Mass formula and entropy of black holes and quasiblack holes

José P. S. Lemos
IST

1. Introduction

Motivation:

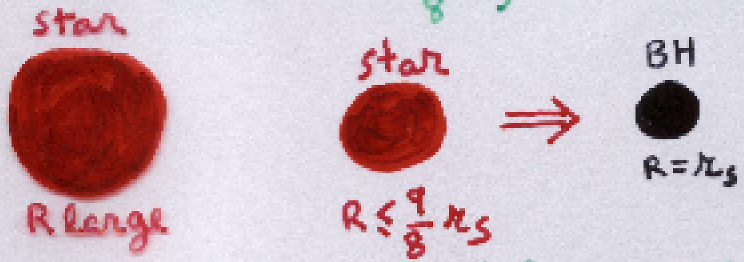
A. Buchdahl limit

- Black hole: region from which there is no escape

Schwarzschild BH $r_s = 2GM$ $(c=1)$

- Buchdahl: perfect fluid, ρ monotonic

when $R \leq \frac{9}{8} r_s \Rightarrow$ collapse of the matter



- Can one bypass this limit? Put charge, repulsion

B. Glendenning limit ("compact stars" p.71)

- electric force on a particle $\lesssim$ gravitational force

$$\frac{(Z_{net}\, e)\, e}{R^2} \lesssim \frac{G \overbrace{M}^{Am} m}{R^2}$$

e - particle's charge
m - particle's mass
Z_{net} - star's net charge
A - star's barion number

$M = Am$ star's mass
R star's radius

so $\frac{Z_{net}}{A} \lesssim \left(\frac{\sqrt{G}\, m}{e}\right)^2$ or in natural units $\frac{Z_{net}}{A} \leq \left(\frac{m}{e}\right)^2$

For protons $\frac{Z_{net}}{A} \lesssim 10^{-36}$ (For $A = 10^{57} \Rightarrow Z_{net} \lesssim 10^{21}$). So no charged stars!

- Way out? Certainly. But not in usual stars.

E.g., ptles with $m = e$.

Implementation:

A. Newtonian gravitation   easy
(Newton - Coulomb)

Two charged massive ptles

$F_g = \frac{Gmm}{r^2}$, $F_e = \frac{e^2}{r^2}$ If $\sqrt{G}\, m = e \Rightarrow F_g = F_e$

system of two ptles in equilibrium

Put another ptle, any number, continuous distribution, any symmetry, any configuration. Result holds.

B. General relativity   history (long)
(Einstein - Maxwell)

• Reissner (1916), Nordström (1918), Jeffery (1921) found

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r^2} + \frac{Q^2}{r^2}} + r^2 d\Omega^2 \quad \text{sph. symm}$$

$Q < M$ BH $Q = M$ ($r_+ = M$) extremal BH $Q > M$ naked singularity

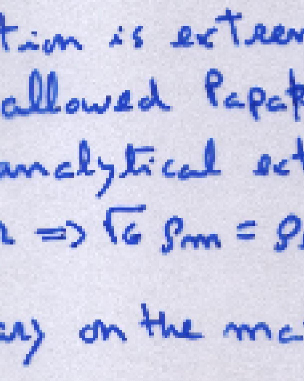
• Weyl (1917) $ds^2 = -V dt^2 + g_{ij} dx^i dx^j$ with axial symm.
seek $V = V(\phi)$ with ϕ the electric potential (in vacuum)

$$\Rightarrow \quad V = a + b\phi + G\phi^2$$

Majumdar (1947) showed it works for any symmetry
Plus: If $V = (a + \sqrt{G}\,\phi)^2$ in vacuum ⇒ Laplace equation
1 ptle solution is extremal RN
Many ptles allowed Papapetrou (1947)
Maximal analytical extension Carter (1966), Hartle Hawking (1972)
In matter ⇒ $\sqrt{G}\, \rho_m = \rho_e$ Majumdar-Papapetrou solutions

Put boundary on the matter: Bonnor stars (1953-1999)

Examples
• Bonnor star
(star of clouds)



Q = M
outside

• star of supersymmetric ptles with e = m.
Outside Q = M
(Tod 1983)

Recent developments:

- When sufficiently compact Bonnor star → quasiblack hole
  QBH neither usual regular spacetime nor BH. It is an object on the verge of becoming extremal BH but distinct

- Properties of Bonnor stars:

1. JPSL, E. Weinberg PRD 2004 "QBHs from extremal charged dust"
2. JPSL, V. Zanchin PRD 2005 "A class of exact solutions of Einstein's equations in $d \geq 4$: Majumdar-Papapetrou solutions"
3. A. Kleber, JPSL, V. Zanchin Grav. Cosm. 2005 "Thick shells and stars in Majumdar-Papapetrou general relativity"
4. JPSL, V. Zanchin JMP 2006 "Gravitational magnetic monopoles and Majumdar-Papapetrou stars"
5. S. Gao, JPSL IJMP 2008 "Collapsing and static thin massive charged shells in $d \geq 4$"
6. JPSL, V. Zanchin PRD 2008 "Bonnor stars in d spacetime dimensions"
7. JPSL, V. Zanchin PRD 2009 "Electrically charged fluids with pressure: new theorems and results for Weyl type systems"

- Properties of QBHs:

1. JPSL, O. Zaslavskii PRD 2007 "QBHs: definition and general properties"
2. JPSL, O. Zaslavskii PRD 2008 "BH mimickers: regular versus singular behavior"
3. JPSL, O. Zaslavskii PRD 2008 "The mass formula for QBHs"
4. JPSL, O. Zaslavskii PRD 2009 "The angular momentum and mass formulas for stationary rotating QBHs"
5. JPSL, O. Zaslavskii arXiv 2009 "Entropy of QBHs"

2. Equations, solutions and properties of QBHs

- $G_{\alpha\beta} = 8\pi\,(T_{\alpha\beta} + E_{\alpha\beta})$ $\qquad \nabla_\alpha F^{\alpha\beta} = j^\beta \qquad (G=1)$

  matter electrom.

$ds^2 = -\dfrac{dt^2}{U^2} + U^2 g_{ik}\,dx^i dx^k \qquad A_\alpha = -\phi\,\delta^0_\alpha \qquad$ Try $U = U(\phi)$ (Weyl 1917)

in vacuum $\Rightarrow U(\phi) = a + b\phi + \phi^2$. If perfect square $U = (a+\phi)^2$ Majumdar-Papapetrou

in vacuum $\to$ extremal RN BHs (many)

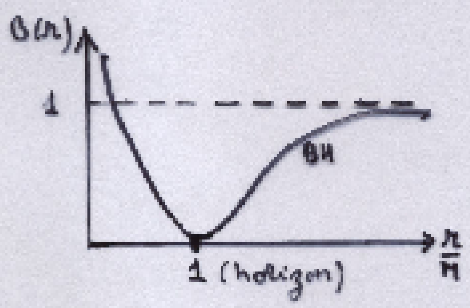
in matter $\to$ $\rho_m = \rho_e$, isotropic coordinates:

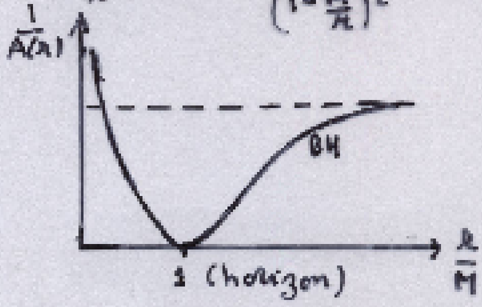
$\nabla^2 U = -4\pi\,\rho_m U^3 \qquad \phi = -\dfrac{1}{U} \qquad \rho_m = \rho_e$

- Put a boundary $\to$ Bonnor star: solve in isotropic coordinates. Interpret in Schw. coordinates $ds^2 = -B\,dt^2 + A\,dr^2 + r^2 d\Omega^2$

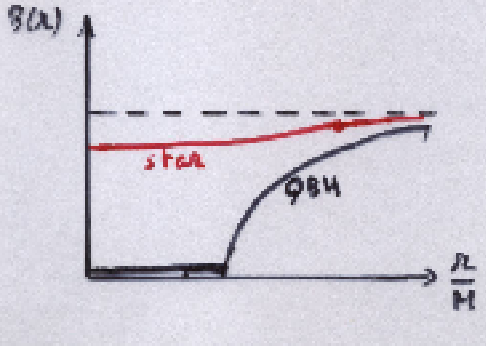
Majumdar Papapetrou matter! extremal RN

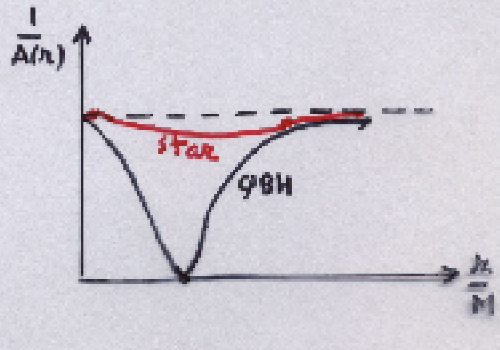
- Vacuum: extremal RN $ds^2 = -\left(1-\dfrac{M}{r}\right)^2 dt^2 + \dfrac{dr^2}{\left(1-\dfrac{M}{r}\right)^2} + r^2 d\Omega^2$





- Bonnor star and quasiblack hole





- Definition of a quasiblack hole (Lemos, Zaslavskii PRD 2007)

(a) The function $1/A(r)$ attains a minimum at some $r^* \neq 0$ such tha $1/A(r^*) = \varepsilon$

(b) For each ε the configuration is regular with a nonvanishing $B(r)$

(c) When $\varepsilon \to 0$, $B(r) \to 0$ for $r \leq r^*$

- Properties
  - There is an infinite redshift 3-region
  - Kretschmann scalar finite everywhere
  - Infinite (impenetrable) barrier from inside to outside
  - shows naked behavior (Horowitz Ross PRD 1998)
    tidal forces on a freely falling frame $\to \infty$
  - other properties: e.g. Lemos-Weinberg solution (PRD 2004)

interior	quasi horizon	exterior
Bertotti-Robinson	extremal BR	extremal RN

- Theorem: If $T_{\mu\nu}$ finite at the boundary ($\Theta_{\mu\nu} \neq \infty$) then QBH is extremal

- Allowing surface stresses $\Theta_{\mu\nu} \to \infty$ yields non-extremal QBHs

- Other QBHs: Yang-Mills-Higgs matter

  $\to$ gravitating magnetic monopoles (Lue and Weinberg PRD 2000)

3. Mass formula for QBHs

(i) static systems

- $ds^2 = -N^2 dt^2 + dl^2 + g_{ab}\, dx^a dx^b$ Gaussian coordinates fine near quasihorizon

- General definition of QBH: consider configuration depending on ε such that (a) for $\varepsilon \neq 0$ metric is regular, $N \neq 0$ (b) $\varepsilon \to 0 \Rightarrow N \leq N_B \to 0$ in the inner region (c) Kretschmann remains finite everywhere

- $Kr = P_{ijkl} P^{ijkl} + 4 C_{ij} C^{ij}$ $\quad P_{ijkl}$ curvature for $t = \text{const}$ $\quad C_{ij} = \dfrac{N_{;ij}}{N}$

  All terms are positive

  so each term should be finite separately

Choose $l = 0$ on surface $\quad$ QBH means $N = N_0(x^a) \to 0$

Now;

$$\lim_{l \to 0} C_{ll} = \lim_{l \to 0} \frac{N''}{N_0} \qquad \lim_{l \to 0} C_{al} = \lim_{l \to 0} \frac{N'_{;a}}{N_0} \qquad ' \equiv \frac{\partial}{\partial l}$$

So finiteness $\Rightarrow \lim_{\varepsilon \to 0} \lim_{l \to 0} N'' = 0 \qquad \lim_{\varepsilon \to 0} \lim_{l \to 0} N'_{;a} = 0$

For small l write

$$N = N_0 + k_1(x^a, \varepsilon)\, l + k_2(x^a, \varepsilon) \frac{l^2}{2} + k_3(x^a, \varepsilon) \frac{l^3}{3!} + O(l^4)$$

Find from finiteness

$$\lim_{\varepsilon \to 0} k_1(x^a, \varepsilon) = \underset{\substack{\text{surface}\\ \text{gravity}\\ \text{of}\\ \text{QBH}}}{\kappa} \qquad \lim_{\varepsilon \to 0} k_2(x^a, \varepsilon) = 0$$

• Mass $M = \int (-T_0{}^0 + T_k{}^k) N \sqrt{g_3}\, d^3x$ (Tolman 1934)

inner mass: $N \leq N_B \to 0 \Rightarrow M_{in} = 0$

surface mass: define $\Theta_\mu{}^\nu = \int T_\mu{}^\nu\, dl$

$$M_{surf} = \int_{surf} \underbrace{(-\Theta_0{}^0 + \Theta_a{}^a)}_{-\frac{1}{4\pi}[K_0{}^0]} N\, d\sigma = \frac{1}{4\pi}\int \left[\left(\frac{\partial N}{\partial l}\right)_+ - \left(\frac{\partial N}{\partial l}\right)_-\right] d\sigma$$

But $\left(\frac{\partial N}{\partial l}\right)_+ = \kappa$ $\left(\frac{\partial N}{\partial l}\right)_- = 0$. so

$$M_{surf} = \frac{\kappa A_h}{4\pi}$$ A_h quasi horizon area

outer mass: $M_{out} = \phi_h Q + M_{out}^{matter}$

• Mass formula $M = \frac{\kappa A_h}{4\pi} + \phi_h Q + M_{out}^{matter}$ if $M_{out}^{matter} = 0$

$$M_h = \frac{\kappa A_h}{4\pi} + \phi_h Q$$ Smarr's formula (PRL 1973)

(Lemos, Zaslavskii PRD 2008) now for QBH

Note: $\left(\begin{matrix}\text{non-extremal QBHs} \\ \Theta_\mu{}^\nu \to \infty \quad M_{surf} \neq 0\end{matrix}\right)$ $\left(\begin{matrix}\text{extremal QBHs} \\ \Theta_\mu{}^\nu \text{ finite} \quad M_{surf} = 0\end{matrix}\right)$

• QBH : mass due to boundary
  BH : only exterior relevant to mass (Bardeen, Carter, Hawking 1973)

(ii) stationary QBHs

- Found by Bardeen and Wagoner (ApJ 1971)
  Rotating disks. Fastest yields extremal Kerr outside
- Meinel (CQG 2006) reconsidered the problem: to be Kerr has to be extremal
- Lemos, Zaslavskii (PRD 2009) full problem

$$M = \frac{\kappa A_h}{4\pi} + 2\omega_h J_h + \phi_h Q + M_{out}^{matter}$$ $J = J_h + J_{out}^{matter}$

4. Entropy of QBHs

Introduction

- For a BH $S = \frac{1}{4}A$ transparent in an Euclidean action approach
  Presence of horizon crucial

- Imagine collapsing body. Surface close to r_+
  no reason for $S = \frac{A}{4}$, it appears as a jump.

- We can restore continuity and trace origin of entropy if instead of a BH consider a QBH.
  Study matter distribution with a boundary at some T in the threshold of QBH

- Use 1st law of thermodynamics
  For spherical shells Davies, Ford, Page (PRD 1986) Martinez (PRD 1996)
  with specific equations of state

  Here model-independent
  Exploit the fact that boundary almost coincides with quasihorizon
  Surface stresses diverge. Price paid! Looks unphysical
  The whole QBH picture turns out to be useful →
  Enables tracing features of BHs.

- Note: QBHs appeared first in gravitating magnetic monopoles. Used to argue that tracing out inner degrees of freedom, outer region describes inner one as a statistical density matrix $S \propto A$ from entanglement.
  (Lue, Weinberg 2000)
  We use a thermodynamic approach and find precisely $S = \frac{A}{4}$.

The entropy formula

- static metric, compact body, not necessarily sph. symm.

$$ds^2 = -N^2 dt^2 + dl^2 + g_{ab}\,dx^a dx^b \qquad \text{Gaussian coord.}$$

$l = $ const at boundary $\quad N, g_{ab}$ different for interior, exterior

- system at local temperature (Tolman)

$$T = \frac{T_0}{N} \qquad T_0 \text{ constante, temp. at } \infty$$

- First law for the boundary

$$T\,d(s\sqrt{g}) = d(\varepsilon\sqrt{g}) + \frac{\Theta^{ab}}{2}\sqrt{g}\,dg_{ab}$$

$g \equiv \det g_{ab}$
s - entropy density
ε - quasilocal energy density
Θ_{ab} - surface stresses

- $\varepsilon = \dfrac{k - k^0}{8\pi}$

$$\Theta_{ab} = \frac{k_{ab} + \left(\frac{N'}{N} - k\right) g_{ab}}{8\pi} - \Theta^0_{ab}$$

k_{ab} 2D extrinsic curvature of boundary embedded in the 3D $t=$ const hypersurface
$k = k_a^a$ $\quad$ ^{0} background reference spacetime

(quasilocal energy: time rate of change of the classical action at boundary. Includes grav. and matter fields Brown, York) (PRD 1993)

Here $k_{ab} = -\frac{1}{2} g_{ab}' \qquad k = -\frac{1}{\sqrt{g}}\sqrt{g}\,'$

- strategy: integrate 1st law to obtain S
  - usual: need eq. of state
  - here:
    - (i) choose sequence of configurations s.t. all members remain on threshold of horizon formation
    - (ii) Integrate over this very subset
    - (iii) Answer model independent (no need of EOS)

Must change simultaneously the radius R and proper mass M to keep it near r_+, in a way that $N \to 0$ for all configurations.

- There are two relevant quantities: R and r_+

r_+ fixed $R \to r_+$

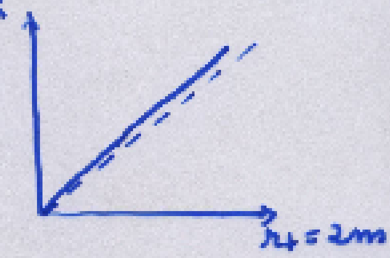
small parameter

$$\varepsilon = \frac{R - r_+}{R}$$

other small parameter $\delta \equiv \frac{dr_+}{r_+}$ since want to consider small variations of thermod. quantities for two close systems that differ in r_+.

So send $\varepsilon \to 0$ first, then $\delta \ll 1$.
Ensures dealing with QBHs having slightly different radii

- For systems kept near quasihorizon allows integration of 1st law along such a sequence



e.g.: spherical shell of radius R

$$\underbrace{m}_{\text{ADM}} = \underbrace{M}_{\text{proper}} - \frac{M^2}{2R} \qquad r_+ = 2m$$

Integrating along a curve where R, m and M change, but $R \simeq 2m$.

- Imposing regularity conditions for exterior (Visser et al. 96 2004)

$$K_{ab} = K^{(1)}_{ab}\, l + O(l^2)$$

$\varepsilon \sim K \sim K^{(1)\,a}_{\;\;a}\, l$ is finite, but $\Theta_{ab} \sim \frac{k}{N}$ diverges.

Main contribution to 1st law $T\, d(s\sqrt{g}) = d(\sqrt{g}\,\varepsilon) + \frac{\Theta^{ab}}{2}\sqrt{g}\,\delta g_{ab}$

is $8\pi\,\Theta_{ab} = \frac{k}{N} g_{ab} \Rightarrow \frac{T_0}{N} d(s\sqrt{g}) = \frac{k/N}{16\pi}\sqrt{g}\, g^{ab}\, \delta g_{ab}$

i.e. $d(s\sqrt{g}) = \frac{k}{16\pi T_0}\sqrt{g}\, g^{ab}\, dg_{ab}$

- Up to now T_0 is arbitrary. Invoke quantum fields and divergence unless $T_0 = T_H = \frac{k}{2\pi}$

Then $d(s\sqrt{g}) = \frac{1}{4} d\sqrt{g}$

Integration over area $\Rightarrow$ $S = \frac{1}{4} A$ up to a constant.

(Lemos, Zaslavskii arXiv 2009)

- Universal: whatever distribution of entropy the body has in appropriate limit all distributions give $S = \frac{A}{4}$

  Manifestation of universality inherent to BH physics in general and BH entropy in particular

  Derivation based on quasilocal approach of Brown, York (PRD 1993)

- Application: spherical distribution of matter
  - Thin shell (Martinez PRD 1996) showed $T_0 = T_0(m)$, $S = S(m)$
    1st law reduces to $dS = \frac{dm}{T_0(m)}$

    Body at quasihorizon $T_0 = T_H = \frac{1}{8\pi m} \Rightarrow S = 4\pi m^2 = \frac{A}{4}$
  - Non-trivial example: distribution of matter up to R

    choose $N^2(r) = V(r)\, e^{2\psi(r)}$ $\quad \frac{dl}{dr} = \frac{1}{\sqrt{V(r)}}$ $\quad$ with $V(r) = 1 - \frac{2m(r)}{r}$

    Einstein's eqs: $m(r) = 4\pi \int_0^r d\bar{r}\, \bar{r}^2 \rho$ $\quad \psi(r) = 4\pi \int_R^r d\bar{r}\, \frac{(p_r + \rho)\bar{r}}{V(\bar{r})}$

    ρ: matter en. density $\quad p_r$: radial pressure

    for $r > R$ $\quad m(r) = m$ $\quad \psi(r) = 0$ Schwarzschild
  - The (integrated) 1st law is

$$T\,dS = dE + 8\pi\, \Theta_\theta^\theta R\, dR$$

can put as $\left(\psi(R) = 0,\ T = \frac{T_0}{\sqrt{V}},\ E(R) = R(1 - \sqrt{V}),\ 8\pi\Theta_\theta^\theta = \frac{1}{R}(V(R) - 1) + \left(\frac{1}{2}\frac{1}{\sqrt{V}}\frac{dV}{dr} + \sqrt{V}\frac{d\psi}{dr}\right)_{r=R}\right)$

$$T_0\, dS = dm + 4\pi\, p_r R^2\, dR$$

To integrate have to know $S(m, R)$

Consider, however, the region $R \simeq 2m = r_+$

Put $T_0 = T_H = \frac{e^{\psi(r_+)}}{4\pi} \frac{dV(r_+)}{dr}$, $\ r \to R \to r_+$, $\ \psi(R) \to 0$, $\quad T_H = \frac{1}{4\pi r_+}(1 - 8\pi \rho(r_+) r_+^2)$

regularity at horizon (Visser PRD 1992) $\quad p_r(r_+) = -\rho(r_+)$

obtain $\quad dS = 2\pi\, r_+\, dr_+$

so $\quad S = \frac{1}{4} A \quad$ non-trivially

- For extremal case: $N \sim \exp(\alpha l)$ $\quad l \to -\infty$

$\frac{\partial N}{\partial l} \sim N$ so $\frac{1}{N}\frac{\partial N}{\partial l}$ is finite and $d(s\sqrt{g}) = 0$

For any $T_0 \Rightarrow S = 0$ in accord with Hawking (PRD 1995) Teitelboim (PRD 1995) Hod (2000)

5. Conclusions

- For a system in which a BH never forms, but a QBH forms S comes from quasihorizon surface, stems from the contribution of states living in a thin layer

(i) Matter properties from inside are irrelevant, final answer for S is insensitive to them. QBH deletes information as in BH physics.

(ii) Along with the results for mass formula for QBHs, for exterior observers QBHs yield smooth transition to BHs

(iii) Role of huge surface stresses important
non-extremal: stresses infinite $\Rightarrow S = \frac{A}{4}$
extremal: stresses finite $\Rightarrow S = 0$
new insights into classical arguments

(iv) Key role of the shell layer supports the view that entropic (quantum) states live on the quasihorizon

(v) Attempt to place degrees of freedom of a BH on horizon's vicinity not new (Zurek, Thorne PRL 1985, Carlip PRD 1997, review Lemos 2005)
Here, hints: degrees in the matter before horizon then on geometry and matter. A tie.
Non-trivial interplay between gravitational and matter degrees of freedom.