THE FUNDAMENTAL GROUP OF S^1-MANIFOLDS

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Abstract. We address the problem of computing the fundamental group of a symplectic S^1-manifold for non-Hamiltonian actions on compact manifolds, and for Hamiltonian actions on non-compact manifolds with a proper moment map. We generalize known results for compact manifolds equipped with a Hamiltonian S^1-action. Several examples are presented to illustrate our main results.

1. Introduction

In this paper we address the problem of computing the fundamental group of a symplectic S^1-manifold. For a compact manifold equipped with a Hamiltonian circle action, a result in [Li] states that this group is equal to the fundamental group of any of its reduced spaces (as topological spaces) and to the fundamental group of its minimum and maximum level sets. We will consider here non-Hamiltonian actions on compact manifolds (Theorem 2.1) and Hamiltonian actions on non-compact manifolds with a proper moment map (Theorem 3.1).

When the action is **non-Hamiltonian**, one can consider a *generalized moment map*[1] introduced by McDuff in [MD1] as follows: first, the symplectic form is deformed to a rational invariant symplectic form making the non-zero class $[\iota(\xi_M)\omega]$ rational, where ξ_M denotes the vector field generating the action; then, for a multiple of this symplectic form, there is a map $\phi : M \longrightarrow S^1$ such that

$$\iota(\xi_M)\omega = \phi^*(d\theta),$$

called *generalized moment map* (or *circle-valued moment map*). This map has many of the properties of an ordinary moment map and can even be used to reduce M. In particular, choosing an invariant pair of a Riemannian metric g and a compatible almost complex structure J on M and identifying S^1 with $\mathbb{R}/\mathbb{Z}$ in the usual way, one may define the gradient of ϕ with respect to g and see that it is equal to $J\xi_M$. Its flow has all the nice properties of the gradient flow of an ordinary moment map. In particular, its critical set is a disjoint union of symplectic submanifolds of M (each of codimension at least 4 since ϕ has no local maxima or minima).

Using the gradient flow of ϕ we prove (Theorem 2.1) that, if M is a connected compact symplectic manifold equipped with a non-Hamiltonian circle action and P is a connected component of an arbitrary level set of the generalized moment map ϕ then, as fundamental groups of topological spaces, $\pi_1(M)$ is a semidirect product

$$\pi_1(M) = \pi_1(P) \rtimes \mathbb{Z},$$

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[1] This generalized moment map is a special case of a Lie group-valued moment map (see [OR] and the references therein).

when the action has no critical points, or

$$\pi_1(M) = \pi_1(M_{\text{red}}) \rtimes \mathbb{Z},$$

where $M_{\text{red}} := P/S^1$ is a connected component of the symplectic quotient

$$\phi^{-1}(a_0)/S^1,$$

where $a_0 = \phi(P)$.

Note that the proof for the Hamiltonian case presented in [Li] relies heavily, at each step, on the existence of a minimum and so it **cannot** be adapted to the non-Hamiltonian case. Nevertheless, since we have a generalized moment map, we can still use (circle-valued) Morse-Bott theory to prove the above result.

When the action is Hamiltonian but M is **not compact** one can again use Morse-Bott theory, provided that the moment map is proper (i.e. the inverse image of a compact set is compact). In this case, we obtain that, if (M, ω) is a connected symplectic manifold (not necessarily compact) with proper moment map $\phi : M \longrightarrow \mathbb{R}$, and P is an arbitrary (compact) level set of ϕ then, as fundamental groups of topological spaces, $\pi_1(M)$ is either, $\pi_1(M) = \pi_1(P)$, when the action has no critical points, or $\pi_1(M) = \pi_1(M_{\text{red}})$, where M_{red} is the symplectic quotient at any value of ϕ. Moreover, if ϕ has a minimum (or a maximum), we recover the referred result for the compact case in [Li]:

$$\pi_1(M) = \pi_1(M_{\text{red}}) = \pi_1(F_{\min}) \text{ (or equal to } \pi_1(F_{\max})),$$

where $F_{\min}$ and $F_{\max}$ are the sets of minimal and maximal points respectively.

Although properness of the moment map is a strong condition which is not verified in many problems in classical mechanics with a global S^1-action, our results may still be relevant when, for instance, one can perform a preliminary reduction or symplectic cutting (cf. [L]) making the induced S^1-moment map proper. Let us remark, however, that the requirement of a proper moment map is essential to our results as can be seen in Examples 6 and 7. Indeed, even the statement in Proposition 2.1 that all reduced spaces have the same fundamental group may fail to hold when the moment map is not proper.

Finally, in Section 4, we present several other examples illustrating our results.

2. Non-Hamiltonian circle actions

In this section we prove our result for non-Hamiltonian actions on compact manifolds:

Theorem 2.1. *Let M be a connected, compact symplectic manifold equipped with a non-Hamiltonian circle action and $\phi : M \longrightarrow S^1$ be the corresponding generalized moment map. Let P be a connected component of an arbitrary level set of ϕ. Then, as fundamental groups of topological spaces, $\pi_1(M)$ is a semidirect product*

$$\pi_1(M) = \pi_1(P) \rtimes \mathbb{Z},$$

when the action has no critical points, or

$$\pi_1(M) = \pi_1(M_{\text{red}}) \rtimes \mathbb{Z},$$

where $M_{\text{red}} := P/S^1$ is a connected component of the (arbitrary) symplectic quotient $\phi^{-1}(a_0)/S^1$, for $a_0 = \phi(P)$.

Throughout, we shall choose an S^1-invariant compatible pair, (J, g), of an almost complex structure and a Riemannian metric and we identify S^1 with $\mathbb{R}/\mathbb{Z}$ in the usual way to define the gradient of ϕ with respect to g. This gradient is equal to $J\xi_M$, where ξ_M is the vector field generating the action.

In order to prove Theorem 2.1 we will need a series of preliminary results. The first one is proved in [O] but we include a sketch of its proof for the sake of completion.

Lemma 1. *(Ono* [O]*) Let M be a symplectic compact connected manifold equipped with a non-Hamiltonian circle action and $\phi : M \longrightarrow S^1$ the corresponding generalized moment map. Then, given any point $y_0 \in M$, there exists a homologically non-trivial loop $\gamma : S^1 \longrightarrow M$ passing through y_0.*

Proof. Since the generalized moment map is locally a function, one can define its Hessian at critical points, their indices and the gradient flow of ϕ. Moreover, since the action is non-Hamiltonian, the critical points cannot have index 0 nor $2n$, where $2n$ is the dimension of M. Let us consider the quotient space

$$X = M/\sim,$$

where $x \sim y$ iff x and y are in the same connected component of a level set of ϕ. As the indices of the critical points of ϕ are even, X has no branch point. Moreover, X has no boundary and is homeomorphic to a circle (cf. [O] for details). Therefore, one can deform the trajectory of the gradient flow of ϕ passing through y_0 to a homologically non-trivial loop $\gamma : S^1 \longrightarrow M$ through y_0. $\square$

Lemma 2. *Let M be a compact symplectic manifold equipped with a non-Hamiltonian circle action and $\phi : M \longrightarrow S^1$ be its generalized moment map. Then, for any regular value a_0 and a point $y_0 \in \phi^{-1}(a_0)$, the inclusion $j : \phi^{-1}(a_0) \longrightarrow M$ induces the following exact sequence of fundamental groups*

$$\pi_1(\phi^{-1}(a_0), y_0) \xrightarrow{j_*} \pi_1(M, y_0) \xrightarrow{\phi_*} \pi_1(S^1, a_0).$$

Proof. Clearly $\mathrm{im}(j_*) \subset \ker(\phi_*)$, so we just need to show that $\ker(\phi_*) \subset \mathrm{im}(j_*)$. Let $[\gamma] \in \ker(\phi_*)$. Then, identifying S^1 with $\mathbb{R}/\mathbb{Z}$, we may assume without loss of generality that there are regular values of ϕ, say a and b, with $0 \leq a \leq a_0 \leq b < 1$, for which γ is homotopic to a loop contained in

$$M^{[a,b]} := \{x \in M : a \leq \phi(x) \leq b\}.$$

Let $M^{[a,b]}_{y_0}$ be the connected component of $M^{[a,b]}$ containing y_0.

If there are no critical points in $M^{[a,b]}_{y_0}$ then

$$\pi_1\left(M^{[a,b]}, y_0\right) = \pi_1(\phi^{-1}(a_0), y_0)$$

(cf. [Mi]) and so $[\gamma] \in \mathrm{im}(j_*)$.

If there is just one critical value c in (a, b) let us assume for simplicity that there is just one component F of the critical set inside $M^{[a,b]}_{y_0}$ where ϕ assumes the value c (if there is more than one component we argue similarly for each one[2]). The normal bundle of F has a complex structure induced by the almost complex structure J and splits as a sum $\nu^- \oplus \nu^+$, where ν^- is tangent to the incoming flow lines of $J\xi_M$

[2] Alternatively, by a slight perturbation of the restriction of ϕ to $M^{[a,b]}$, we could assume that no two critical components assume the same critical value and proceed as in the case where there is more than one critical value in (a, b) which is described at the end of the proof.

(that is, tangent to the stable manifold). Let D_F^- be the negative disk bundle of ν^- and $S(D_F^-)$ its sphere bundle. By Morse-Bott theory (see [B]) we have

$$M_{y_0}^{[a,b]} = M_{y_0}^{[a,\tilde{a}]} \cup_{S(D_F^-)} D_F^-,$$

for any regular value $\tilde{a}$ in (a,c). Hence, by the Van-Kampen theorem, $\pi_1\left(M_{y_0}^{[a,b]}\right)$ is the free product with amalgamation[3]

$$\begin{aligned} \pi_1\left(M_{y_0}^{[a,b]}\right) &= \pi_1\left(M_{y_0}^{[a,\tilde{a}]}\right) *_{\pi_1(S(D_F^-))} \pi_1(D_F^-) \\ &= \pi_1(\phi^{-1}(\tilde{a}),\tilde{y}) *_{\pi_1(S(D_F^-))} \pi_1(F), \end{aligned} \tag{2.1}$$

where $\tilde{y}$ is a point in the appropriate component of $\phi^{-1}(\tilde{a})$.

If index$(F) > 2$, then $\pi_1(S(D_F^-))$ is isomorphic to $\pi_1(F)$ and so, since we also have $\pi_1(D_F^-) = \pi_1(F)$, we get

$$\pi_1\left(M_{y_0}^{[a,b]}\right) = \pi_1(\phi^{-1}(\tilde{a}),\tilde{y}).$$

If index$(F) = 2$, we consider the principal circle bundle

$$S^1 \overset{\tilde{i}}{\hookrightarrow} S(D_F^-) \xrightarrow{\tilde{p}_S} F$$

and its homotopy exact sequence

$$\cdots \longrightarrow \pi_1(S^1) \xrightarrow{\tilde{i}_*} \pi_1(S(D_F^-)) \xrightarrow{(\tilde{p}_S)_*} \pi_1(F) \longrightarrow \{1\}.$$

Note that $S(D_F^-)$ can be identified with the restriction of the circle bundle

$$\phi^{-1}(\tilde{a}) \longrightarrow M_{\tilde{a}}$$

to F, where $M_{\tilde{a}}$ is the reduced space $\phi^{-1}(\tilde{a})/S^1$ (there is an embedding of F in $M_{\tilde{a}}$ as it is shown in [Li]), and so we also have an inclusion

$$S(D_F^-) \overset{\kappa}{\hookrightarrow} \phi^{-1}(\tilde{a}).$$

In the amalgamation (2.1), the elements of

$$(\tilde{p}_S)_*(\pi_1(S(D_F^-))) = \pi_1(F)$$

(the map $(\tilde{p}_S)_*$ is surjective) are identified with the corresponding elements in

$$\kappa_*(\pi_1(S(D_F^-))) \subset \pi_1(\phi^{-1}(\tilde{a}),\tilde{y}),$$

implying that $\pi_1\left(M_{y_0}^{[a,b]}\right)$ can be identified with the quotient

$$\pi_1(\phi^{-1}(\tilde{a}),\tilde{y})/N,$$

where N is the normal subgroup generated by all the elements of $\kappa_*(\ker((\tilde{p}_S)_*))$.

Repeating this argument using $-\phi$ instead of ϕ (corresponding to reversing the direction of the circle action), we can substitute $\tilde{a}$ by any value $\hat{a} \in (c,b]$ in the above argument. However, the relevant critical points will no longer be the index-2 critical points but the ones with index equal to $2n-2$, where $2n$ is the dimension of M.

[3] The term amalgamation in $G_1 *_A G_2$ is usually used for the quotient group of the free product of G_1 by G_2 obtained by identifying the two subgroups that correspond to A under two monomorphisms $A \longrightarrow G_i$ (see for example [CGKZ]). Here we slightly abuse this notation since we do not require these maps to be one-to-one.

If $[a,b]$ has more than one critical value or if there is more than one component of the critical set assuming the value c, let n_1 be the number of index-2 components of the critical set inside $M_{y_0}^{[a,b]}$ assuming values in $(a_0,b]$. Similarly, let n_2 be the number of index-$(2n-2)$ components of the critical set inside $M_{y_0}^{[a,b]}$ assuming values in $[a,a_0)$. By induction on n_1 and n_2 and by using the Van-Kampen Theorem as in (2.1), possibly more than once each time we cross one of the corresponding critical levels (using ϕ or $-\phi$ accordingly), we see that $\pi_1\left(M_{y_0}^{[a,b]}, y_0\right)$ can be obtained from $\pi_1(\phi^{-1}(a_0),y_0)$ by taking a sequence of n_1+n_2 quotients as explained above, and the result follows.

Indeed, if $a<c_1<\cdots<c_m<b$ are the critical values of ϕ in (a,b), we first prove by induction that, for $a_0\in(a,c_1)$, the fundamental group of $M_{y_0}^{[a,b]}$ can be obtained from $\pi_1(\phi^{-1}(a_0),y_0)$ by a sequence of n_1 quotients as above where n_1 is the number of index-2 critical components of ϕ inside $M_{y_0}^{[a,b]}$. Similarly, if $a_0\in(c_m,b)$ we prove by induction that the fundamental group of $M_{y_0}^{[a,b]}$ can be obtained from $\pi_1(\phi^{-1}(a_0),y_0)$ by a sequence of n_2 quotients as above where n_2 is now the number of index-$(2n-2)$ critical components of ϕ inside $M_{y_0}^{[a,b]}$. Finally, if we consider any other regular value $a_0\in(c_1,c_m)$, the Van-Kampen theorem yields

$$\pi_1\left(M_{y_0}^{[a,b]}\right)=\pi_1\left(M_{y_0}^{[a,a_0]}\right)*_{\pi_1(\phi^{-1}(a_0),y_0)}\pi_1\left(M_{y_0}^{[a_0,b]}\right).$$

Here we know that $\pi_1\left(M_{y_0}^{[a_0,b]}\right)$ can be obtained from $\pi_1(\phi^{-1}(a_0),y_0)$ by a sequence of n_1 quotients, where n_1 is the number of index-2 critical components of ϕ inside $M_{y_0}^{[a_0,b]}$ and $\pi_1\left(M_{y_0}^{[a,a_0]}\right)$ is obtained from $\pi_1(\phi^{-1}(a_0),y_0)$ by a sequence of n_2 quotients, where n_2 is the number of index-$(2n-2)$ critical components of ϕ inside $M_{y_0}^{[a,a_0]}$. Then, since the map

$$\pi_1(\phi^{-1}(a_0),y_0)\longrightarrow\pi_1\left(M_{y_0}^{[a_0,b]}\right)$$

induced by the composition of the n_1 quotient maps is surjective, we conclude that $\pi_1\left(M_{y_0}^{[a,b]}\right)$ can be obtained from $\pi_1(\phi^{-1}(a_0),y_0)$ by taking a sequence of n_1+n_2 quotients. □

Note that the level sets of the generalized moment map ϕ may not be connected leading to non-connected reduced spaces (cf. Example 1). Nevertheless, we will show that all their connected components have the same fundamental group. For that, we first consider the equivalence relation $\sim$ defined in the proof of Lemma 1 and take

$$M/{\sim}\cong S^1.$$

The map ϕ descends to the quotient $M/{\sim}$ (since $y\sim x$ implies $\phi(x)=\phi(y)$) giving us a finite covering of S^1,

$$\tilde{\phi}: M/{\sim}\cong S^1\longrightarrow S^1$$

(i.e. $\tilde{\phi}(z)=z^k$ for $z\in S^1$ and some $k\in\mathbb{Z}$). Hence, we have the following decomposition of ϕ

$$\begin{tikzcd} M \arrow[r, "\phi^c"] \arrow[rr, bend right, "\phi"'] & M/{\sim}\cong S^1 \arrow[r, "\tilde{\phi}"] & S^1, \end{tikzcd} \tag{2.2}$$

where the map $\phi^c : M \longrightarrow S^1$ is surjective and has connected level sets. Moreover, considering the gradient flow of ϕ^c with respect to the metric g, it is easy to check that it has the same critical set as ϕ, as well as all the nice properties of its gradient flow. In particular, the indices of the critical submanifolds are all even. Moreover, the k connected components of the reduced space $M_a := \phi^{-1}(a)/S^1$ are the reduced spaces

$$M^c_{a_j} := (\phi^c)^{-1}(a_j)/S^1$$

of ϕ^c, where the a_j's ($j = 1, \ldots, k$) are such that $\tilde{\phi}(a_j) = a$, that is, M_a is the disjoint union

$$M_a = \bigsqcup_{j=1}^{k} (\phi^c)^{-1}(a_j)/S^1.$$

Proposition 2.1. *Let M be a manifold satisfying the hypotheses of Lemma 1. Then, the fundamental group of all connected components of all reduced spaces $M_a := \phi^{-1}(a)/S^1$ is always the same, even for critical values of the generalized moment map.*

Proof. Let us consider the map $\phi^c : M \longrightarrow S^1$ defined above. If the action has no fixed points then all "reduced spaces" $(\phi^c)^{-1}(a)/S^1$ are diffeomorphic and we are done. If that is not the case, let us again identify S^1 with $\mathbb{R}/\mathbb{Z}$ and assume that 0 is a regular value of ϕ^c (if not, we just break up the circle at another point). Let c_1 be the smallest critical value of ϕ^c in $[0, 1]$ and let us again assume that there is only one connected component F of the critical set assuming this value (see Footnote 2). Let D^-_F be the negative disk bundle of ν^- and $S(D^-_F)$ its sphere bundle. Then, by Morse-Bott theory, $(\phi^c)^{-1}(c_1)$ has the same homotopy type as

$$(\phi^c)^{-1}(a) \cup_{S(D^-_F)} D^-_F,$$

where a is any regular value in $[0, c_1)$. This implies that $M^c_{c_1} := (\phi^c)^{-1}(c_1)/S^1$ has the same homotopy type as

$$\left((\phi^c)^{-1}(a)/S^1\right) \cup_{S(D^-_F)/S^1} \left(D^-_F/S^1\right) = M^c_a \cup_{S(D^-_F)/S^1} \left(D^-_F/S^1\right),$$

and so $\pi_1(M^c_{c_1})$ is the free product with amalgamation

$$\pi_1(M^c_{c_1}) = \pi_1(M^c_a) *_{\pi_1(S(D^-_F)/S^1)} \pi_1(D^-_F/S^1).$$

However, the local normal form for ϕ (and consequently for ϕ^c) on a neighborhood of F is the same as the local normal form of a neighborhood of a critical set of an ordinary moment map, implying that $S(D^-_F)/S^1$ is a weighted projectivized bundle over F and so, since we also have that D^-_F is homotopy equivalent to F, we conclude that

$$\pi_1(S(D^-_F)/S^1) = \pi_1(D^-_F/S^1)$$

and so

$$\pi_1(M^c_{c_1}) = \pi_1(M^c_a).$$

Using $-\phi$ instead of ϕ (corresponding to reversing the direction of the circle action) we obtain that

$$\pi_1(M^c_{c_1}) = \pi_1(M^c_b)$$

for $b \in (c_1, c_2)$ where c_2 is a critical value of ϕ^c (if it exists) and the interval (c_1, c_2) contains only regular values. Repeating this for every critical component of ϕ^c we conclude that all connected components of all reduced spaces (even critical ones) have the same fundamental group. □

Note that an alternative proof of this Proposition could be done using the fact that, for circle actions, when passing a critical value of the moment map the reduced spaces change by a weighted blow-down followed by a weighted blow-up (cf. [G] and [BP]) and so, by [MD2], their fundamental group does not change.

Lemma 3. *Let M be a manifold satisfying the hypotheses of Lemma 1 equipped with a circle action with a non-empty fixed point set. Let $a_0 \in S^1$ be a regular value of the generalized moment map ϕ, and consider the principal circle bundle*

$$S^1 \overset{i}{\hookrightarrow} \phi^{-1}(a_0) \overset{p}{\longrightarrow} M_{a_0},$$

where $M_{a_0} := \phi^{-1}(a_0)/S^1$ is the reduced space at a_0. Then, for $y_0 \in \phi^{-1}(a_0)$, the kernel of the map $p_* : \pi_1(\phi^{-1}(a_0), y_0) \longrightarrow \pi_1(M_{a_0}, p(y_0))$ is equal to the kernel of the map

$$j_* : \pi_1(\phi^{-1}(a_0), y_0) \longrightarrow \pi_1(M, y_0),$$

defined in Lemma 2.

Proof. Clearly $\ker(p_*) \subset \ker(j_*)$. Indeed, if $[\gamma] \in \ker(p_*)$ then, the gradient flow of ϕ gives us a homotopy between γ and a constant path contained in some critical level set and so $[\gamma] = 1$ in $\pi_1(M)$.

Let us now see that $\ker(j_*) \subset \ker(p_*)$. Take $[\gamma] \in \ker(j_*)$. Then, γ is homotopic to a nullhomotopic loop in $M_{y_0}^{[a,b]}$ for some regular values a, b with $0 \le a \le a_0 \le b < 1$. Indeed, if that were not the case, there would exist a homotopy

$$H : [0,1] \times [0,1] \longrightarrow M$$

between γ and the constant path based at y_0 for which ϕ restricted to $D := \mathrm{im}(H)$ would be surjective. Hence, there would exist a loop $\alpha : S^1 \longrightarrow D$ in D such that $(\phi \circ \alpha)(S^1) = S^1$ and then, since $\pi_1(D) = \{1\}$, we would have $[\alpha] = 1$ in $\pi_1(D)$ while $\phi_*[\alpha] = [\phi \circ \alpha] \neq 1$, which is impossible.

If the critical points in $M_{y_0}^{[a,b]}$ have index greater than 2 and smaller than $2n-2$ (where $2n$ is the dimension of M), or if there are no critical points at all in this set, then, as we saw in the proof of Lemma 2,

$$\pi_1\left(M_{y_0}^{[a,b]}, y_0\right) = \pi_1(\phi^{-1}(a_0), y_0),$$

implying that γ is nullhomotopic in $\phi^{-1}(a_0)$ and so $p_*([\gamma]) = 1$.

If there are components of the critical set inside $M_{y_0}^{[a,b]}$ with index equal to 2 or equal to $2n-2$ then, again like in the proof of Lemma 2, $\pi_1\left(M_{y_0}^{[a,b]}, y_0\right)$ can be obtained from $\pi_1(\phi^{-1}(a_0), y_0)$ by taking a sequence of quotients. Indeed, for each index-2 component F with $\phi(F) = c_i > a_0$, we consider the maps

$$\kappa_i : S(D_F^-) \longrightarrow \phi^{-1}(a_0), \quad p_{S_i} : S(D_F^-) \longrightarrow F \quad \text{and} \quad p : \phi^{-1}(a_0) \longrightarrow M_{a_0}$$

defined as in the proof of Lemma 2; then we take a sequence of quotients of $\pi_1(\phi^{-1}(a_0), y_0)$ by N_i, the normal subgroups generated by all the elements of

$$(\kappa_i)_*(\ker(p_{S_i})_*);$$

we repeat this procedure for each index-$(2n-2)$ component F with $\phi(F) < a_0$, this time using $-\phi$ instead of ϕ. We conclude that, if $[\gamma] = 1$ in $\pi_1\left(M_{y_0}^{[a,b]}, y_0\right)$, then $[\gamma] \in N_i$ for one of the groups N_i considered above. However,

$$(\kappa_i)_*(\ker(p_{S_i})_*) \subset \ker(p_*)$$

and so, since $\ker(p_*)$ is normal, we conclude that $[\gamma] \in N_i \subset \ker(p_*)$ and the result follows. $\square$

With these results we can now prove Theorem 2.1.

Proof. (*of Theorem 2.1*) First, let us assume that the action has no fixed points that is, the generalized moment map ϕ has no critical points. In this case, since M is connected, all the level sets of ϕ are equivariantly diffeomorphic, since we can use the flow of $J\xi_M$ to identify the level sets. Moreover, since we are assuming that there are no fixed points, the map ϕ^c defined in (2.2) is a fibration with connected fiber

$$P := (\phi^c)^{-1}(a)$$

(a fixed level set of ϕ^c) which is a connected component of the level set $\phi^{-1}(a^k)$ for some $k \in \mathbb{Z}$. Hence, the long exact homotopy sequence for $P \longrightarrow M \longrightarrow S^1$ gives us that the sequence

$$\{1\} \longrightarrow \pi_1(P) \xrightarrow{j_*} \pi_1(M) \xrightarrow{(\phi^c)_*} \pi_1(S^1) \longrightarrow \{1\} \tag{2.3}$$

is exact, implying that j_* is injective. Moreover, the homologically non-trivial loop $\gamma : S^1 \longrightarrow M$ given by Lemma 1 is a section of the above fibration, and so $\phi^c_* \circ \gamma_* = \mathrm{id}$. Hence, $\pi_1(M)$ is a semidirect product $\pi_1(M) = G_1 \rtimes G_2$ where G_1 is the kernel of ϕ^c_* and $G_2 := \mathrm{im}(\gamma_*)$. Moreover, we have

$$G_1 = \mathrm{im}(j_*) \cong \pi_1((\phi^c)^{-1}(a))/\ker(j_*) = \pi_1((\phi^c)^{-1}(a))$$

(since j_* is injective), and so, as $(\phi^c)_*$ maps G_2 isomorphically onto $\pi_1(S^1) = \mathbb{Z}$, the result follows.

If the action has fixed points they cannot be local maxima nor minima. Taking a fixed point F, we consider the homologically non-trivial loop $\gamma : S^1 \longrightarrow M$ through F given by Lemma 1. By Lemmas 2 and 3, we have the following exact sequence, where a is a regular value of ϕ^c:

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] & \pi_1\left((\phi^c)^{-1}(a)\right) \arrow[r, "j_*"] & \pi_1(M) \arrow[r, "(\phi)^c_*"] & \pi_1(S^1) \arrow[l, bend left, "\gamma_*"] \end{tikzcd} \tag{2.4}$$

with $(\phi)^c_* \circ \gamma_* = \mathrm{id}$. Hence, taking $G_1 := \ker((\phi^c)_*)$ and $G_2 := \mathrm{im}(\gamma_*)$ we have that $\pi_1(M)$ is a semidirect product of G_1 and G_2. Moreover, considering the map

$$p : (\phi^c)^{-1}(a) \longrightarrow (\phi^c)^{-1}(a)/S^1 =: M^c_a,$$

we have, by Lemma 3, that $\ker(j_*) = \ker(p_*)$, and so

$$G_1 = \mathrm{im}(j_*) = \pi_1((\phi^c)^{-1}(a))/\ker(p_*) = \pi_1(M^c_a),$$

where the "reduced space" M^c_a is a connected component of the reduced space $\phi^{-1}(a^k)/S^1$ of ϕ. On the other hand, ϕ^c_* maps G_2 isomorphically onto $\pi_1(S^1) = \mathbb{Z}$. Hence, $\pi_1(M)$ contains two subgroups G_1 and G_2 such that G_1 is normal and isomorphic to $\pi_1(M^c_a)$ and G_2 is isomorphic to $\mathbb{Z}$. Moreover, each element of $\pi_1(M)$ is uniquely represented as the product of an element of G_1 by an element of G_2.

Indeed, $\pi_1(M)$ is a semidirect product of $\pi_1(M_a^c)$ by $\mathbb{Z}$ and then, by Proposition 2.1, the result follows. □

Remark 1. To be able to completely determine the semidirect product above one must know how the elements of $\mathbb{Z}$ act by conjugation on the fundamental group of a connected component P of a level set of the moment map (when the action has no fixed points) or on the fundamental group of a connected component of a reduced space, $M_{\text{red}} = P/S^1$. Indeed, one needs to establish the homomorphism

$$\Psi : \mathbb{Z} \longrightarrow Aut(\pi_1(P)) \quad \text{or} \quad \Psi : \mathbb{Z} \longrightarrow Aut(\pi_1(M_{\text{red}}))$$

given by $\Psi(j)(g) = jgj^{-1}$, where $j \in \mathbb{Z}$ (since $\mathbb{Z}$ is cyclic it suffices to know the image of the generator). This will of course depend on the manifold M. Nevertheless, this action of $\mathbb{Z}$ is independent of the choice of the level set P.

3. Non-Compact Manifolds

We consider in this section Hamiltonian circle actions with a proper moment map on non-compact manifolds M. The proof that the fundamental group of M is equal to the fundamental group of its reduced spaces does not follow from the proof for the compact case presented in [Li] since, in this case, we do not necessarily have a maximum or a minimum.

Just as in the compact case, the image of the proper moment map is an interval $I \subset \mathbb{R}$ but now not necessarily compact. However, the level sets of ϕ are still connected [LMTW]. Moreover, note that the indices of the critical submanifolds are all even. In particular, there are no critical submanifolds of index 1 or $2n-1$. As noted by Atiyah in [A] this implies that the number of connected components of the level sets of ϕ can only change when passing a local maximum or a minimum. Since, in this case, the level sets of ϕ are connected it follows that ϕ has at most a unique local maximal manifold and a unique local minimal manifold. The values of ϕ at these critical manifolds (if they exist) cannot be points in the interior of I and so they are in fact global extrema. Our result is then the following.

Theorem 3.1. *Let S^1 act on a connected symplectic manifold (M, ω) (not necessarily compact) with proper moment map $\phi : M \longrightarrow \mathbb{R}$, and let P be an arbitrary (compact) level set of ϕ. Then, as fundamental groups of topological spaces, the fundamental group of M is either $\pi_1(M) = \pi_1(P)$, when the action has no critical points, or $\pi_1(M) = \pi_1(M_{\text{red}})$, where M_{red} is the symplectic quotient at any value of ϕ.*

Moreover, if ϕ has a minimum (or a maximum), then

$$\pi_1(M) = \pi_1(M_{\text{red}}) = \pi_1(F_{\min}) \ (\textit{or} = \pi_1(F_{\max})),$$

where $F_{\min}$ and $F_{\max}$ are the sets of minimal and maximal points respectively.

Proof. Recall that like in the compact case the image of the moment map ϕ is an interval $I \subset \mathbb{R}$ (not necessarily compact) and that the level sets of ϕ are still connected [LMTW]. If ϕ has no critical points, then, as in classical Morse-Bott theory, M is diffeomorphic to $\phi^{-1}(a) \times I$ for any value a of ϕ, and so

$$\pi_1(M) = \pi_1(\phi^{-1}(a)).$$

If the action has fixed points, then, considering a regular value of ϕ, say a_0, we can adapt the proof of Lemma 2 to show that the sequence

$$\pi_1(\phi^{-1}(a_0)) \overset{j_*}{\hookrightarrow} \pi_1(M) \xrightarrow{\phi_*} \{1\}$$

is exact (i.e. j_* is surjective). Indeed, if $[\gamma] \in \pi_1(M)$, then γ is homotopic to some loop contained in a compact set (ϕ is proper)

$$M^{[a,b]} := \{x \in M : a \leq \phi(x) \leq b\}$$

for some values $a, b \in \mathbb{R}$ with $a \leq b$.

The proof of Proposition 2.1 can also be adapted to show that all reduced spaces (even critical ones) have the same fundamental group. Similarly, we can use the proof of Lemma 3 to show that the kernel of the map

$$p_* : \pi_1(\phi^{-1}(a_0)) \longrightarrow \pi_1(M_{a_0})$$

is equal to the kernel of the map

$$j_* : \pi_1(\phi^{-1}(a_0)) \longrightarrow \pi_1(M),$$

induced respectively by the quotient and the inclusion maps (here M_{a_0} denotes the reduced space $\phi^{-1}(a_0)/S^1$). Consequently, since j_* and p_* are surjective,

$$\pi_1(M_{a_0}) = \pi_1(\phi^{-1}(a_0))/\ker(p_*) = \pi_1(\phi^{-1}(a_0))/\ker(j_*) = \pi_1(M).$$

Hence, to finish our proof, we just need to show that, when ϕ has either a minimum at $F_{\min}$, or a maximum at $F_{\max}$, we have

$$\pi_1(M_{a_0}) = \pi_1(F_{\min}) \quad \text{or} \quad \pi_1(M_{a_0}) = \pi_1(F_{\max}).$$

Let us consider the case where ϕ has a minimum (the other case is similar). Here we can use the following argument used in [Li] for the compact case: let m be the minimum value of ϕ and consider an interval (m, b) formed by regular values of ϕ. For $a \in (m, b)$ we have, by the equivariant symplectic embedding theorem, that $\phi^{-1}(a)$ is a sphere bundle over $F_{\min}$. Let S^{2l+1} be its fiber, where

$$\dim(F_{\min}) = 2(n - l - 1).$$

Then, the reduced space M_a is diffeomorphic to an orbibundle over $F_{\min}$ with fiber a weighted projective space $\mathbb{C}P^l_w := S^{2l+1}/S^1$, and we have the exact sequence

$$\pi_1(\mathbb{C}P^l_w) \longrightarrow \pi_1(M_a) \rightarrow \pi_1(F_{\min}) \longrightarrow \{1\}.$$

Since $\mathbb{C}P^l_w$ is simply connected (cf. Remark 2), we have

$$\pi_1(F_{\min}) = \pi_1(M_a)$$

and the result follows. □

Remark 2. We have used above the fact that the fundamental group of a weighted projective space (as a topological space) is trivial. Indeed, weighted projective spaces $\mathbb{C}P^l_w$ are compact symplectic toric orbifolds [LT] and can be given the structure of an algebraic toric variety with fan equal to the fan defined by the moment map image which is a simple rational nonsmooth polytope in $(\mathbb{R}^l)^*$ (see for instance [Au] for details). This fan, which is spanned by the faces of the dual polytope, is complete meaning that the union of all its cones is the whole space $\mathbb{R}^l$. Then we know by [F] that the fundamental group of the associated toric variety (the weighted projective space) is trivial.

Alternatively, we can see $\mathbb{C}P^l_w$ as a quotient of $S^{2l+1} \subset \mathbb{C}^{l+1}$ by a diagonal circle action acting with different weights on each factor. The corresponding quotient map $S^{2l+1} \longrightarrow \mathbb{C}P^l_w$ induces a surjection in π_1 since the fibers are connected and so we conclude that weighted projective spaces are simply connected.

4. Examples

4.1. **Non-Hamiltonian actions on compact manifolds.**

Example 1. Let us begin with a very simple example of a non-Hamiltonian circle action with an empty fixed point set. Let M be the 2-torus $\mathbb{T}^2 = S^1 \times S^1$ (and so $\pi_1(M) = \mathbb{Z}^2$) with symplectic form $\sigma = d\theta_1 \wedge d\theta_2$, and consider the S^1-action given by

$$e^{i\beta} \cdot (e^{i\theta_1}, e^{i\theta_2}) = (e^{i(2\beta+\theta_1)}, e^{i\theta_2}).$$

The generalized moment map $\phi : M \longrightarrow S^1$ is just

$$\phi(e^{i\theta_1}, e^{i\theta_2}) = e^{2i\theta_2}.$$

All the level sets of ϕ are equal to two disjoint copies of S^1. We can decompose $\phi = \tilde{\phi} \circ \phi^c$ in the following way:

$$\begin{array}{ccccc} M & \xrightarrow{\phi^c} & S^1 & \xrightarrow{\tilde{\phi}} & S^1 \\ (e^{i\theta_1}, e^{i\theta_2}) & \mapsto & e^{i\theta_2} & \mapsto & e^{2i\theta_2}, \end{array}$$

where the level sets of ϕ^c are the connected components of the level sets of ϕ (they are all equal to S^1), and we get the result in Theorem 2.1. That is, $\pi_1(\mathbb{T}^2)$ is a semidirect product of

$$\pi_1((\phi^c)^{-1}(a), y_0)$$

by $\mathbb{Z}$. Indeed, for a point $y_0 \in M$ and $a = \phi^c(y_0)$, the two subgroups of $\pi_1(M, y_0)$ isomorphic to

$$\pi_1((\phi^c)^{-1}(a), y_0) = \mathbb{Z}$$

and to $\pi_1(S^1) = \mathbb{Z}$ are both normal, implying that their semidirect product is just the regular direct product of the two groups.

Example 2. Let us consider the example of a 6-manifold M with a free symplectic circle action with **contractible orbits** constructed in [K]. Here, we take Y, the smooth oriented simply-connected 4-manifold underlying a $K3$ surface (see for example [BHPV]), and consider the mapping torus[4] X of an orientation-preserving diffeomorphism $\Phi : Y \longrightarrow Y$ obtained as follows. First, knowing that the intersection form of Y is

$$Q = 3H \oplus 2E_8,$$

where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the so-called (rank-2) hyperbolic plane quadratic form and E_8 is the unique unimodular even and positive definite quadratic form of rank 8 (see [BHPV] for details), we consider an automorphism f of $H \oplus H$, such that $f(x) = x + c$ and $f(c) = c$, where x and c are two non-zero primitive classes in $H^2(Y, \mathbb{Z})$. Then, we extend f to all of Q, preserving the orientation of a maximal positive-definite subspace and find, by a result of Matumoto [M], an orientation-preserving diffeomorphism $\Phi : Y \longrightarrow Y$ with $\Phi^* = f$.

[4]The mapping torus of a map $h : Y \longrightarrow Y$ is the identification space $T(h) = Y \times [0,1]/\{(x,0) = (h(x),1)) \mid x \in Y\}$ (cf. [R]).

Finally, from X, we obtain M as the total space of the circle bundle $\pi : M \longrightarrow X$ with Euler class c (since $\Phi^* c = c$, we can choose a lift to a cohomology class on the mapping torus which we also denote by c). Since this bundle has contractible fibers, its homotopy long exact sequence gives us that $\pi_1(M) = \pi_1(X)$. Moreover, since X is a mapping torus and Y is simply connected (implying that $\Phi_* : \pi_1(Y) \longrightarrow \pi_1(Y)$ is trivially an isomorphism), we have that $\pi_1(X)$, a semidirect product $\pi_1(Y) \rtimes \mathbb{Z}$ (see [R]), is equal to $\mathbb{Z}$.

Let us now obtain the same result using Theorem 2.1. Let ω be the S^1-invariant symplectic form in M (we omit its construction for simplicity but the details can be found in [K]) and let ξ_M be the vector field generating the action. Since the closed 1-form $\iota(\xi_M)\omega$ vanishes on the tangents to the S^1-action, it can be written as $\pi^*\alpha$ where α is a closed 1-form on the quotient X. Moreover, it is shown in [FGM] that there is a map $\nu : X \longrightarrow S^1$ for which $\alpha = \nu^*(d\theta)$. Hence

$$\iota(\xi_M)\omega = \pi^* \nu^*(d\theta)$$

and so $\phi := \nu \circ \pi$ is the generalized moment map for this action. However, X is a mapping torus implying that there is a natural map ν^c from X to S^1 with (connected) level sets equal to Y and, as is shown in [FGM], $\nu = \tilde{\nu} \circ \nu^c$ where the map

$$\tilde{\nu} : S^1 \longrightarrow S^1$$

is a finite covering of the circle. Hence, the connected components of the level sets of ν are equal to Y and so, denoting by P a connected component of an arbitrary level set of ϕ, we get $\pi_1(P) = \pi_1(Y) = \{1\}$ since both the orbits of the circle action and Y are contractible. Therefore, Theorem 2.1 also gives

$$\pi_1(M, y_0) = \pi_1(P, y_0) \rtimes \mathbb{Z} = \mathbb{Z},$$

where $y_0 \in P$.

Example 3. The only known example of a manifold M equipped with a non-Hamiltonian circle action with fixed points was constructed by McDuff in [MD1]. Theorem 2.1 will allow us to compute its fundamental group. This 6-dimensional manifold, M, is obtained by first considering a special manifold with boundary, X, equipped with a Hamiltonian circle action with moment map

$$\nu : X \longrightarrow [0, 7],$$

having two boundary components (lying over the endpoints 0 and 7), and then gluing them together.

This manifold X, which has 4 critical levels at $s = 1$, 2, 5 and 6 with zero sets of codimension 4, is constructed as follows (for simplicity we will not keep track of symplectic forms). Considering coordinates $\theta_1, \ldots, \theta_4$ on T^4 and letting σ_{ij} be the form $d\theta_i \wedge d\theta_j$, we construct five regular pieces $\nu^{-1}(I)$, where

- $\nu^{-1}(I) = T^4 \times S^1 \times I$, for $I = (0, 1)$ and $I = (6, 7)$,
- $\nu^{-1}(I) = P_I \times I$, with P_I a principal circle bundle over T^4 of Chern class c_I, where
  - i) $c_I := -[\sigma_{42}]$ for $I = (1, 2)$;
  - ii) $c_I := -[\sigma_{31} + \sigma_{42}]$ for $I = (2, 5)$;
  - iii) $c_I := -[\sigma_{31}]$ for $I = (5, 6)$.

Then, we construct four additional pieces Q_λ ($\lambda = 1, 2, 5, 6$), lying over the intervals

$$[\lambda - \varepsilon, \lambda + \varepsilon],$$

which are then glued to the already defined parts. The singularity as s increases through 1 is diffeomorphic to the singularity as s decreases through 6, and similarly for 2 and 5, so X will be completely determined with the description of Q_1 and Q_2.

The piece Q_1 is of the form $T^2 \times Y$, where Y is a 4-manifold obtained from $S^2 \times S^2$ with symplectic form $2\rho \oplus \rho$ (where ρ is a symplectic form on S^2 with total area 1) and eqipped with the standard diagonal circle action in the following way. considering the moment map for this action $H = 2\mu_1 + \mu_2$ where μ_i is the moment map for the i-th factor with respect to ρ (note that $\mu_i(S^2) = [0, 1]$) we take

$$V := H^{-1}([2 - \varepsilon, 2 + \varepsilon]),$$

which fibers over S^2. Then we cut the inverse image of a disc avoiding the unique critical value of this projection (which is an S^1-invariant set diffeomorphic to $D^2 \times S^1 \times [-\varepsilon, \varepsilon]$) and glue back a copy of

$$(T^2 - \mathrm{Int}(D^2)) \times S^1 \times [-\varepsilon, \varepsilon]$$

(cf. Figure 1).

The piece Q_2 is of the form $S \times_{S^1} Y$ where S is the total space of the circle bundle

$$S^1 \hookrightarrow S \longrightarrow T^2$$

with Euler characteristic -1, so that Q_2 fibers over T^2 with fiber Y.

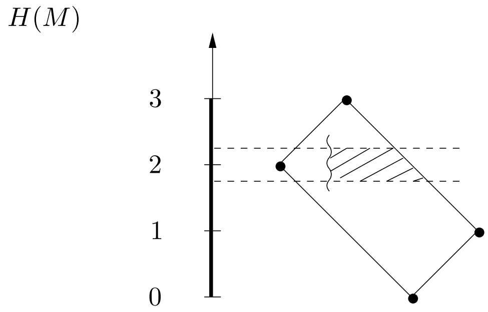

FIGURE 1. Obtaining Y from $S^2 \times S^2$ in Example 3.

The manifold M is then obtained from X by gluing $\nu^{-1}(0)$ to $\nu^{-1}(7)$ by the diffeomorphism of T^4 that interchanges θ_1 with θ_3 and θ_2 with θ_4.

Any reduced space M_a at a regular value a of the generalized moment map is diffeomorphic to T^4, implying that $\pi_1(M_a) = \mathbb{Z}^4$, and so, by Proposition 2.1 $\pi_1(M_c) = \mathbb{Z}^4$, for every reduced space at a critical value c. We conclude from Theorem 2.1 that

$$\pi_1(M) = \mathbb{Z}^4 \rtimes \mathbb{Z}.$$

The action of $\mathbb{Z}$ on $\mathbb{Z}^4$ is determined by the diffeomorphism of T^4 used to glue the boundary components of X.

4.2. **Hamiltonian actions on non-compact manifolds.**

The first two examples below satisfy the hypotheses of Theorem 3.1. On Example 4, the proper moment map has no minima nor maxima while, on Example 5, such type of critical points do exist. The last two examples (6 and 7) illustrate that the properness of the moment map is essential to our results on the fundamental group. In particular, in Example 6, there are no critical points and

$$\pi_1(M) \neq \pi_1(\phi^{-1}(a))$$

for some values a of the moment map ϕ and, in Example 7, there is a critical point (a minimum) and $\pi_1(M_{\text{red}})$ is not always the same for all values of the moment map.

Example 4. We can construct a non-compact symplectic manifold X with a Hamiltonian S^1-action with no minima or maxima from Example 3 above in the following way. Taking the manifold X from McDuff's example we attach two pieces to its boundary of the form $T^4 \times S^1 \times I$ where $I = (-\infty, 0]$ and $[7, \infty)$, extending its symplectic form and moment map ν in the natural way. The resulting moment map is proper and has no minimum nor maximum. Then, since the fundamental group of its reduced spaces is $\mathbb{Z}^4$, so is $\pi_1(X)$.

Example 5. Consider $M = S^2 \times \mathbb{R}^2$ with symplectic form $\omega = \rho_0 \oplus \omega_0$ (where ρ_0 and ω_0 are the standard symplectic forms on S^2 and on $\mathbb{R}^2$), and the following S^1-action: take the S^1-action on S^2 by rotations about the vertical z-axis and the standard S^1-action on $\mathbb{R}^2$ by rotations around the origin. The moment map on M is just the sum of the height function z with the map

$$\nu(u, v) = \frac{u^2 + v^2}{2}.$$

Physically, we have a classical spin and a harmonic oscillator ν.

This moment map $\phi = \nu + z$ is proper, has a minimum at $S \times \{0\}$ and a critical point of index 2 at $N \times \{0\}$, where S and N are respectively the south and north poles of the sphere. This circle action extends to a Hamiltonian $\mathbb{T}^2 = S^1 \times S^1$ torus action, where the action of the second circle on the sphere is by clockwise rotations and on $\mathbb{R}^2$ is the standard one. The moment map for this extended $\mathbb{T}^2$-action is $(\nu + z, \nu - z)$ and its image is pictured in Figure 2. All regular reduced spaces of ϕ are homeomorphic to S^2 and so,

$$\pi_1(M) = \pi_1(M_{\text{red}}) = \pi_1(F_{\min}) = \{1\}.$$

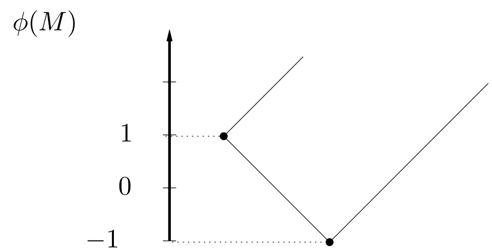

FIGURE 2. Moment map image for the $\mathbb{T}^2$-action on $S^2 \times \mathbb{R}^2$.

Example 6. Let us now give an example which shows that the requirement for properness of the moment map is essential to our results. Let us consider

$$M = S^2 \times \mathbb{C} \setminus \{0\}$$

with the same symplectic form and the same circle action as in Example 5 above. The image of the moment map $\phi = \nu + z$ is now the interval $(-1, \infty)$. This map **has no critical points** on M and is **no longer proper** (note for instance that the level sets $\phi^{-1}(a)$ for values $a \in (-1, 1)$ are not compact). We can easily check that Theorem 3.1 is no longer valid. In fact, the fundamental group of the manifold, $\pi_1(M) = \mathbb{Z}$, is no longer the fundamental group of the level sets of ϕ. Indeed, considering, for instance, a value $a \in (-1, 1)$, the level sets $\phi^{-1}(a)$ are diffeomorphic to $S^3 \setminus \{pt\}$. Note also that the level sets of this moment map are no longer all diffeomorphic (as they would be for a proper moment map with no critical points) since, for $a \in (1, \infty)$, they are diffeomorphic to $S^2 \times S^1$.

Example 7. We end this section with an example of a non-compact Hamiltonian S^1-space with a **non-proper** moment map **with a critical point** for which the conclusion on the fundamental groups in Theorem 3.1 fails to hold. Let us consider

$$M = \big(S^2 \setminus N\big) \times \big(S^2 \setminus N\big)$$

with symplectic form $2\rho \oplus \rho$, where ρ is a symplectic form on S^2 with total area 1 and N is the north pole, equipped with the standard diagonal circle action (cf. Figure 3). This action is Hamiltonian and its moment map ϕ has a unique critical value at 0 corresponding to the fixed point (S, S), where S is the south pole of the sphere (see Example 3 for the moment map expression and compare Figures 1 and 3). We can see that the reduced spaces have different fundamental groups. Indeed, for $a \in (0, 1)$, the reduced spaces

$$M_a = \phi^{-1}(a)/S^1$$

are spheres while, for $a \in (2, 3)$, they are spheres minus two points. That is, $\pi_1(M_a) = \{1\}$ for $a \in (0, 1)$, and $\pi_1(M_a) = \mathbb{Z}$ for $a \in (2, 3)$.

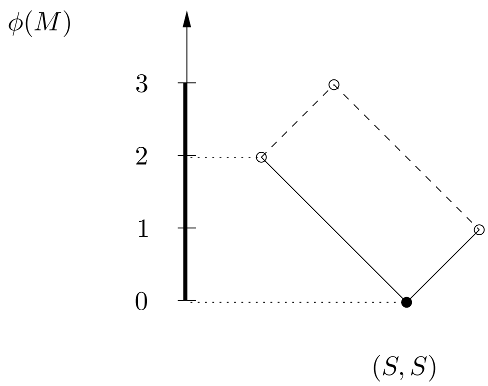

FIGURE 3. Moment map image for an extended T^2-action on $\big(S^2 \setminus \{N\}\big) \times \big(S^2 \setminus \{N\}\big)$.

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References

[A] M. F. Atiyah *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), 1-15.

[Au] M. Audin, *Torus actions on symplectic manifolds*, Progress in Mathematics, **93**, Birkhuser Verlag, Basel, (2004).

[B] R. Bott *Nondegenerate critical manifolds*, Ann. of Math. **60** (1954), 248-261.

[BHPV] W. Barth, K. Hulek, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, **4**, Springer-Verlag, Berlin, (2004).

[BP] M. Brion and C. Procesi, *Action d'un tore dans une varieté projective*, Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory, Progress in Mathematics, **93**, Birkhäuser, Boston, (1991).

[CGKZ] D.J. Collins, R.I. Grigorchuk, P.F. Kurchanov and H. Zieschang, *Combinatorial Group Theory and Applications to Geometry*, Encyclopaedia of mathematical sciences, **58**, Springer, (1998).

[F] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, **131**, Princeton University Press, Princeton, (1993).

[FGM] M. Fernandez, A. Gray and J.W. Morgan, *Compact symplectic manifolds with free circle actions and Massey products*, Michigan Math. J. **38** (1991), 271-283.

[G] L. Godinho, *Blowing up symplectic orbifolds*, Ann. Global Anal. Geom. **20** (2001), 117-162.

[L] E. Lerman, *Symplectic cuts*, Math. Res. Lett. **2** (1995), 247-258.

[LT] E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, Trans. Amer. Math. Soc. **349** (1997), 4201–4230.

[LMTW] E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, *Non abelian convexity by symplectic cuts*, Topology **37** (1998), 245-259.

[Li] H. Li, π_1 *of Hamiltonian* S^1*-actions*, Proc. Amer. Math. Soc. **131** (2003), 3579-3582.

[K] D. Kotschick, *Free circle actions with contractible orbits on symplectic manifolds*, Math. Z. **252** (2006), 19-25.

[M] T. Matumoto, *On diffeomorphisms of a K3 surface*, Algebraic and Topological Theories (Kinosaki, 1984), 616-621, Kinokuniya, Tokyo (1986).

[MD1] D. McDuff, *The moment map for circle actions on symplectic manifolds*, Journal of Geometry and Physics **5** (1998), no 2, 149-160.

[MD2] D. McDuff, *Examples of simply-connected symplectic non-Kählerian manifolds*, Differential Geometry **20** (1984), 267-277.

[Mi] J. Milnor, *Morse Theory*, Princeton University Press, Princeton, N.J., (1963).

[O] K. Ono, *Obstruction to circle group actions preserving symplectic structure*, Hokkaido Math. Journal **21** (1992), 99-102.

[OR] J.P. Ortega and T. Ratiu, *Momentum Maps and Hamiltonian Reduction*, Progress in Mathematics, **222**, Birkhäuser, Boston, (2003).

[R] A. Ranicki, *High Dimensional Knot theory, Algebraic Surgery in Codimension 2*, Springer Monographs in Mathematics, **26**, Springer-Verlag, Berlin, (1998).

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