COMBINATÓRIA E TEORIA DE CÓDIGOS

Homework 1 (deadline 1/3/2013, in class)

1. How many integer solutions to $x_1 + x_2 + x_3 + x_4 = 21$ are there if:
   (a) $x_i \geq 0$, $i = 1, 2, 3, 4$;
   (b) $0 \leq x_i \leq 8$, $i = 1, 2, 3, 4$;
   (c) $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 6$, $3 \leq x_3 \leq 8$, $4 \leq x_4 \leq 9$.

2. Determine the number of monic polynomials of degree n in $\mathbb{F}_q[t]$ without roots in $\mathbb{F}_q$, where $\mathbb{F}_q$ is a field with q elements.

3. Using generating functions, solve the following recurrence relation: $a_n = 2a_{n-2}$, for $n \geq 2$, and $a_0 = 1$, $a_1 = 2$.

4. An *order* k *homogeneous recurrence relation with constant coeficients* is of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \quad (n \geq k) ,$$

   where $c_0, c_1, \ldots, c_k \in \mathbb{R}$ are constants, and $c_0 \neq 0$. The *characteristic polynomial* of the recurrence relation is defined by

$$p(x) = c_0 x^k + c_1 x^{k-1} + \cdots + c_{k-1} x + c_k \ \in \mathbb{R}[x],$$

   and its roots are called *characteristic roots*.

   (a) Show that the general solution of a first order recurrence relation is $a_n = a_0 r^n$, $n \geq 0$, where $r = -\frac{c_1}{c_0}$, i.e., r is the root of the associated characteristic polynomial.
   (b) Study the homogeneous quadratic (of second order) case by proving the following statements:
      (i) If the characteristic roots r_1 and r_2 are real and distinct, then the general solution is

$$a_n = A(r_1)^n + B(r_2)^n ,$$

      where $A, B \in \mathbb{R}$ are constants, i.e., $(r_1)^n$ e $(r_2)^n$ are two linearly independent solutions.
      (ii) If there is only one characteristic root $r \in \mathbb{R}$ (of multiplicity 2), then the general solution is

$$a_n = Ar^n + Bnr^n ,$$

      where $A, B \in \mathbb{R}$ are constants.
      (iii) If there are two complex roots $r_1, r_2 \in \mathbb{C}$, then r_1 and r_2 are complex conjugates and the general solution is

$$a_n = A(r_1)^n + B(r_2)^n ,$$

      where $A, B \in \mathbb{C}$ are constants (as in the real case). Show also that, if $a_0, a_1 \in \mathbb{R}$, then A and B are complex conjugates and $a_n \in \mathbb{R}$, for all $n \geq 0$.
      [Sugestion: recall that any $z \in \mathbb{C} \setminus \{0\}$ can be written as $z = \rho(\cos(\theta) + i\,\mathrm{sen}(\theta))$ and $(\cos(\theta) + i\,\mathrm{sen}(\theta))^n = \cos(n\theta) + i\,\mathrm{sen}(n\theta)$.]

(c) Generalize part (b) for relations of order k:

(i) Show that, if $r \in \mathbb{R}$ is a characteristic root with multiplicity m, then it contributes with

$$a_n^{(r)} = A_0 r^n + A_1 n r^n + A_2 n^2 r^n + \cdots + A_{m-1} n^{m-1} r^n \ ,$$

for the general solution, where $A_0, A_1, \ldots, A_{m-1} \in \mathbb{R}$ are constants.

(ii) If $r \in \mathbb{C}$ is a complex characteristic root with multiplicity m, what is the contribution of r and of its conjugate $\bar{r}$ to the general solution?

5. Using the previous exercise, solve the following recurrence relations:

(a) $a_n = 2a_{n-1} + 3a_{n-2}$, $n \geq 2$, and $a_0 = 3$, $a_1 = 5$;

(b) $4a_n - 4a_{n-1} + a_{n-2} = 0$, $n \geq 2$, and $a_0 = 5$, $a_1 = 4$;

(c) $a_n - 2a_{n-1} + 2a_{n-2} = 0$, $n \geq 2$, and $a_0 = a_1 = 4$;

(d) $a_n = a_{n-1} + 5a_{n-2} + 3a_{n-3}$, $n \geq 3$, and $a_0 = a_1 = 3$, $a_2 = 7$.

6. The following binary word

$$01111000000?001110000?00110011001010111000000000?01110$$

encodes a date. The encoding method used consisted in writing the date in 6 decimal digits (e.g. 290296 means February 29th, 1996), then converting it to a number in base 2 (e.g. 290296 becomes 1000110110111111000), and enconding the binary number using the rule

$$\begin{aligned} \{0,1\}^2 &\longrightarrow \mathcal{C} \subseteq \{0,1\}^6 \\ 00 &\longmapsto 000000 \\ 01 &\longmapsto 001110 \\ 10 &\longmapsto 111000 \\ 11 &\longmapsto 110011 \end{aligned}$$

The received word contains 3 unknown digits (which were deleted) and it may also contain some switched digits.

(a) Find the deleted bits;

(b) How many, and in which positions, are the wrong bits?

(c) Which date is it?

(d) Repeat the problem switching the bits in positions 15 and 16.

7. What is the capacity of a code for correcting erasure errors, and for correcting erasure and symbol errors simultaneously? Prove your statements carefully and ilustrate with examples.