Análise Matemática IV
1º semestre de 2001/2002

Exercício-teste 2

Determine as partes real e imaginária da função

$$f(z) = e^{e^z}$$

e verifique que estas satisfazem as equações de Cauchy-Riemann.

Resolução

Seja $z = x + iy$. Então tem-se que $e^z = e^x \cdot (\cos y + i \operatorname{sen} y)$ e

$$\begin{aligned} e^{e^z} &= e^{e^x \cdot (\cos y + i \operatorname{sen} y)} \\ &= e^{e^x \cos y} \cdot e^{i(e^x \operatorname{sen} y)} \\ &= e^{e^x \cos y} \cdot \big(\cos(e^x \operatorname{sen} y) + i \operatorname{sen}(e^x \operatorname{sen} y)\big). \end{aligned}$$

Logo, as partes real e imaginária de f são

$$\begin{aligned} u(x,y) &= e^{e^x \cos y} \cdot \cos(e^x \operatorname{sen} y) \\ v(x,y) &= e^{e^x \cos y} \cdot \operatorname{sen}(e^x \operatorname{sen} y) \end{aligned}$$

respectivamente.
As equações de Cauchy-Riemann são:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Sejam $\varphi(x,y) = e^x \cos y$ e $\psi(x,y) = e^x \operatorname{sen} y$. Então,

$$u(x,y) = e^{\varphi} \cos \psi, \quad v(x,y) = e^{\varphi} \operatorname{sen} \psi \quad \text{e} \quad e^{x+iy} = \varphi(x,y) + i\psi(x,y).$$

Logo,

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad\qquad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

e então,

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^{\varphi} \frac{\partial \varphi}{\partial x} \cos \psi - e^{\varphi} \frac{\partial \psi}{\partial x} \operatorname{sen} \psi = e^{\varphi} \frac{\partial \psi}{\partial y} \cos \psi + e^{\varphi} \frac{\partial \varphi}{\partial y} \operatorname{sen} \psi, \\ \frac{\partial u}{\partial y} &= e^{\varphi} \frac{\partial \varphi}{\partial y} \cos \psi - e^{\varphi} \frac{\partial \psi}{\partial y} \operatorname{sen} \psi = -e^{\varphi} \frac{\partial \psi}{\partial x} \cos \psi - e^{\varphi} \frac{\partial \varphi}{\partial x} \operatorname{sen} \psi, \\ \frac{\partial v}{\partial x} &= e^{\varphi} \frac{\partial \varphi}{\partial x} \operatorname{sen} \psi + e^{\varphi} \frac{\partial \psi}{\partial x} \cos \psi \qquad \frac{\partial v}{\partial y} = e^{\varphi} \frac{\partial \varphi}{\partial y} \operatorname{sen} \psi + e^{\varphi} \frac{\partial \psi}{\partial y} \cos \psi, \end{aligned}$$

verificando-se as equações de Cauchy-Riemann para as funções dadas.