgraded

Fix some logic $\mathcal{L} = \langle \mathcal{S}, \Vdash \rangle$ in what follows.

Call the quasi-partition $\langle \Gamma, \Delta \rangle \in \mathsf{QPart}(\mathcal{S})$ closed
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- $\mathcal{S} = \mathcal{V}$, $\mathcal{D} = \Gamma$, $\mathcal{U} = \Delta$, $\mathsf{Sem} = \{\mathsf{Id}_\mathcal{V}\}$

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The Lindenbaum Bundle of $\mathcal{L}$ will now be the set

$\{\mathcal{L}_\Xi : \Xi \in \mathsf{QPart}(\mathcal{S})$ and Ξ is closed$\}$. Then, again:

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Any fiber from the Lindenbaum Bundle is sound for a $\mathbf{T}$-logic $\mathcal{L}$:

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A fundamental lemma, reconsidered

LA-Extension Lemma: [Scott 1971, Segerberg 1982]

A fundamental lemma, reconsidered

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Any pair of sets Γ and Δ such that $\Gamma \not\Vdash \Delta$ of a logic $\mathcal{L}$ that respects (C3) and (CC) can be extended to sets $\Gamma_{\mathsf{cqp}} \supseteq \Gamma$ and $\Delta_{\mathsf{cqp}} \supseteq \Delta$ that define a closed quasi-partition $\langle \Gamma_{\mathsf{cqp}}, \Delta_{\mathsf{cqp}} \rangle$ of $\mathcal{S}$.

A fundamental lemma, reconsidered

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Proof. Similar to the one before, now using (C2$\mathbf{Lc}$) and (C2$\mathbf{Rc}$).

Obviously, by compactness, in a multiple-conclusion environment, one means:

(CC) $\Gamma \Vdash \Delta \quad \Rightarrow \quad (\exists \Gamma_\Phi \in \mathsf{Fin}(\Gamma))(\exists \Delta_\Phi \in \mathsf{Fin}(\Delta))\ \Gamma_\Phi \Vdash \Delta_\Phi$

Multiple-Conclusion $\mathbb{T}$-logics are many-valued

[W-Reduction]

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Given some $\mathbf{T}$-logic $\mathcal{L}$, consider again the superlogic $\mathcal{L}_{\mathcal{F}}$ of its Lindenbaum Bundle $\mathcal{F} = \{\mathcal{L}_{\Xi} : \Xi \in \mathsf{QPart}(\mathcal{S})$ and Ξ is closed$\}$.

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Soundness is obvious.

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$$\Sigma \vDash_{\mathcal{F}} \Pi \;\Rightarrow\; \Sigma \Vdash \Pi, \text{ where } \vDash_{\mathcal{F}} = \bigcap_{\mathcal{F}}(\vDash_\Xi).$$

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Suppose $\Sigma \nVdash \Pi$. By (C2), there is some quasi-partition $\Xi = \langle \Gamma, \Delta \rangle$ of $\mathcal{S}$ such that $\Sigma, \Gamma \nVdash \Delta, \Pi$.

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By (C1), we must have $\Sigma \subseteq \Gamma$ and $\Pi \subseteq \Delta$.

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Suppose $\Sigma \nVdash \Pi$. By (C2), there is some quasi-partition $\Xi = \langle \Gamma, \Delta \rangle$ of $\mathcal{S}$ such that $\Sigma, \Gamma \nVdash \Delta, \Pi$.

From (C3), Ξ must be closed: $\Gamma \nVdash \Delta$.

By (C1), we must have $\Sigma \subseteq \Gamma$ and $\Pi \subseteq \Delta$. By definition of $\vDash_{\Xi}$, we conclude that $\Sigma \nvDash_{\Xi} \Pi$.

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Soundness is obvious. Now, for completeness:

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Suppose $\Sigma \nVdash \Pi$. By (C2), there is some quasi-partition $\Xi = \langle\Gamma, \Delta\rangle$ of $\mathcal{S}$ such that $\Sigma, \Gamma \nVdash \Delta, \Pi$.

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So: Every multiple-conclusion $\mathbf{T}$-logic is κ-valued, for $\kappa = |\mathcal{S}|$.

Multiple-Conclusion $\mathbb{T}$-logics are many-valued

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Tarskian, or Scottian Logics?

Given some $\mathbf{T}$-logic $\mathcal{L}$, consider again the superlogic $\mathcal{L}_{\mathcal{F}}$ of its Lindenbaum Bundle $\mathcal{F} = \{\mathcal{L}_{\Xi} : \Xi \in \text{QPart}(\mathcal{S})$ and Ξ is closed$\}$.

Soundness is obvious. Now, for completeness:

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So: Every multiple-conclusion $\mathbf{T}$-logic is κ-valued, for $\kappa = |\mathcal{S}|$.

Multiple-Conclusion $\mathbb{T}$-logics are 2-valued

[S-Reduction]

Exactly like before...

Multiple-Conclusion $\mathbb{T}$-logics are 2-valued

[S-Reduction]

For any many-valued valuation $\S : \mathcal{S} \rightarrow \mathcal{V}_{\S}$ for a $\mathbf{T}$-logic $\mathcal{L}$, with semantics Sem(κ), consider its 'binary print':

Let $\mathcal{V}(2) = \{T, F\}$ and $\mathcal{D}(2) = T$, and

define a bivaluation $b^{\S} : \mathcal{S} \rightarrow \mathcal{V}(2)$ such that

$$b^{\S}(\varphi) = T \text{ iff } \S(\varphi) \in \mathcal{D}.$$

Multiple-Conclusion $\mathbb{T}$-logics are 2-valued

[S-Reduction]

For any many-valued valuation $\S : \mathcal{S} \longrightarrow \mathcal{V}_\S$ for a $\mathbf{T}$-logic $\mathcal{L}$, with semantics $\mathsf{Sem}(\kappa)$, consider its 'binary print':

Let $\mathcal{V}(2) = \{T, F\}$ and $\mathcal{D}(2) = T$, and define a bivaluation $b^\S : \mathcal{S} \longrightarrow \mathcal{V}(2)$ such that

$$b^\S(\varphi) = T \;\text{ iff }\; \S(\varphi) \in \mathcal{D}.$$

Collect such $b^\S$'s into $\mathsf{Sem}(2)$. Note that:

$$\boxed{\Sigma \vDash_{\mathsf{Sem}(2)} \Pi \;\text{ iff }\; \Sigma \vDash_{\mathsf{Sem}(\kappa)} \Pi.}$$

Q.E.D.

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Collect such $b^\S$'s into $\mathsf{Sem}(2)$. Note that:

$$\boxed{\Sigma \vDash_{\mathsf{Sem}(2)} \Pi \;\text{ iff }\; \Sigma \vDash_{\mathsf{Sem}(\kappa)} \Pi.}$$

Q.E.D.

More importantly, as we will see:

The binary print of a multiple-conclusion logic is unique!

Categoricity of multiple-conclusion CRs

Recall that single-conclusion CRs are **not** categorical,

neither for many-valued tarskian interpretations

nor for 2-valued tarskian interpretations. . .

Categoricity of multiple-conclusion CRs

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Is it possible that $\mathsf{Sem}_1 \neq \mathsf{Sem}_2$ yet $\models_1 = \models_2$,
in a multiple-conclusion environment?

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Recall that single-conclusion CRs are **not** categorical,
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Is it possible that $\text{Sem}_1 \neq \text{Sem}_2$ yet $\models_1 = \models_2$,
in a multiple-conclusion environment?

The answer is **NO** if we are talking about
bivaluation semantics**!!**

Categoricity of multiple-conclusion CRs

Lemma [Uniqueness of 2-valued counter-examples]

Let b and c be two bivaluations on $\mathcal{S}$.
Let $\langle \Sigma, \Pi \rangle$ be a quasi-partition of $\mathcal{S}$.
Then, $\Sigma \not\models_b \Pi$ and $\Sigma \not\models_c \Pi \;\Rightarrow\; b = c$.

Categoricity of multiple-conclusion CRs

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What is that supposed to mean, in practice??

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Fix some $\mathcal{S}$ in what follows.

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Given a quasi-partition $\Theta = \langle \Gamma, \Delta \rangle$, say that
a bivaluation b: $\mathcal{S} \to \{T, F\}$ respects Θ
if $b(\Gamma) \nsubseteq \{T\}$ or $b(\Delta) \nsubseteq \{F\}$.

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Given a collection of quasi-partitions $\mathcal{P}$, let Biv$(\mathcal{P})$ be the set of all bivaluations that respect some $\Theta \in \mathcal{P}$.

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Biv($\mathcal{P}$) is the set of all bivaluations that respect some $\Theta \in \mathcal{P}$.

Call CQPart($\mathcal{S}$) the set of all closed quasi-partitions of $\mathcal{S}$.

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Biv($\mathcal{P}$) is the set of all bivaluations that respect some $\Theta \in \mathcal{P}$.
Call CQPart($\mathcal{S}, \mathcal{L}$) the set of all closed quasi-partitions of $\mathcal{S}$ in $\mathcal{L}$.

Then, for a multiple-conclusion logic $\mathcal{L}$:

Biv($\mathcal{P}$) is adequate for $\mathcal{L}$ **iff** $\mathcal{P} =$ CQPart($\mathcal{S}, \mathcal{L}$)

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In this sense, **categoricity** is the ‘dual’ to **adequacy**!

Having the right connections

Fix some $\mathcal{S}$ in what follows.

Let $\mathcal{T}^{\mathcal{A}}$ be the collection of all abstract $\mathbf{T}$-logics over $\mathcal{S}$,
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Given some $\Vdash \in \mathcal{T}^{\mathcal{A}}$,
let $\mathsf{Biv}_{\Vdash}$ be the collection of all bivaluations
that respect every $\langle \Gamma, \Delta \rangle$, where $\Gamma \Vdash \Delta$.

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Consider the mappings $\mathbf{BA} : \mathcal{T}^{\mathcal{B}} \longrightarrow \mathcal{T}^{\mathcal{A}}$ and $\mathbf{AB} : \mathcal{T}^{\mathcal{A}} \longrightarrow \mathcal{T}^{\mathcal{B}}$ such that:

$$\mathsf{Biv} \overset{\mathbf{BA}}{\longmapsto} \Vdash_{\mathsf{Biv}}$$

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Consider: $\mathsf{Biv} \overset{\mathbf{BA}}{\mapsto} \Vdash_{\mathsf{Biv}}$ $\Vdash \overset{\mathbf{AB}}{\mapsto} \mathsf{Biv}_{\Vdash}$

Observe that: [Dunn & Hardegree 2001]

$\langle \mathbf{BA}, \mathbf{AB} \rangle$ is a **Galois connection**

between the posets $\langle \mathcal{T}^{\mathcal{A}}, \supseteq \rangle$ and $\langle \mathcal{T}^{\mathcal{B}}, \subseteq \rangle$, that is:

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Question: When can the converses of 1(a) and 1(b) be proven?

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- The converse to 1(a) amounts to completeness, and can be attained in either single- or multiple-conclusion $\mathbf{T}$-logics.

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As a matter of fact:

- The converse to 1(a) amounts to completeness, and can be attained in either single- or multiple-conclusion $\mathbf{T}$-logics.
- The converse to 1(b) amounts to categoricity, and can only be attained in multiple-conclusion $\mathbf{T}$-logics.

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- The converse to 1(b) amounts to categoricity, and can *only* be attained in multiple-conclusion $\mathbf{T}$-logics.

So, here is a further good reason to go multiple-conclusion:

To **reconciliate** most logics with their **intended models**!!