Multiple-Conclusion Logics

PART 3: “Is that all that there is?”

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Introductory (and Motivational) Course

$\Gamma \Vdash \Delta$

More specific kinds of consequence

Recall, again, the axioms of $\mathbf{T}$-logics:

(C1) $\Gamma, \beta \Vdash \beta, \Delta$ overlap

(C2) $(\exists \Theta \subseteq \mathcal{S})(\forall \langle \Sigma, \Pi \rangle \in \mathsf{QPart}(\Theta))\ \Sigma, \Gamma \Vdash \Delta, \Pi \ \Rightarrow\ \Gamma \Vdash \Delta$ full cut

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- a many-valued semantics $\mathsf{Sem} = \{\S_k : \mathcal{S} \to \mathcal{V}_{\S_k}\}_{k \in K}$
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But this last axiom corresponds to:

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  where $\mathsf{Sem}[\alpha] = \{\S(\alpha) : \S \in \mathsf{Sem}\}$, and $\mathcal{S}$ is a free algebra etc

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How can such logics be characterized abstractly?

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- Łukasiewicz's $Ł_{\aleph_0}$ is genuinely $2^{\aleph_0}$-valued [Shoesmith & Smiley 1971]
- Intuitionistic Logic is not genuinely finitely-valued [Gödel 1932]
- most usual Normal Modal Logics are not genuinely finitely-valued [Dugundji 1940]

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But then again, which logics *have* adequate matrix semantics?

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Say that sets Γ and Δ are *disconnected* if $\mathsf{At}[\Gamma] \cap \mathsf{At}[\Delta] = \varnothing$.

Recall that a theory Λ of $\mathcal{L}$ is called $\mathcal{L}$-trivializing in case $(\forall \Upsilon \subseteq \mathcal{S})\ \Lambda \Vdash \Upsilon$.

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Consider now the following axiom:

(C4) $[\Gamma_k]_{k\in K} \Vdash [\Delta_k]_{k\in K} \quad \Rightarrow \quad \Gamma_k \Vdash \Delta_k$, for any $k \in K$,
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Or, in a single-conclusion version:

($\mathsf{C4^s}$) $[\Gamma_k]_{k\in K}, \Gamma \Vdash \beta \quad \Rightarrow \quad \Gamma \Vdash \beta$, s-cancellation
whenever $\Gamma \cup \{\beta\}, [\Gamma_k]_{k\in K}$ are pairwise disconnected,
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Examples:

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- Johánsson's Minimalkalkül does *not* respect cancellation

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Theorem: Suppose $|\text{At}| = |\mathcal{S}|$. Then: [Shoesmith & Smiley 1971]
A single-conclusion logic $\mathcal{L}$ has an adequate matrix semantics
iff
$\mathcal{L}$ is a substitutional $\mathbf{T}$-logic that respects cancellation [(C4)].

[check also Wójcicki 1969–1970]

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- The above theorem is also valid for compact multiple-conclusion logics [Shoesmith & Smiley 1978]
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Logics with matrix semantics

Theorem: Suppose $|\mathsf{At}| = |\mathcal{S}|$. Then: [Shoesmith & Smiley 1971]
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- The above theorem is also valid for compact multiple-conclusion logics [Shoesmith & Smiley 1978]
- Which property would play the role of cancellation, *in general*, for multiple-conclusion logics?
- How can genuinely finite-valued logics be abstractly characterized? (they are all *compact* and *decidable*, but $Ł_{\aleph_0}$ also is. . .)

Generalizing truth-functionality

Recall that logics with matrix semantics are based on fixed sets $\mathcal{V}$, $\mathcal{D}$, $\mathcal{U}$, and a family of mappings $\mathsf{Sem} = \{\S_k : \mathcal{S} \to \mathcal{V}\}$ s.t.:

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Some alternative kinds of consequence

The $\mathbf{T}$-axioms:

(C1) $\Gamma, \beta \Vdash \beta, \Delta$ overlap

(C2) $(\exists \Theta \subseteq \mathcal{S})(\forall \langle \Sigma, \Pi \rangle \in \mathsf{QPart}(\Theta))\ \Sigma, \Gamma \Vdash \Delta, \Pi \ \Rightarrow\ \Gamma \Vdash \Delta$ full cut

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And now some variations on the $\mathbf{T}$-axioms:

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Illustration: Inferential Many-Valuedness

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SC-*interpretations.* Similar to tarskian, but: [G. Malinowski 1990s]

- besides $\mathcal{D}$ and $\mathcal{V}$, there is also a set of rejected values $\mathcal{R} \subseteq \mathcal{U}$ (inspired on Łukasiewicz's *refutation*)

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An interesting *consequence:* (against 'Suszko's Thesis')

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SC-*axioms:*

(C2c) $\Gamma, \alpha \Vdash \beta$ and $\Gamma \Vdash \alpha \;\Rightarrow\; \Gamma \Vdash \beta$ — cautious cut

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- besides $\mathcal{D}$ and $\mathcal{V}$, there is also a set of rejected values $\mathcal{R} \subseteq \mathcal{U}$
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The canonical way of producing an ‘S-Reduction’
would now result in a 3-valued semantics:

$$\mathcal{D} \mapsto T \qquad \mathcal{R} \mapsto F \qquad \mathcal{U} \setminus \mathcal{R} \mapsto I$$

where I is some sort of ‘intermediary **logical** value’.

Thou shalt not trivialize!

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Examples? Recall the indecent logics:

(i)	(ii)	(iii)	(iv)
dadaistic	**nihilistic**	**semitrivial**	**trivial**
$(\forall\beta\Gamma\Delta)$	$(\forall\alpha\Gamma\Delta)$	$(\forall\alpha\beta\Gamma\Delta)$	$(\forall\Gamma\Delta)$
$\Gamma \Vdash_{\mathsf{i}} \beta, \Delta$	$\Gamma, \alpha \Vdash_{\mathsf{ii}} \Delta$	$\Gamma, \alpha \Vdash_{\mathsf{iii}} \beta, \Delta$	$\Gamma \Vdash_{\mathsf{iv}} \Delta$

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Observe that (i)–(iv) consist in compact substitutional $\mathbf{T}$-logics.

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Note also that (iii) is the only one that
does **not** have a matrix semantics.

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How to avoid these, and restore minimal decency ? Through:

(PNO) $\neg$(C0.1.1) **Principle of Non-Overcompleteness**

Case study: NEGATION: Pure local rules

Here is a typical subclassical rule of negation:

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$$(\Gamma \Vdash {\sim}\beta, \beta, \Delta) \qquad \textit{casus judicans}$$

Case study: **NEGATION**: Pure local rules

Here is a typical subclassical rule of negation:

$(\Gamma \Vdash {\sim}\beta, \beta, \Delta)$ *casus judicans*

And here is its **dual**:

pseudo-scotus $(\Gamma, \alpha, {\sim}\alpha \Vdash \Delta)$

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ex contradictione sequitur quodlibet $(\Gamma, \alpha, {\sim}\alpha \Vdash \beta, \Delta)$

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Notice that *ex contradictione* and *pseudo-scotus* are distinct rules!

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$$(\Gamma, \beta \Vdash {\sim}\beta, \Delta) \;/\; (\Gamma \Vdash {\sim}\beta, \Delta)$$ *consequentia mirabilis*

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➡ *causa mirabilis* $$(\Gamma, {\sim}\alpha \Vdash \alpha, \Delta) \;/\; (\Gamma, {\sim}\alpha \Vdash \Delta)$$

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Here is a typical subclassical rule of negation:

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Here are the pure rules making up a classical negation:

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CRR : 'Consistency-Related Rules'

dextro-levo $(\Gamma \Vdash \alpha, \Delta)$ **/** $(\Gamma, \sim\alpha \Vdash \Delta)$

symmetry $(\Gamma \Vdash \sim\alpha, \Delta)$ **/** $(\Gamma, \alpha \Vdash \Delta)$

("α and $\sim\alpha$ are not both true")

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'Determinedness-Related Rules' : **DRR**

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Interlude on Paranormality:

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Interlude on Paranormality:

Some **CRR** must be failed by paraconsistent logics.

Some **DRR** must be failed by paracomplete logics.

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reductio ad absurdum

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And its so far unsuspected **dual**:

reductio ex evidentia

$(\Gamma, \beta \Vdash \alpha, \Delta$ and $\Gamma', \sim\beta \Vdash \alpha, \Delta')$ **/** $(\Gamma', \Gamma, \sim\alpha \Vdash \Delta, \Delta')$

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Some have claimed that *reductio* (*ad absurdum*) rules are enough so as to characterize classical negation. . .

[See, e.g., Béziau 1994]

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reductio ad absurdum	$\not\Rightarrow$	*pseudo-scotus*	$(\alpha, {\sim}\alpha \Vdash)$

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Double negation introduction

$(\Gamma, \gamma \Vdash \sim\sim\gamma, \Delta)$

Double negation elimination

$(\Gamma, \sim\sim\gamma \Vdash \gamma, \Delta)$

But WHAT is negation, after all?

There are several other usual rules for negation, such as:

Double negation manipulation

$$(\Gamma, \gamma \Vdash \delta, \Delta) \;/\; (\Gamma, {\sim}{\sim}\gamma \Vdash {\sim}{\sim}\delta, \Delta)$$
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(Contextual) Contraposition

$(\Gamma, \gamma \Vdash \delta, \Delta)$ **/** $(\Gamma, {\sim}\delta \Vdash {\sim}\gamma, \Delta)$

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Contextual Replacement (for negation)

$$(\Gamma, \gamma \dashv\Vdash \delta, \Delta) \;/\; (\Gamma, {\sim}\gamma \dashv\Vdash {\sim}\delta, \Delta)$$
$$(\Gamma, {\sim}\gamma \dashv\Vdash \delta, \Delta) \;/\; (\Gamma, \gamma \dashv\Vdash {\sim}\delta, \Delta)$$
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So: What gives an operator the right to be called **negation**?

Is there a set of indisputable rules for negation**??**

A semantic intuition

A ‘binary print’ of negation: [Béziau 1996]

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	$\circledcirc^3_2$
T	T
T	F
F	F

	$\circledcirc^2_2$
T	T
F	T
F	F

	$\circledcirc^1_2$
T	T
F	F

kinds of affirmation

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T	T
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	$\circledcirc_2^2$
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kinds of affirmation

. . . .

kinds of negation

	$\circledcirc^1_1$
T	F
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kinds of affirmation

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kinds of negation

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(N1.m) $(\Gamma, \sim^{m+1}\varphi \not\Vdash \sim^m\varphi, \Delta)$ *m-verificatio*

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m-falsificatio $(\Gamma, \sim^m\varphi \nVdash \sim^{m+1}\varphi, \Delta)$ (N2.m)

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Then:

A **decent negation** should respect (N1.m) and (N2.m), for every m.

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Some particular *consequences:*

A proposal

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m-falsificatio $(\Gamma, \sim^m\varphi \not\Vdash \sim^{m+1}\varphi, \Delta)$ (N2.m)

Then:

A **decent negation** should respect (N1.m) and (N2.m), for every m.

Some particular *consequences:*

- (*m-nonbot*) $(\Gamma, \sim^{m+1}\varphi \not\Vdash \Delta)$

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Let $\sim^0\varphi \stackrel{\text{def}}{=} \varphi$, and $\sim^{n+1}\varphi \stackrel{\text{def}}{=} \sim^n{\sim}\varphi$.

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The operator $\smile\varphi \stackrel{\text{def}}{=} \Diamond\neg\varphi$ defines a decent (paraconsistent) negation.

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But the above negative rules might not be enough...

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These should also be **avoided**!!

Ineffable Inconsistencies, revisited

"Every logic has an **inconsistent counterpart** that coincides with it from the point of view of single-conclusion."

Given any consistent tarskian logic $\mathcal{L}$, one can always find an inconsistent logic $\mathcal{IL}$ such that:

$$\Gamma \vDash^{\mathsf{m}}_{\mathcal{IL}} \beta, \Delta \;\text{ iff }\; \Gamma \vDash^{\mathsf{m}}_{\mathcal{L}} \beta, \Delta$$

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Now, what about $\sim$-inconsistency**??**

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Here are some descriptions of paraconsistency sometimes to be found in the literature:

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Suppose $\mathcal{L}$ has a negation symbol $\sim$.

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Suppose $\mathcal{L}$ has a negation symbol $\sim$.

Say that $\mathcal{L}$ is **$\sim$-inconsistent** if:

$$(\exists \S \in \mathsf{Sem}_{\mathcal{L}})(\exists \alpha \in \mathcal{S}_{\mathcal{L}})$$

$$\S(\alpha) \in \mathcal{D}_{\S} \text{ and } \S({\sim}\alpha) \in \mathcal{D}_{\S}$$

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Then, using the same trick as before:

Given any $\sim$-consistent tarskian logic $\mathcal{L}$, one can always find a $\sim$-inconsistent logic $\mathcal{IL}$ such that:

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So, $\mathcal{IL}$ is $\sim$-inconsistent, but it still respects *ex contradictione*:

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$$\Gamma \Vdash \Delta$$

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