SYMPLECTIC FORM ON HYPERPOLYGON SPACES

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ABSTRACT. In [GM], a family of parabolic Higgs bundles on $\mathbb{CP}^1$ has been constructed and identified with a moduli space of hyperpolygons. Our aim here is to give a canonical alternative construction of this family. This enables us to compute the Higgs symplectic form for this family and show that the isomorphism of [GM] is a symplectomorphism.

1. Introduction

Hyperpolygon spaces, $X(\alpha)$, with $\alpha \in (\mathbb{R}_{\geq 0})^n$, were introduced by Konno in [Ko] as the hyper-Kähler analogue of polygon spaces. They are defined as the hyper-Kähler quotients of the cotangent bundle $T^*\mathbb{C}^{2n}$ by the group

$$K := \Big(\mathrm{U}(2) \times \mathrm{U}(1)^n\Big)/\mathrm{U}(1) = \Big(\mathrm{SU}(2) \times \mathrm{U}(1)^n\Big)/(\mathbb{Z}/2\mathbb{Z})\,; \tag{1}$$

in the definition of K, the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts diagonally as multiplication by -1 on each factor (see [Ko] for details). It is shown in [GM] there exists an isomorphism between $X(\alpha)$ and a family of parabolic Higgs bundles over $\mathbb{CP}^1$ that depends on α. More precisely, $X(\alpha)$ is isomorphic to the family of parabolic Higgs bundles (E, Φ) on $(\mathbb{CP}^1, \{x_i\}_{i=1}^n)$, where x_i are fixed marked points, E is a holomorphically trivial vector bundle over $\mathbb{CP}^1$ of rank two with a weighted complete flag structure over each of the marked points

$$E_{x_i,1} \supsetneq E_{x_i,2} \supsetneq 0\,,$$
$$0 \leq \beta_1(x_i) < \beta_2(x_i) < 1\,,$$

with $\beta_2(x_i) - \beta_1(x_i) = \alpha_i$.

Here we give an alternative canonical construction of these families of parabolic Higgs bundles. This construction enables us to prove that the pullback, by the above isomorphism, of the Higgs symplectic form for each family to the corresponding hyperpolygon space $X(\alpha)$ coincides with the symplectic form on $X(\alpha)$ obtained by reduction from the Liouville symplectic form on $T^*\mathbb{C}^{2n}$. As a consequence, the isomorphism constructed in [GM] is a symplectomorphism.

2. A family of parabolic Higgs bundles on $\mathbb{CP}^1$

Fix n distinct ordered points $D := \{x_1, \cdots, x_n\} \subset \mathbb{CP}^1$, with $n \geq 3$. For each $i \in [1, n]$, fix a real number $\alpha_i \in (0, 1)$. Let $\mathcal{M}_P^H$ be the moduli stack of parabolic Higgs

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bundles over $\mathbb{CP}^1$ of rank two and degree zero with parabolic divisor D and parabolic weights $\{\alpha_i\,,0\}$ at x_i, for $i \in [1\,,n]$.

Let

$$U := (\mathbb{C}^2\setminus\{0\})^n \subset (\mathbb{C}^2)^{\oplus n}$$

be the Zariski open subset. The algebraic cotangent bundle T^*U is the trivial vector bundle over U with fiber $((\mathbb{C}^2)^{\oplus n})^*$. So the total space of T^*U is identified with the Cartesian product $((\mathbb{C}^2)^{\oplus n})^* \times U$. For each $i \in [1\,,n]$, let

$$\overline{f}_i : U \longrightarrow \mathbb{C}^2\setminus\{0\} \text{ and } \overline{g}_i : ((\mathbb{C}^2)^{\oplus n})^* \longrightarrow (\mathbb{C}^2)^* \tag{2}$$

be the projections to the i–th factor of the Cartesian products. For $v \in \mathbb{C}^2$ and $w \in (\mathbb{C}^2)^*$, we have $v\otimes w \in \mathbb{C}^2\otimes(\mathbb{C}^2)^* = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$, and $w(v) \in \mathbb{C}$. Define the Zariski closed subscheme of $((\mathbb{C}^2)^{\oplus n})^* \times U$

$$\mathcal{Z} := \left\{(y,z) \in ((\mathbb{C}^2)^{\oplus n})^* \times U \mid \sum_{i=1}^n \overline{f}_i(z)\otimes\overline{g}_i(y) = 0 \text{ and } \overline{g}_i(y)\left(\overline{f}_i(z)\right) = 0\ \forall\, i\right\}. \tag{3}$$

(Note that $\overline{f}_i(z)\otimes\overline{g}_i(y) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$.)

The section of the trivial rank two vector bundle over $\mathbb{C}^2\setminus\{0\}$

$$\mathbb{C}^2\times(\mathbb{C}^2\setminus\{0\}) \longrightarrow \mathbb{C}^2\setminus\{0\}$$

defined by $v \longmapsto (v\,,v)$ will be denoted by s_0. Let

$$L_0 \subset \mathbb{C}^2\times(\mathbb{C}^2\setminus\{0\}) \longrightarrow \mathbb{C}^2\setminus\{0\} \tag{4}$$

be the line subbundle generated by the section s_0. Let f_i be the composition

$$\mathcal{Z} \longrightarrow U \xrightarrow{\overline{f}_i} \mathbb{C}^2\setminus\{0\}\,, \tag{5}$$

where $\overline{f}_i$ and $\mathcal{Z}$ are constructed in (2) and (3) respectively, and the map $\mathcal{Z} \longrightarrow U$ is the natural projection.

We will construct a morphism from $\mathcal{Z}$ to the moduli stack $\mathcal{M}^H_P$. This amounts to constructing a parabolic Higgs vector bundle over $\mathbb{CP}^1\times\mathcal{Z}$ of the given type.

The vector bundle underlying the parabolic bundle will be the trivial vector bundle of rank two

$$\mathcal{V} := \mathbb{C}^2\times(\mathbb{CP}^1\times\mathcal{Z}) \longrightarrow \mathbb{CP}^1\times\mathcal{Z} \tag{6}$$

over $\mathbb{CP}^1\times\mathcal{Z}$. For each point $x_i \in D$, we have the line subbundle over $\mathcal{Z}$

$$\mathcal{L}_i := f_i^*L_0 \subset \mathcal{V}|_{\{x_i\}\times\mathcal{Z}} = \mathbb{C}^2\times\mathcal{Z}\,, \tag{7}$$

where f_i and L_0 are constructed in (5) and (4) respectively. The quasiparabolic filtration over $\{x_i\}\times\mathcal{Z}$ is given by the line subbundle $\mathcal{L}_i$. The parabolic weight of $\mathcal{L}_i$ is α_i and the parabolic weight of $\mathcal{V}|_{\{x_i\}\times\mathcal{Z}}$ is 0.

Let us denote the holomorphic cotangent bundle of $\mathbb{CP}^1$ by $K_{\mathbb{CP}^1}$. Consider the short exact sequence of coherent sheaves

$$0 \longrightarrow K_{\mathbb{CP}^1} \longrightarrow K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D) \longrightarrow (K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D))\,|_D \cong \mathcal{O}_D \longrightarrow 0$$

over $\mathbb{CP}^1$; the above identification of $(K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D))|_D$ with $\mathcal{O}_D$ is given by the Poincaré adjunction formula [GH, p. 146]. Tensoring this exact sequence with $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$, and then taking the corresponding long exact sequence of cohomologies, we get

$$0 \longrightarrow H^0(\mathbb{CP}^1,\, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\otimes K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D)) \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\otimes H^0(D,\,\mathcal{O}_D) \tag{8}$$

$$= \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)^{\oplus n} \stackrel{\phi}{\longrightarrow} H^1(\mathbb{CP}^1,\, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\otimes K_{\mathbb{CP}^1})$$

because $H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}) = 0$. Using Serre duality, we have

$$H^1(\mathbb{CP}^1,\, K_{\mathbb{CP}^1}) = H^0(\mathbb{CP}^1,\, \mathcal{O}_{\mathbb{CP}^1})^* = \mathbb{C}\,.$$

Therefore,

$$H^1(\mathbb{CP}^1,\, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\otimes K_{\mathbb{CP}^1}) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\otimes H^1(\mathbb{CP}^1,\, K_{\mathbb{CP}^1}) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\,.$$

The homomorphism

$$\phi\,:\, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)^{\oplus n} \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$$

in the exact sequence (8) coincides with the one defined by

$$(A_1\,,\cdots,A_n) \longmapsto \sum_{i=1}^n A_i.$$

Now we will generalize these over a family.

Let $p\,:\,\mathbb{CP}^1\times\mathcal{Z}\longrightarrow\mathbb{CP}^1$ be the natural projection. Let $\mathcal{A}:=\mathbb{C}[\mathcal{Z}]$ be the $\mathbb{C}$-algebra defined by the algebraic functions on the scheme $\mathcal{Z}$. For notational convenience, the vector bundle $\mathrm{End}(\mathcal{V})\longrightarrow\mathbb{CP}^1$, where $\mathcal{V}$ is defined in (6), will be denoted by $\widetilde{\mathcal{V}}$. Consider the short exact sequence of coherent sheaves

$$0\longrightarrow\widetilde{\mathcal{V}}\otimes p^*K_{\mathbb{CP}^1}\longrightarrow\widetilde{\mathcal{V}}\otimes p^*(K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D))\longrightarrow\widetilde{\mathcal{V}}|_{D\times\mathcal{Z}}\longrightarrow 0 \tag{9}$$

over $\mathbb{CP}^1\times\mathcal{Z}$, where the line bundle $(p^*(K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D)))\,|_{D\times\mathcal{Z}}$ is identified with $\mathcal{O}_{D\times\mathcal{Z}}$ using the Poincaré adjunction formula (as done before). We have

$$H^0(\mathbb{CP}^1\times\mathcal{Z},\,\widetilde{\mathcal{V}}\otimes p^*K_{\mathbb{CP}^1}) = 0 \quad\text{and}\quad H^1(\mathbb{CP}^1\times\mathcal{Z},\,\widetilde{\mathcal{V}}\otimes p^*K_{\mathbb{CP}^1}) = \mathrm{End}(\mathbb{C}^2)\otimes_{\mathbb{C}}\mathcal{A}$$

because $H^1(\mathcal{Z},\,\mathcal{O}_{\mathcal{Z}}) = 0$ (recall that $\mathcal{Z}$ is a Zariski open subset of an affine variety). Also,

$$H^0(\mathbb{CP}^1\times\mathcal{Z},\,\widetilde{\mathcal{V}}|_{D\times\mathcal{Z}}) = \mathrm{End}(\mathbb{C}^2)\otimes(\oplus_{i=1}^n\mathcal{A}) = (\mathrm{End}(\mathbb{C}^2)\otimes_{\mathbb{C}}\mathcal{A})^{\oplus n}\,.$$

Therefore, the long exact sequence of cohomologies associated to the exact sequence in (9) gives

$$0\longrightarrow H^0\Big(\mathbb{CP}^1\times\mathcal{Z},\,\widetilde{\mathcal{V}}\otimes p^*(K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D))\Big)\longrightarrow(\mathrm{End}(\mathbb{C}^2)\otimes\mathcal{A})^{\oplus n}\stackrel{\overline{\phi}}{\longrightarrow}\mathrm{End}(\mathbb{C}^2)\otimes\mathcal{A}\,,$$

where the above homomorphism $\overline{\phi}$ sends any $(A_1\,,\cdots,A_n)\in(\mathrm{End}(\mathbb{C}^2)\otimes\mathcal{A})^{\oplus n}$ to $\sum_{i=1}^n A_i$.

Consequently, from the condition $\sum_{i=1}^n\overline{f}_i(z)\otimes\overline{g}_i(y) = 0$ in (3) it follows that there is a unique algebraic section

$$\theta_0\in H^0\left(\mathbb{CP}^1\times\mathcal{Z},\,\mathrm{End}(\mathcal{V})\otimes p^*(K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D))\right)$$

such that for any $(y,z)\in\mathcal{Z}$ and $x_i\in D$,

$$\theta_0(x_i\,,y,z) = \overline{f}_i(z)\otimes\overline{g}_i(y)\in\mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)\,.$$

From the condition $\overline{g}_i(y)\left(\overline{f}_i(z)\right) = 0$ in (3) it follows immediately that $\theta_0(x_i, y, z)$ is nilpotent with respect to the quasiparabolic filtration (constructed in (7)). Therefore, θ_0 defines a Higgs field for the family of parabolic vector bundles. In other words,

$$(\mathcal{V}, \{\mathcal{L}_i\}_{i=1}^n, \theta_0)$$

is a family of parabolic Higgs bundles on $\mathbb{CP}^1$ parametrized by $\mathcal{Z}$. Hence we get a morphism

$$\varphi_0 : \mathcal{Z} \longrightarrow \mathcal{M}_P^H . \tag{10}$$

Let $\mathcal{M}_P^{H,S} \subset \mathcal{M}_P^H$ be the moduli stack of stable parabolic Higgs bundles. We choose $\{\alpha_i\}_{i=1}^n$ in such a way that the image of φ_0 constructed in (10) intersects the stable locus $\mathcal{M}_P^{H,S}$. Then from the openness of the stability condition, [Ma], it follows that there is a nonempty Zariski open subset

$$\mathcal{U}_S \subset \mathcal{Z} \tag{11}$$

such that image of $\varphi_0|_{\mathcal{U}_S}$ is in $\mathcal{M}_P^{H,S}$, in other words,

$$\varphi_0|_{\mathcal{U}_S} : \mathcal{U}_S \longrightarrow \mathcal{M}_P^{H,S} \subset \mathcal{M}_P^H . \tag{12}$$

Let M_P^H be the moduli space of stable parabolic Higgs bundles over $\mathbb{CP}^1$ of rank two and degree zero with parabolic divisor D and parabolic weights $\{\alpha_i, 0\}$ at x_i, $i \in [1, n]$. Consider the morphism $\mathcal{M}_P^{H,S} \longrightarrow M_P^H$ to the coarse moduli space. Let

$$\varphi : \mathcal{U}_S \longrightarrow M_P^H \tag{13}$$

be its composition with the morphism in (12).

Let G be the complexification $K_{\mathbb{C}} = \Big(\mathrm{SL}(2,\mathbb{C}) \times (\mathbb{C}^*)^n\Big)/(\mathbb{Z}/2\mathbb{Z})$, of K defined in (1). Then G acts on $((\mathbb{C}^2)^{\oplus n})^* \times U$ by

$$\begin{aligned}[A, \lambda_1, \cdots, \lambda_n] \cdot ((y_1, \cdots, y_n),(z_1, \cdots, z_n)) \\ = ((\lambda_1^{-1} y_1 A, \cdots, \lambda_n^{-1} y_n A), (A^{-1} z_1 \lambda_1, \cdots, A^{-1} z_n \lambda_n)),\end{aligned} \tag{14}$$

keeping the space $\mathcal{Z}$ invariant.

There is a notion of α-stability for hyperpolygons coming from hyper-Kähler quiver varieties [N, Ko]. In [GM] it is shown that $X(\alpha)$, obtained as a GIT quotient using this α-stability, coincides with the GIT quotient, $\mathcal{U}_S/\!\!/G$, of $\mathcal{U}_S$ by G. Indeed, the space $\mathcal{Z}$ is precisely the 0-level set of the complex moment map for the hyper-Kähler action of G on $((\mathbb{C}^2)^{\oplus n})^* \times U$ as considered in [Ko, GM] and the open dense subset $\mathcal{U}_S$ coincides with the set of α-stable elements in this level set (see [GM, Theorem 3.1]). This also implies that the morphism φ is G-invariant.

Let $\mathcal{H}(\alpha) \subset M_P^H$ be the image of this morphism. Note that $\mathcal{H}(\alpha)$ is the space of parabolic Higgs bundles in M_P^H whose underlying vector bundle is holomorphically trivial. We have shown the following result.

Proposition 2.1. *The morphism φ in (13) is G-invariant, inducing the isomorphism*

$$\overline{\varphi} : \mathcal{U}_S/\!\!/G = X(\alpha) \longrightarrow \mathcal{H}(\alpha)$$

constructed in [GM, Theorem 3.1].

3. Symplectic structure on the moduli of parabolic Higgs bundles

3.1. **A natural 1–form.** The moduli space M_P^H has a natural algebraic 1–form [Hi], [BR]; we will recall its construction.

Take a stable parabolic Higgs bundle

$$\mathbf{E} := (E, \{\ell_i\}_{i=1}^n, \theta) \in M_P^H \tag{15}$$

over $\mathbb{CP}^1$, where ℓ_i is a line in E_{x_i} giving the quasiparabolic filtration over $x_i \in D$. Let

$$\mathrm{End}_P(E) \subset \mathrm{End}(E) := E \otimes E^*$$

be the subsheaf given by locally defined endomorphisms s such that $s(x_i)(\ell_i) \subset \ell_i$ for all x_i in the domain of definition of s. Let

$$\mathrm{End}_P^0(E) \subset \mathrm{End}_P(E)$$

be the subsheaf given by the locally defined endomorphisms s such that $s(x_i)(\ell_i) = 0$ and $s(x_i)(E_{x_i}) \subset \ell_i$ for all x_i in the domain of definition of s. So both $\mathrm{End}_P(E)|_{\mathbb{CP}^1\setminus D}$ and $\mathrm{End}_P^0(E)|_{\mathbb{CP}^1\setminus D}$ are identified with $\mathrm{End}(E)|_{\mathbb{CP}^1\setminus D}$. For holomorphic sections s and t of $\mathrm{End}_P(E)$ and $\mathrm{End}_P^0(E)\otimes\mathcal{O}_{\mathbb{CP}^1}(D)$ respectively, both defined over an open subset U_0 of $\mathbb{CP}^1$, the composition $s \circ t$ is a holomorphic section of $\mathrm{End}(E)\otimes\mathcal{O}_{\mathbb{CP}^1}(D)$ over U_0 which is nilpotent over the points of $D \cap U_0$; here "$\circ$" is the composition of endomorphisms of E. Therefore, $\mathrm{trace}(s\circ t)$ is a holomorphic function on U_0. The pairing

$$\mathrm{End}_P(E) \otimes \big(\mathrm{End}_P^0(E)\otimes\mathcal{O}_{\mathbb{CP}^1}(D)\big) \longrightarrow \mathcal{O}_{\mathbb{CP}^1}$$

defined by $s\otimes t \longmapsto \mathrm{trace}(s\circ t)$ is nondegenerate. Hence we get an isomorphism

$$\mathrm{End}_P^0(E)\otimes\mathcal{O}_{\mathbb{CP}^1}(D) \xrightarrow{\sim} \mathrm{End}_P(E)^*\,. \tag{16}$$

For any locally defined section s of $\mathrm{End}_P(E)$, note that $s\circ\theta - \theta\circ s$ is a locally defined section of $\mathrm{End}_P^0(E)\otimes K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D)$. Consider the two-term complex

$$\mathcal{C}^\bullet : \mathcal{C}^0 := \mathrm{End}_P(E) \xrightarrow{[\cdot,\theta]} \mathcal{C}^1 := \mathrm{End}_P^0(E)\otimes K_{\mathbb{CP}^1}\otimes\mathcal{O}_{\mathbb{CP}^1}(D)\,.$$

The tangent space of M_P^H at $\mathbf{E}$, defined in (15) has the following description in terms of hypercohomology:

$$T_{\mathbf{E}}M_P^H = \mathbb{H}^1(\mathcal{C}^\bullet) \tag{17}$$

(see [BR, Section 6]).

Consider the short exact sequence of complexes

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{End}^0_P(E) \otimes K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(D) \\
\downarrow & & \downarrow \\
\mathrm{End}_P(E) & \xrightarrow{[\cdot,\theta]} & \mathrm{End}^0_P(E) \otimes K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(D) \\
\downarrow & & \downarrow \\
\mathrm{End}_P(E) & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

over $\mathbb{CP}^1$. It produces the following long exact sequence of hypercohomologies:

$$H^0(\mathbb{CP}^1,\, \mathrm{End}^0_P(E) \otimes K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(D)) \longrightarrow \mathbb{H}^1(\mathcal{C}^\bullet) \xrightarrow{\eta} H^1(\mathbb{CP}^1,\, \mathrm{End}_P(E))\,. \tag{18}$$

Using (16) and Serre duality,

$$H^1(\mathbb{CP}^1,\, \mathrm{End}_P(E))^* \cong H^0\left(\mathbb{CP}^1,\, \mathrm{End}^0_P(E) \otimes K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(D)\right)\,. \tag{19}$$

Now consider the composition

$$T_{\mathbf{E}} M^H_P = \mathbb{H}^1(\mathcal{C}^\bullet) \xrightarrow{\eta} H^1(\mathbb{CP}^1,\, \mathrm{End}_P(E)) \xrightarrow{\theta\cdot} \mathbb{C}\,,$$

where $\theta \cdot v = \theta(v)$ (see (19) for the duality pairing), and the homomorphism η is constructed in (18). This composition defines an algebraic 1-form

$$\lambda \in H^0(M^H_P,\, \Omega^1_{M^H_P})\,. \tag{20}$$

The de Rham differential $d\lambda$ is the natural symplectic form on M^H_P, which we refer to as the Higgs symplectic form [BR].

It should be mentioned that $H^1(\mathbb{CP}^1,\, \mathrm{End}_P(E))$ parametrizes the infinitesimal deformations of the parabolic vector bundle $(E\,, \{\ell_i\}_{i=1}^n)$. The homomorphism η in (18) is the forgetful homomorphism that sends infinitesimal deformations of the parabolic Higgs bundle $(E\,, \{\ell_i\}_{i=1}^n, \theta)$ to the corresponding infinitesimal deformations of $(E\,, \{\ell_i\}_{i=1}^n)$ obtained by forgetting the Higgs field.

3.2. **The pullback of λ.** The total space $((\mathbb{C}^2)^{\oplus n})^* \times U$ of the cotangent bundle of U is equipped with the Liouville symplectic form. Let ω_0 be the restriction to $\mathcal{U}_S$ (see (11)) of this Liouville symplectic form. Recall that $d\lambda$, where λ is constructed in (20), is the canonical holomorphic symplectic form on M^H_P.

Theorem 3.1. *Consider φ constructed in* (13)*. The pulled back form $\varphi^* d\lambda = d\varphi^*\lambda$ on $\mathcal{U}_S$ coincides with ω_0.*

Proof. Take any point

$$\underline{y} = ((y_1\,, \cdots, y_n)\,, (z_1\,, \cdots, z_n)) \in \mathcal{U}_S \subset ((\mathbb{C}^2)^{\oplus n})^* \times (\mathbb{C}^2)^{\oplus n} = (\mathbb{C}^2)^{\oplus n} \times (\mathbb{C}^2)^{\oplus n}\,, \tag{21}$$

where $y_i \in (\mathbb{C}^2)^*$, $z_i \in \mathbb{C}^2$. Let

$$\varphi(\underline{y}) = (E\,,\{\ell_i\}_{i=1}^n\,,\theta) \tag{22}$$

be the parabolic Higgs bundle.

Let $d\varphi : T\mathcal{U}_S \longrightarrow TM_P^H$ be the differential of the morphism φ, where $T\mathcal{U}_S$ and TM_P^H are holomorphic tangent bundles. We will compute the composition

$$T_{\underline{y}}\mathcal{U}_S \xrightarrow{d\varphi} T_{\varphi(\underline{y})}M_P^H \cong \mathbb{H}^1(\mathcal{C}^\bullet) \xrightarrow{\eta} H^1(\mathbb{CP}^1,\, \mathrm{End}_P(E)) \tag{23}$$

(see (18) and (17) for η and the isomorphism respectively). Take any

$$\underline{v} \,=\, ((u_1\,,\cdots\,,u_n)\,,(v_1\,,\cdots\,,v_n)) \,\in\, T_{\underline{y}}\mathcal{U}_S\,, \tag{24}$$

where $u_i \in (\mathbb{C}^2)^*$ and $v_i \in \mathbb{C}^2$. We want to describe the image of $\underline{v}$ under the composition in (23).

We recall from Section 2 that E is the trivial vector bundle $\mathbb{C}^2 \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$. For each $i \in [1\,,n]$, let

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}^2) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\cdot z_i\,,\mathbb{C}^2/(\mathbb{C}\cdot z_i))$$

be the surjective homomorphism obtained by restricting endomorphisms of $\mathbb{C}^2$ to the line $\mathbb{C}\cdot z_i$ generated by z_i, and then projecting the image of this line to $\mathbb{C}^2/(\mathbb{C}\cdot z_i)$. We will consider $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\cdot z_i\,,\mathbb{C}^2/(\mathbb{C}\cdot z_i))$ as the torsion sheaf on $\mathbb{CP}^1$ supported at the point $x_i \in D$. The kernel of the homomorphism of coherent sheaves

$$E \,=\, \mathbb{C}^2 \times \mathbb{CP}^1 \,\longrightarrow\, \bigoplus_{i=1}^n \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\cdot z_i\,,\mathbb{C}^2/(\mathbb{C}\cdot z_i))$$

coincides with $\mathrm{End}_P(E)$, because the quasiparabolic line $\ell_i \subset E_{x_i} = \mathbb{C}^2$ at x_i coincides with $\mathbb{C}\cdot z_i$. In other words, we get a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathrm{End}_P(E) \longrightarrow \mathrm{End}(E) \longrightarrow \bigoplus_{i=1}^n \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\cdot z_i\,,\mathbb{C}^2/(\mathbb{C}\cdot z_i)) \longrightarrow 0$$

over $\mathbb{CP}^1$. Let

$$\bigoplus_{i=1}^n \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}\cdot z_i\,,\mathbb{C}^2/(\mathbb{C}\cdot z_i)) \xrightarrow{\xi} H^1(\mathbb{CP}^1,\, \mathrm{End}_P(E)) \tag{25}$$

be the map obtained by the long exact sequence of cohomologies associated to this short exact sequence.

Consider the homomorphism $\mathbb{C}\cdot z_i \longrightarrow \mathbb{C}^2$ defined by $w\cdot z_i \longmapsto w\cdot v_i$ (see (24) for v_i). Its composition with the the natural projection of $\mathbb{C}^2 \longrightarrow \mathbb{C}^2/(\mathbb{C}\cdot z_i)$ yields a homomorphism

$$\overline{v}_i \,:\, \mathbb{C}\cdot z_i \longrightarrow \mathbb{C}^2/(\mathbb{C}\cdot z_i)\,.$$

Now it is straightforward to check that

$$(\eta\circ d\varphi)\,(\underline{v}) \,=\, \sum_{i=1}^n \xi(\overline{v}_i) \,\in\, H^1(\mathbb{CP}^1,\, \mathrm{End}_P(E))$$

(see (23) and (25) for η and ξ respectively), where $\overline{v}_i$ is constructed above.

Consider the Higgs field θ in (22). Using the isomorphism in (19), it defines a functional on $H^1(\mathbb{CP}^1, \mathrm{End}_P(E))$. We want to calculate $\theta(\xi(\overline{v}_i)) \in \mathbb{C}$.

Now we will recall a property of the Serre duality pairing. Let V be an algebraic vector bundle over an irreducible smooth complex projective curve X. Fix a point $x \in X$. Let S be a subspace of the vector space $(V \otimes \mathcal{O}_X(x))_x$. Let $\widetilde{V}$ be the vector bundle on X that fits in the short exact sequence

$$0 \longrightarrow \widetilde{V} \longrightarrow V \otimes \mathcal{O}_X(x) \longrightarrow (V \otimes \mathcal{O}_X(x))_x/S \longrightarrow 0 .$$

Therefore, we have a short exact sequence of coherent sheaves

$$0 \longrightarrow V \longrightarrow \widetilde{V} \longrightarrow S \longrightarrow 0 . \tag{26}$$

Let

$$S \stackrel{\beta}{\longrightarrow} H^1(X, V)$$

be the homomorphism in the long exact sequence of cohomologies associated to the short exact sequence in (26). Then for any

$$\gamma \in H^0(X, V^* \otimes K_X) ,$$

and any $w \in S$, the Serre duality pairing $\gamma(\beta(w)) \in \mathbb{C}$ coincides with $\gamma(x)(w)$; note that since $\gamma(x) \in (V^* \otimes K_X)_x = ((V \otimes \mathcal{O}_X(x))_x)^*$ (Poincaré adjunction formula), we can evaluate $\gamma(x)$ on w.

From the above property of the Serre duality pairing it follows immediately that

$$\theta(\xi(\overline{v}_i)) = y_i(v_i) ;$$

recall from (21) and (24) that $y_i \in (\mathbb{C}^2)^*$ and $v_i \in \mathbb{C}^2$ respectively.

Consequently, the form $\varphi^*\lambda$ on $\mathcal{U}_S$ coincides with the following 1-form λ' on $\mathcal{U}_S$: for any point $\underline{y}$ as in (21) and any tangent vector $\underline{v}$ at $\underline{y}$ as in (24),

$$\lambda'(\underline{v}) := \sum_{i=1}^{n} y_i(v_i) .$$

It is straightforward to check that $\lambda'(\underline{v})$ is the evaluation at $\underline{v}$ of the tautological one-form on the total space of the holomorphic cotangent bundle $((\mathbb{C}^2)^{\oplus n})^* \times U \longrightarrow U$. Therefore, $d\lambda'$ coincides with the restriction to $\mathcal{U}_S$ of the Liouville symplectic form on $((\mathbb{C}^2)^{\oplus n})^* \times U$. Consequently, $d\varphi^*\lambda$ coincides with ω_0. □

The holomorphic symplectic form ω_0 on $\mathcal{U}_S$ is easily seen to be G-invariant (see (14)). This defines a natural holomorphic symplectic form ω on the hyperpolygon space $X(\alpha) = \mathcal{U}_S /\!\!/ G$ and we obtain:

Corollary 3.2. *The isomorphism* $\overline{\varphi} : (X(\alpha), \omega) \longrightarrow (M_P^H, d\lambda)$ *from Proposition 2.1 is a symplectomorphism.*

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