CONTACT GEOMETRY AND TOPOLOGY

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Two general remarks:

(i) **Why "topology" in the title?**: Differential topology studies smooth manifolds. Manifolds of the same dimension look locally the same, $\text{Diff}(M)$ is very large, properties that can tell two manifolds apart are global. Contact geometry retains features from differential topology: there are no contact invariants, the group of isomorphisms of a contact structure is very large and almost all problems are global.

(ii) **Contact geometry is not an isolated branch of geometry:** Quite on the contrary, it has strong links with symplectic geometry, complex geometry and CR geometry; ideas, techniques from the latter give hints about how to proceed in contact geometry. And the other way around, solutions to problems in contact geometry provide tools essentially in symplectic geometry.

A brief overview of the topics we are going to cover:

(1) **Introduction:** A bit of history, basic definitions and properties, and basic examples.
(2) **Isotopies, contact transformations and Gray's stability:** The group of contact transformations, with emphasis on the infinitesimal level.
(3) **Symplectic geometry and contact geometry I:** Liouville vector fields and pre-quantum line bundles.
(4) **Complex geometry and contact geometry I:** Convexity in complex geometry; Levi form and plurisubharmonic functions.
(5) **(Semi)-local normal forms:** Darboux' theorem and neighborhood theorems.
(6) **Symplectic cobordisms and contact geometry:** Weinstein's symplectic handles.
(7) **Symplectic and complex geometry and contact geometry II:** Open book decompositions.

Bibliography:

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- McDuff D., Salamon D., Introduction to symplectic topology. Clarendon Press, Oxford (1995).

1. Introduction

Notation:

- All manifolds, maps,... will be smooth.
- Manifolds will be connected and will have empty boundary unless otherwise stated (boundaries will be considered only when dealing with fillings and cobordisms). When the boundary is not open, structures on the manifolds are assumed to be restriction of structures in the open manifold (up to diffeomorphism) obtained by adding a small collar.

Definition 1. *A contact structure on a manifold M is given by a field of hyperplanes ξ (i.e. a codimension 1 distribution of TM) which is* maximally non-integrable. *The pair (M,ξ) is called a contact manifold.*

Integralibity: For each $x \in M$ there exist a local 1-form $\alpha \in \Omega^1(U)$ so that $\xi = \ker\alpha$ in U and

$$\alpha \wedge d\alpha = 0 \Leftrightarrow d\alpha_{|\xi} = 0$$

Maximal non-integrability: For each $x \in M$ there exist a local 1-form $\alpha \in \Omega^1(U)$ so that $\xi = \ker\alpha$ in U and

$$d\alpha_{|\xi_x} \text{ has no kernel,} \tag{1}$$

or equivalently the induced map

$$\begin{array}{rcl} d\alpha^{\#}_{|\xi_x} : \xi_x & \longrightarrow & \xi_x^* \\ u & \longmapsto & d\alpha_x(u,\cdot) \end{array}$$

is invertible.

Remark 1. *Maximal non-integrability does not depend on the 1-form whose kernel is ξ. Indeed any other 1-form must be $\alpha' = f\alpha$, f never vanishing,*

$$d(f\alpha) = fd\alpha + df \wedge \alpha,$$

so

$$d\alpha'_{|\xi_x} = d(f\alpha)_{|\xi_x} = fd\alpha_{|\xi_x}$$

Remark 2. *Being contact is a local notion, meaning that (i) the definition involves a computation that uses data in a neighborhood of any point and, (ii) it can be checked on any open cover of the manifold. Complex structures, symplectic structures are local. Riemannian structures are given by pointwise data (apart from the smoothness of the tensor).*

Remark 3. *If (M,ξ) is a contact manifold, it must have odd dimension. If we fix $u_1,\ldots,u_d$ a basis of ξ_x, and the corresponding dual basis on ξ_x^*, then $d\alpha^{\#}_{|\xi_x}$ is represented by an anti-symmetric matrix A. Since* $\det A = \det A^t$, *we conclude*

$$\det A = \det(-A^t) = (-1)^d \det A,$$

and therefore the result follows.

Following Klein, contact geometry is the study of those quantities/magnitudes, properties which remain invariant under the group of **contact transformations or contactomorphisms**, i.e. those $\phi \in \mathrm{Diff}(M)$ such that $\phi_*\xi = \xi$.

Definition 2. *Let (M,ξ) be a 2n+1-dimensional contact manifold. An submanifold $N \hookrightarrow M$ is called isotropic if*

$$TN \subset \xi$$

If in addition its dimension is n then it is called Legendrian.

The study of isotropic/Legendrian submanifolds of a contact manifold is one example of the problems dealt with in contact geometry.

1.1. **A bit of history.** The main reference is this subsection is [8].

Contact geometry did not appeared as the result of definition 1 being given out of nowhere and then its properties explored.

The first example of a contact manifold, together with the study of its contact properties, appeared in the work of Lie in 1872, linked to *the geometry of first order differential equations.*

Let us work the O.D.E. case: We seek for a function $u(x)$ such that

$$F(x, u(x), \dot{u}(x)) = 0, \tag{2}$$

where $F \in C^\infty(\mathbb{R}^3)$, $\mathbb{R}^3$ with coordinates x, u, p.

If we are to have a solution $u(x)$ then in particular it can be extended to a curve $\gamma(x) = (x, u(x), p(x))$, such that

$$\gamma \subset \{F \equiv 0\}$$

So we would like to know among the curves $\gamma \subset \{F \equiv 0\}$ which ones are of the form

$$\gamma(x) = (x, u(x), \dot{u}(x))$$

Notice that the way to detect that is to compute the tangent vector of the curve

$$\dot{\gamma}(x) = (1, \dot{u}(x), \dot{p}(x)) \in T_{(x,u,p)}\mathbb{R}^3$$

to get the equation

$$\dot{u}(x) = p(x) \text{ in } T_{(x,u,p)}\mathbb{R}^3, \tag{3}$$

Equation 3 makes sense without any reference to F, therefore it can be written for any $(x, u, p) \in \mathbb{R}^3$.

The closure of all directions $\dot{\gamma}(x)$ solving equation 3 defines a hyperplane

$$D_{(x,u,p)} \subset T_{(x,u,p)}\mathbb{R}^3$$

There is a second description: $\mathbb{R}^3 \simeq$ "contact elements (on $\mathbb{R}^2$)" $\ni l_{(x,u,p)}$ line through (x, u) with slope p.

Consider the projection

$$\begin{aligned} \pi \colon \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, u, p) &\mapsto (x, u) \end{aligned}$$

We define for each (x, u, p) the hyperplane

$$\xi_{(x,u,p)} := \pi_*^{-1}(l_{(x,u,p)})$$

$$l_{(x,u,p)} = \mathrm{Ker}(du - pdx), \;\; du - pdx \in T^*_{(x,u)}\mathbb{R}^2$$

therefore

$$\xi_{(x,u,p)} = \mathrm{Ker}(du - pdx), \;\; du - pdx \in T^*_{(x,u,p)}\mathbb{R}^3$$

Then ones sees that $\mathrm{Ker}(du - pdx)$ are the closure of the solutions of equation 3.

In particular $D = \xi$ is a contact distribution: indeed, $d(du - pdx) = -dp \wedge dx$. This is a 2-form on $\mathbb{R}^3$ no-where vanishing, therefore it has a 1-dimensional kernel spanned by $\partial/\partial u \in \mathfrak{X}(\mathbb{R}^3)$. It does not belong to ξ because its projection onto $T\mathbb{R}^2$ is the line with infinite slope.

Why is this geometric point of view useful? Because we have a correspondence between

(1) Curves $\gamma \subset \{F \equiv 0\}$, γ tangent to D, so that $\gamma(x) = (x, u(x), p(x))$ and
(2) solutions u of 2

Assume that $F \pitchfork \xi$ (genericity condition), then the intersection $TF \cap \xi$ defines a line field in F tangent to ξ. Its trajectories -when transversal to the projection onto the x-axis- give rise to (unique) solutions.

For O.D.E.'s this brings a geometric point of view, but we trade the initial O.D.E. by another one in F, so the difficulty is the same. Besides, the contact character of ξ does not enter at all. In any case it is useful when describing global solutions.

For 1st order P.D.E.'s

$$F(x_1, \ldots, x_n, u(x_1, \ldots, x_n), \frac{\partial u}{\partial x_1}(x_1, \ldots, x_n), \ldots, \frac{\partial u}{\partial x_n}(x_1, \ldots, x_n)) = 0, \qquad (4)$$

$F \in C^\infty(\mathbb{R}^{2n+1})$, $\mathbb{R}^{2n+1}$ with coordinates $x_1, \ldots, x_n, u, p_1, \ldots, p_n$

- There is an associated distribution ξ by hyperplanes on $\mathbb{R}^{2n+1}$ so that for any solution u,
$$L_u := \{(x_1, \ldots, x_n, u(x_1, \ldots, x_n), \frac{\partial u}{\partial x_1}(x_1, \ldots, x_n), \ldots, \frac{\partial u}{\partial x_n}(x_1, \ldots, x_n))\}$$
is an n-dimensional manifold tangent to ξ.
- The distribution ξ has an alternative description by pulling back the "contact elements in $\mathbb{R}^{n+1}$" $\ni H_{x,u,p}$ is the hyperplane through (x, u) not containing $\partial/\partial u \in \mathfrak{X}(\mathbb{R}^{n+1})$. One checks in this way that (i) ξ is given by the kernel of $du - \sum_{i=1}^n p_i dx_i$, (ii) it is a contact distribution and therefore L_u is a Legendrian submanifold. So we get a correspondence
  (1) Legendrians $L \subset \{F \equiv 0\}$, so that $L = (x, u(x), p(x))$ and
  (2) solutions u of 4
- In order to build the former, if $F \pitchfork \xi$ then the is a characteristic flow in $F \equiv 0$ defined up to scalar, so that any I^{n-1} (i) *isotropic submanifold*, (i) I transverse to the characteristic flow, gives rise to a Legendrian submanifold $L_I \subset F \equiv 0$.
- For any $S^{n-1} \subset \mathbb{R}^n$, and any $f \colon S \to \mathbb{R}$ generic, one can associate I_s a unique (n-1)-dimensional isotropic submanifold in F transverse to the characteristic flow, so that L_{I_s} comes from a function u, such that $u_{|S} = f$.

As a result solving the P.D.E. -under the genericity assumption- reduces to "algebraic operations" (implicit function theorem) plus solving an O.D.E. (and there is local uniqueness for the initial condition f).

Even more, Lie was interested in transformations $\phi \in \mathrm{Diff}(\mathbb{R}^{2n+1})$ sending solutions of (any) F to solutions of some F_ϕ. As we will see, those are necessarily transformations preserving the contact distribution, i.e.

$$\phi^*(du - \sum_{i=1}^n p_1 dx_i) = f(du - \sum_{i=1}^n p_1 dx_i),$$

f a no-where vanishing function.

Research on (global) contact geometry started in the 50's with

- Homotopic information (Chern),
- Deformations (Gray),
- Examples (Boothby-Wang),
- Exact contact manifolds all whose Reeb trajectories are closed (Reeb).

In the 70's and early 80's

- h-principle (Gromov, reducing existence of contact structures on open manifolds to a homotopic question),
- Surgeries (Lutz and Meckert),
- Examples related to links of isolated singularities (Thomas).

Late 80's, explosion, motivated by the renewed interest on symplectic geometry (contact geometry is the odd dimensional counterpart of symplectic geometry), and now there are two trends:

(1) Three dimensional contact topology (Eliashberg, Giroux, Honda, Etnyre,...).
(2) Higher dimensional contact topology (Eliahberg, Giroux, Weinstein, Geiges, Bourgeois, Thomas,...).

Contact geometry also appears in the formulation of themodynamics (late 19th century) and optics.

1.2. **Back to basic notions on contact geometry.**

Lemma 1. *(M^{2n+1}, ξ) contact iff for each $x \in M$ there exist a local 1-form α so that $\xi = \ker\alpha$ and $\alpha \wedge d\alpha^n(x) \neq 0$.*

Proof. $d\alpha_{|\xi_x}$ has no kernel $\Leftrightarrow d\alpha_{|\xi_x} \neq 0 \Leftrightarrow \alpha \wedge d\alpha^n(x) \neq 0$ □

Example 1. *On $\mathbb{T}^3 = \mathbb{R}/\mathbb{Z}^3$ with coordinates $\theta_1, \theta_2, \theta_3$ consider*

$$\alpha_n := \cos(n\theta_3)d\theta_1 + \sin(n\theta_3)d\theta_1, \ n \in \mathbb{Z}\backslash\{0\}$$

Then

$$\alpha_n \wedge d\alpha_n = -nd\theta_1 \wedge d\theta_2 \wedge \theta_3,$$

So each 1-forms induces a contact structure.

Maximal non-integrability geometrically for (M^3, ξ): About $x \in M^3$ there exist coordinates x_1, x_2, x_3 so that along vertical lines $x_1 = c_1, x_2 = c_2$ the planes rotate.

To construct the coordinates

- Pick Σ_x a (germ of) surface so that $\Sigma_x \pitchfork \xi$.
- Take local coordinates x_1, x_2 on Σ_x, with $\partial/\partial x_2(x) \in \xi$.
- Take and a local flow ϕ_t, $t \in [0, \epsilon]$, such that its integral curves are contained in ξ and are transversal to D_x. To construct the flow pick a plane Π_x transversal to D_x, and extend it locally. The intersection with ξ gives the desired 1-dimensional distribution.
- Extend x_1, x_2 using the flow and use the time as third coordinate

Choose a local 1-form

$$\alpha = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$$

Then

$$\alpha' = \frac{1}{f}\alpha = dx_1 + g dx_2$$

and lemma 1 implies

$$\frac{\partial g}{\partial x_3} \neq 0,$$

so the hyperplanes rotate along the flow lines, all in the same direction.

Exercise 1. *Show that by using $-g$ as third coordinate we make the rotation speed independent of x_3, and the contact form beomes the one described by Lie.*

Corollary 1. *For three dimensional contact structures we have a local normal form.*

1.2.1. *Co-orientability and exact contact structures.* Let D a codimension 1 distribution of TM. Then we have the exact sequence of vector bundles

$$0 \longrightarrow D \longrightarrow TM \longrightarrow TM/D \longrightarrow 0$$

Definition 3. *A codimension one distribution D of TM is co-orientable if the 1-dimensional distribution TM/D is trivial, i.e. if it has a no-where vanishing section V.*

Using a metric g on TM we can identify

$$D^{\perp} \simeq TM/D$$

and co-orientability is equivalent to the existence of a vector field X which is no-where tangent to D (in particular it never vanishes).

Definition 4. *(M, ξ) a contact manifold is co-orientable if ξ is co-orientable. A co-orientation is a choice of orientation for TM/ξ, i.e. a choice of positive transverse direction to ξ.*

Lemma 2. *(M, ξ) is co-orientable iff there exists $\alpha \in \Omega^1(M)$ so that $\xi = \ker\alpha$. Such an α is called a contact form, and it is defined up to multiplication by a no-where vanishing function. The pair (M, α) is called an exact contact manifold.*

Proof. The metric g gives a map $g^{\#} \colon TM \to T^*M$, and $g^{\#}(X)$ -X perpendicular to ξ- is a 1-form with the required properties.

Conversely out of α we define X solving

$$\alpha(X) = 1, \; X \perp \xi$$

□

So we have:

(M, α) exact $\Rightarrow$ (M, ξ) co-oriented $\Rightarrow$ (M, ξ) co-orientable.

Now if (M, α) is an exact contact manifold, then $\alpha \wedge d\alpha^n$ is a volume form. Then we have

Corollary 2.

- *If (M, ξ) is co-orientable then M is orientable and a choice of co-orientation induces an orientation.*
- *If (M, ξ) is co-orientable and has dimension $4k+3$ then it carries a canonical orientation by choosing the volume form associated to any contact form for ξ.*

Proof. If $\alpha' = f\alpha$ then

$$\alpha' \wedge d\alpha'^n = f^{n+1} \alpha \wedge d\alpha^n$$

□

We prefer to work with co-orientable contact manifolds because 1-forms can be added, multiplied by a function...so we are to have more tools available. At any rate, if (M, ξ) is not co-orientable then we can lift the contact structure to a co-orientable one ξ^{co} on the co-orientable double cover $M^{\mathrm{co}} \xrightarrow{\kappa} M$.

Indeed, fix a base point x and consider the homomorphism

$$\pi_1(M, x) \to \mathbb{Z}_2$$

sending $[c]$ to 1 if it preserves the orientation and -1 otherwise. This is a homomorphism whose kernel is a normal subgroup, since $\mathbb{Z}_2$ is abelian. Therefore its kernel determines up to isomorphism a covering space. We define $\xi^{\mathrm{co}} := \kappa^*\xi$.

Exercise 2. *Check that $(M^{\mathrm{co}}, \xi^{\mathrm{co}})$ is co-orientable.*

Remark 4. *The way to proceed in contact geometry is proving results for co-orientable contact manifolds, and to extend them we seek for equivariant constructions.*

1.3. **First examples.**

Example 2. *In $\mathbb{R}^{2n+1}$ with coordinates $x_1, \dots, x_{2n+1}$ we consider the 1-form*

$$\alpha_{\text{std}} = dx_{2n+1} + \frac{1}{2}\sum_{j=1}^{n}(x_{2j-1}dx_{2j} - x_{2j}dx_{2j-1}), \tag{5}$$

which is linear in the given coordinates. Another linear contact form is

$$\alpha_{\text{std}'} = dx_{2n+1} + \sum_{j=1}^{n} x_{2j-1}dx_{2j}, \tag{6}$$

Notice that $d\alpha_{\text{std}} = d\alpha_{\text{std}'}$.

Exercise 3. *Check that $\alpha_{\text{std}}, \alpha_{\text{std}'}$ are contact forms. More precisely, check that*

$$\alpha_{\text{std}} \wedge d\alpha_{\text{std}} = n!dx_1 \wedge \cdots \wedge d_{2n+1}$$

Observe that the contact form in Lie's example is the pullback of $\alpha_{\text{std}'}$ by the linear isomorphism which reverses all odd coordinates but the last one.

Given M, the manifold of contact elements (of M) is defined

$$\text{Ct}(M) := \{H_x \subset T_xM \text{ hyperplane}, x \in M\}$$

It is the total space of a fiber bundle

$$\mathbb{P}(T_x^*M) \hookrightarrow \text{Ct}(M) \xrightarrow{\pi} M$$

Notice that the fiber bundle

$$T_x^*M\backslash\{0\} \hookrightarrow T^*M\backslash\{\underline{0}\} \to M$$

projects onto $\text{Ct}(M) \xrightarrow{\pi} M$.

The contact distribution ξ is defined

$$\xi_{H_x} := \pi_{*H_x}^{-1}(H_x)$$

Proposition 1.

(1) $(\text{Ct}(M), \xi)$ *is a contact manifold.*

We use a cover that reduces everything to exercise 3. Take U_l a cover of M, so that $\pi^{-1}U_l \simeq U_l \times \mathbb{RP}^{m-1}$. Then take a further cover of projective space by affine charts to get

$$U_{lj} = \{H \in T_xU_i \mid \frac{\partial}{\partial x_j} \notin H\}$$

Coordinates $x_1, \dots, x_m, p_1, \dots, \hat{p}_j, \dots, p_m$ represent H with slopes $p_1, \dots, \hat{p}_j, \dots, p_m$: for $i \neq j$, the plane Π_{ij} spanned by $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}$ is transverse to H, therefore they intersect on a line on Π_{ij} spanned by $\frac{\partial}{\partial x_i} + p_i\frac{\partial}{\partial x_j}$.

It is clear from this description that the 1-form

$$\alpha = dx_j - \sum_{i \neq j} p_i dx_i$$

has ξ as kernel. By exercise 3 the 1-form is contact.

Each chart is nothing but with the manifold of "contact elements of $\mathbb{R}^n$" defined by Lie.

(2) *If $\phi \in \text{Diff}(M)$, then ϕ naturally indices a contact transformation on* $(\text{Ct}(M), \xi)$.

(3) *The contact manifolds* $\mathrm{Ct}(M)$ *are never co-orientable. Indeed, it is enough to show that the restriction to any fiber of* $T\mathrm{Ct}(M)/\xi$ *is not co-orientable. That is done locally. Take coordinates about* $x \in M$. *Pick a hyperplane* $H \in T_xM$. *Next take the geodesic* $v(\theta)$, $\theta \in [0, 2\pi]$ *orthogonal to it in* S^{n-1} *(use the Euclidean metric given by the coordinates). At each point it determines a hyperplane* H_θ *by taking the one orthogonal to* $\dot{v}(\theta)$. *Therefore, we get* (x, H_θ), $\theta \in [0, \pi]$ *a loop on the fiber. Then* $(\dot{v}(\theta), 0)$, $\theta \in [0, \pi]$ *is transverse to the contact distribution. Since* $\dot{v}(0) = -\dot{v}(\pi)$, *the coorientation is reversed.*

Since for m *odd the manifolds* $\mathrm{Ct}(\mathbb{R}^m) = \mathbb{R}^m \times \mathbb{RP}^{m-1}$ *are orientable (actually for any orientable* M*), we deduce that co-orientability and orientability are not equivalent.*

(4) $(\mathrm{Ct}(M), \xi)$ *is universal for codimension 1 distributions. Indeed, a codimension one distribution* D *is equivalent to a section*

$$\begin{array}{rcl} D\colon M & \longrightarrow & \mathrm{Ct}(M) \\ x & \longmapsto & D_x \end{array}$$

and then

$$D = D^*\xi$$

There is an index 2:1 cover from the manifold of oriented contact elements onto $\mathrm{Ct}(M)$

$$\kappa\colon \mathrm{Ct}^{\mathrm{or}}(M) \to \mathrm{Ct}(M),$$

endowing the former with a contact structure $\kappa^*\xi$.

By definition a fiber of $\mathrm{Ct}^{\mathrm{or}}(M)$ is $\mathbb{S}(T_x^*M)$, the result of identifying half lines in $T_x^*M\backslash\{0\}$.

Proposition 2.

(1) $(\mathrm{Ct}^{\mathrm{or}}(M), \kappa^*\xi)$ *is the co-orientable double cover* $(\mathrm{Ct}(M), \xi)$.

(2) *A co-oriented distribution is the same a section of* $\mathrm{Ct}^{\mathrm{or}}(M) \to M$. *A distribution* $D\colon M \to \mathrm{Ct}(M)$ *is co-orientable iff it admits a lift to*

$$\kappa\colon \mathrm{Ct}^{\mathrm{or}}(M) \to \mathrm{Ct}(M)$$

There is a second description of the contact structure $\kappa^*\xi$:

Fix a metric on g and let $S_g(T^*M)$ be the unit sphere bundle of T^*M. Then we have the maps

$$S_g(T^*M) \hookrightarrow T^*M\backslash\{\underline{0}\} \to \mathrm{Ct}^{\mathrm{or}}(M)$$

and the composition is a diffeomorphism Φ. In T^*M consider the Liouville 1-form which in dual coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$ is

$$\lambda_{\mathrm{liouv}} = \sum_{j=1}^{n} p_j dx_j$$

Recall that $(T^*M, \lambda_{\mathrm{liouv}})$ has the universal property for sections of T^*M (1-forms on M).

Exercise 4. *Check that* $\ker \Phi^*\lambda_{\mathrm{liouv}} = \kappa^*\xi$, *where* $\lambda_{\mathrm{liouv}} \in \Omega^1(T^*M\backslash\{\underline{0}\})$.

Example 3. *Consider* $\mathcal{J}^1M = T^*M \oplus \mathbb{R}$. *In other words*

$$\mathcal{J}^1_xM := [f]^1_x,$$

where $[f]^1_x = 0$ *if* $f(x) = 0$ *and* $df_x = 0$.

Let z *denote the coordinate of the* $\mathbb{R}$ *factor, and let* λ *denote the pullback by* $\mathcal{J}^1M \to T^*M$ *of the Liouville 1-form. Then*

$$dz - \lambda$$

defines a contact 1-form.

1.4. **Almost contact manifolds.** What are the obstructions to the existence of (co-orientable) contact structures on an odd dimensional manifold?

We will study what kind of additional structure we get.

Lemma 3. *A contact form α on M gives a reduction of the structural group of TM to $1 \oplus Sp(2n)$, and since the later summand can be reduced to $U(n)$ (by putting an almost complex structure along ξ compatible with $d\alpha$), we conclude that the structural group of a co-orientable contact manifold reduces to $1 \oplus U(n)$.*

Proof. Recall that for a given manifold M, a smooth structure gives rise to P the principal $\mathrm{Gl}(m, \mathbb{R})$-bundle of frames of the tangent bundle, the fiber over $x \in M$ being the frames of T_xM. It is the principal bundle associated to TM. A reduction to a subgroup $H < \mathrm{Gl}(m, \mathbb{R})$ amounts to a choice of trivializations so the the transition functions take values of H.

Recall as well that a reduction is a monomorphism from a principal H bundle into P. This is seen to be equivalent to a section of the G associated bundle $P \times_G G/H$ with classes gH, and right G-action

$$g * g'H = g^{-1}g'H$$

Therefore, one can speak of reductions being homotopic (as sections of the aforementioned associated bundle).

The 1-form α, together with a metric say, gives the splitting $TM = \underline{\mathbb{R}} \oplus \xi$ (where the trivialization is such that $\alpha(X) = 1$). Thus we obtain a reduction to $1 \oplus \mathrm{Gl}(2n, \mathbb{R})$ (where $1 = \mathrm{SO}(1)$).

The contact form endows ξ with a symplectic vector bundle structure $(\xi, d\alpha)$; in appropriate trivializations each fiber becomes $(\mathbb{R}^{2n}, d\alpha_{\mathrm{std}})$, and therefore we get a reduction to $1 \oplus \mathrm{Sp}(2n, \mathbb{R})$.

An almost complex structure on a vector bundle E with fiber $\mathbb{R}^{2n}$ is an isomorphism $J : E \to E$ so that $J^2 = -\mathrm{Id}$. On appropriate trivializations fibers become $(\mathbb{R}^{2n}, i) = \mathbb{C}^n$, so we get a reduction from $\mathrm{Gl}(2n, \mathbb{R})$ to $\mathrm{Gl}(n, \mathbb{C})$.

An almost complex structure J compatible with $(\xi, d\alpha)$ is one such that

$$d\alpha(J\cdot, J\cdot) = d\alpha(\cdot, \cdot),\ d\alpha(\cdot, J\cdot) > 0,$$

and suitable trivializations locally identify fibers with $(\mathbb{R}^{2n}, d\alpha_{\mathrm{std}}, i)$. Since

$$\mathrm{Gl}(n, \mathbb{C}) \cap \mathrm{Sp}(2n, \mathbb{R}) = \mathrm{U}(n, \mathbb{C}),$$

the existence of the c.a.c.s. is equivalent to the sought for reduction.

Compatible almost complex structures always exist: indeed there is a retraction

$$r : \mathrm{Sp}(2n, \mathbb{R}) \to \mathrm{U}(n, \mathbb{C})$$

given by shrinking the non-unitary factor in polar decomposition. Therefore, the inclusion $\mathrm{U}(n, \mathbb{C}) \hookrightarrow \mathrm{Sp}(2n, \mathbb{R})$ is a homotopy equivalence, so by the long exact sequence the homogeneous space $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n, \mathbb{C})$ is contractible.

Obstruction theory tells us that if a fiber bundle has contractible fiber then it always has a global section, and all such sections are homotopic (actually we only need vanishing of the homotopy up to m-1 (m for the homotopy result), where m is the dimension of the base. □

Once we have a (compatible) almost complex structure J on ξ, we can define its total Chern class

$$c(\xi; J) \in H^*(M; \mathbb{Z})$$

Since space of c.a.c.s. is contractible the total Chern class is independent of J.

Definition 5. *An odd dimensional manifold M with a reduction of the structural group of TM to $1 \oplus U(n)$ is called almost contact.*

Being almost contact is the only known obstruction for a manifold to be contact. Moreover, the main conjecture in contact geometry asserts:

Conjecture 1. *Every almost contact manifold is (exact) contact, and the reduction given by the almost contact structure is homotopic to the one given by the contact structure.*

Theorem 1 (Gromov)**.** *The conjecture holds true for open manifolds* [9] *(or see the more readable* [4]*).*

2. Isotopies, contactomorphisms and Gray's stability

Given a manifold M with at least a co-oriented contact structure, we want to have a better understanding of the set of co-oriented contact structures $\mathcal{C}^{\rm co}(M)$. Of course, we want to introduce and study the relevant question for the corresponding topological space.

Notice

$$\mathcal{C}^{\rm co}(M) \subset \Gamma(M, {\rm Ct}^{\rm or}(M)) \subset C^\infty(M, {\rm Ct}^{\rm co}(M))$$

For the latter we have the C^r topologies, $r \in \mathbb{N} \cup \infty$. We will not distinguish between weak and strong, for at some point all manifolds will be compact.

We will endow $\mathcal{C}^{\rm co}(M)$ and $\Gamma(M, {\rm Ct}^{\rm or}(M))$ with the topology induced by the C^1-topology on $C^\infty(M, {\rm Ct}^{\rm co}(M))$, and call it the C^1-topology.

We claim:

Lemma 4. *$\mathcal{C}^{\rm co}(M)$ is an open subset of $\Gamma(M, {\rm Ct}^{\rm or}(M))$ (for the C^1-topology).*

Proof. It can be seen that to preserve the contact condition exactly C^1-control on the distribution is needed. But as usual working with 1-form is more convenient. Consider the sets $\Omega^1_{\rm cont}(M) \subset \Omega^1_{\rm nw}(M)$ of contact and no-where vanishing 1-forms respectively. Again, we make them into topological spaces by putting the topology coming from the C^1-topology on $C^\infty(M, T^*M)$.

Then we have the commutative diagram

$$\begin{array}{ccc} \Omega^1_{\rm cont}(M) & \xrightarrow{\ker} & \mathcal{C}^{\rm co}(M) \\ \downarrow & & \downarrow \\ \Omega^1_{\rm nw}(M) & \xrightarrow{\ker} & \Gamma(M, {\rm Ct}^{\rm or}(M)) \end{array}$$

where the horizontal arrows are surjective.

We claim that the lower row is a continuous open map for the C^1-topologies (go to charts to check this).

Therefore, the C^1-topology on $\Gamma(M, {\rm Ct}^{\rm or}(M))$ is the quotient topology, meaning that many topological problems for the latter can be lifted to equivalent problems on $\Omega^1_{\rm nw}(M)$.

In particular one checks easily that $\Omega^1_{\rm cont}(M)$ is open, since the contact condition

$$\alpha \wedge d\alpha^n \neq 0$$

involves exactly C^1-information on α. Thus $\ker(\Omega^1_{\rm cont}(M)) = \mathcal{C}^{\rm co}(M)$ is an open subset. □

We want to understand the path connected components

$$\mathcal{C}^{\rm co}(M)_i, \ i \in I$$

of $\mathcal{C}^{\rm co}(M)$.

Definition 6. *We say that $\xi, \xi' \in \mathcal{C}^{\rm co}(M)$ are contact homotopic if there exist $\xi_t \in \Gamma(M, \mathrm{Ct}^{\rm or}(M))$ a*
homotopy so that $\xi_0 = \xi$, $\xi_1 = \xi'$.

Observe that a homotopy between ξ, ξ' can be deformed to be smooth (by approximation results), so in particular it becomes a continuous path in the C^1-topology.

Some homotopies come from "deformations" of M, i.e. from isotopies, and hence they should not be taken into account.

Definition 7. *We say that $\xi, \xi' \in \mathcal{C}^{\rm co}(M)$ are contact isotopic if there exist $\phi \in \mathrm{Diff}_0(M)$ such that $\xi' = \phi_*\xi$. This is equivalent to saying that $\phi_t, t \in [0,1]$ a path of diffeomorphisms starting at the identity so that*

$$\phi_*\xi = \xi'$$

In particular contact isotopic forms are contact homotopic.

For contact forms the above condition becomes

$$\phi^*\alpha' = f\alpha,$$

where f is strictly positive.

In other words, on $\mathcal{C}^{\rm co}(M)$ we have the action of $\mathrm{Diff}_0(M)$. Notice that for $\xi \in \mathcal{C}^{\rm co}(M)$ the stabilizer of the action is

$$\mathrm{Stb}_\xi = \mathrm{Cont}(M, \xi) \cap \mathrm{Diff}_0(M)$$

And we want to study the orbit space $\mathcal{C}^{\rm co}(M)/\sim$.

Theorem 2 (Gray)**.** *Let M be compact and $\mathcal{C}^{\rm co}(M)_i \neq \varnothing$, then*

$$\mathcal{C}^{\rm co}(M)_i/\sim = \{\mathrm{point}\}$$

In other words, the $\mathrm{Diff}_0(M)$-orbit of ξ is all $\mathcal{C}^{\rm co}(M)_i$, i.e. if $\xi, \xi' \in \mathcal{C}^{\rm co}(M)$ can be joined by a continuous path in $\mathcal{C}^{\rm co}(M)$, then there exist $\phi \in \mathrm{Diff}_0(M)$ such that $\xi' = \phi_*\xi$ (so they are the same contact structure up to a "global change of coordinates" coming from a global deformation).

Corollary 3. *If M is compact co-oriented contact structures (resp. exact contact forms) are stable, i.e. about any such structure there exist an open neighborhood (in the C^1-topology) so that any other co-oriented distribution (resp. no-where vanishing 1-form) in the neighborhood can be conjugated to the original one.*

Proof. Given $\xi \in \mathcal{C}^{\rm co}(M)$ take $\alpha \in \Omega^1_{\rm cont}(M)$ with $\xi = \ker\alpha$.

We claim that there exist $\mathcal{N}_\alpha \subset \Omega^1_{\rm nw}(M)$ a small open neighborhood of α made of contact forms. If so, $\ker(\mathcal{N}_\alpha)$ solves the problem.

To prove the claim choose $\mathcal{N}_\alpha$ so that for any $\alpha' \in \mathcal{N}_\alpha$ the convex combination $(1-t)\alpha' + t\alpha$ is by contact forms. This is always possible. Then α' is contact isotopic to α, and by the previous theorem contact isotopic. □

2.1. **Contact Hamiltonians.** Let fix a contact 1-form α for (M, ξ), M compact.

We will see that $\mathrm{Cont}(M, \xi)$ is an (infinite dimensional) "Lie group", in the same sense as $\mathrm{Diff}(M)$ is a "Lie group".

In a Lie group G with Lie algebra $\mathfrak{g}$, left multiplication gives a 1 to 1 correspondence between

- geodesics on $\mathfrak{g}$ with velocity v (i.e. $v \in \mathfrak{g}$) and
- curves $\exp(tv)$ through origin with speed v which are left invariant.

The latter is the integral curve of the left invariant vector field V with $V(e) = v$. Equivalently,

$$\exp(\cdot v)\colon \mathbb{R} \to G$$

is the unique homomorphism integrating $t \mapsto tv \in \mathfrak{g}$.

We know vector fields are the "Lie algebra" of $\mathrm{Diff}(M)$ in the same sense. Given $X \in \mathfrak{X}(M)$, we get a homomorphism $t \mapsto \phi_t^X$ determined by

$$\frac{d}{dt}\phi_t(x) = X(\phi_t(x))$$

In a Lie group, left multiplication further identifies

- curves on $\mathfrak{g}$ and
- curves $c(t)$ through origin

We also have an identification between 1-parameter families of vector fields and maps

$$\mathbb{R} \to \mathrm{Diff}(M)$$

sending 0 to the identity.

The identification can be given by the equation formula

$$\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x)) \tag{7}$$

More conceptually, a 1-parameter family of vector fields X_t is identified with the vector field

$$\hat{X} := X_t + \frac{\partial}{\partial t} \in \mathfrak{X}(M \times \mathbb{R})$$

Hence it can be identified with a flow $\Psi_t^{\hat{X}}$ in $M \times \mathbb{R}$, which together with the projection gives rise to ϕ_t.

We want to find out conditions which identify vector fields giving rise to contactomorphisms. If we have a path ϕ_t in $\mathrm{Cont}(M, \xi)$, $\phi_0 = \mathrm{Id}$, then we have

$$\phi_t^*\alpha = f_t\alpha,$$

and infinitesimally

$$\phi_t^* L_{X_t}\alpha = \frac{d}{dt}\phi_t^*\alpha = \frac{d}{dt}f_t\alpha = \frac{d}{dt}f_t\frac{1}{f_t}\phi_t^*\alpha = \phi_t^* h_t\alpha,$$

with $h_t = \frac{d}{dt}f_t\frac{1}{f_t} \circ \phi_t^{-1}$.

So we are led to

Proposition 3. *X_t integrates into a 1-parameter family of contactomorphisms iff*

$$L_{X_t}\alpha = h_t\alpha \tag{8}$$

Exercise 5. *Finish the proof of proposition 3 showing that if $L_{X_t}\alpha = h_t\alpha$, then the flow satisfies*

$$\phi_t^*\alpha = e^{\phi_t^* h_t}\alpha$$

Definition 8. *A vector field $X \in \mathfrak{X}(M)$ is contact (for (M, ξ)) if $L_X\alpha = h\alpha$, $h \in C^\infty(M)$.*

From proposition 3 we deduce

Proposition 4. *Contact vector fields are those whose flow preserves ξ.*

Exercise 6. *Show that contact vector fields are closed under the Lie bracket, so they form a "Lie subalgebra" of $\mathfrak{X}(M)$.*

Recall that if α is a contact form $d\alpha$ is non-degenerate on the hyperplanes ξ_x. Since T_xM is odd dimensional, $d\alpha$ must have a kernel, which in light of the maximal non-degeneracy of ξ has to be 1-dimensional.

Definition 9. *The Reeb vector field R is the unique vector field determined by the conditions*

$$i_R\alpha = 1,\ i_R d\alpha = 0$$

In other words is a no-where vanishing vector field on $\ker d\alpha$ *suitably normalized.*

Notice that

(1) the flow of R preserves the contact form:

$$L_R\alpha = di_R\alpha + i_R\alpha = 0,$$

so in particular the Reeb vector field is contact.

(2) A change in the contact form to $\alpha' = f\alpha$ changes the Reeb vector field.

Therefore, each no-where vanishing function f produces a contact vector field X so that

$$i_X\alpha = \frac{1}{f}$$

More generally if we decompose $X \in \mathfrak{X}(M)$,

$$X = -hR + Z, \quad Z \in \Gamma(\xi)$$

The contact condition is $L_X\alpha = g\alpha$ can be expanded

$$-dh + iZd\alpha = g\alpha \tag{9}$$

If in equation 9 h is given, there exist only one solution with

$$g = -i_R dh\,,\; i_Z d\alpha = dh_{|\xi}$$

Proposition 5.

(1) *There is a 1 to 1 correspondence between $C^\infty(M)$ and contact vector fields. The correspondence sends each vector field to its* **contact Hamiltonian***. It is linear, homeomorphism w.r.t C^∞-topology and support preserving.*

(2) *There exists no contact vector field everywhere tangent to ξ.*

(3) *A contact vector field is Reeb for some rescaled form iff it is everywhere transversal to ξ.*

Exercise 7. *Proof proposition 5. Show also that using proposition we can identify paths in* $\mathrm{Cont}(M,\xi)$ *with paths in $C^\infty(M)$.*

Remark 5. *Point 1 in proposition 5 and exercise 7 are extremely useful to extend isotopies of contact transformations defined in domains of contact manifolds (for example in tubular neighborhoods of compact submanifolds), also keeping control of the support of the isotopy.*

Remark 6. *Exercise 6 together with proposition 5 implies that the contact form can be used to endow $C^\infty(M)$ with a* **local Lie algebra** *structure in the sense of Kirillov, that is a map of sheaves*

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

which does not increase support, is anti-symmetric and satisfies Jacobi identity [11, 12].

Remark 7. *Reeb vector fields/positive functions define a positive cone. They give rise to a notion of positive path of contact transformation* [5]*: These are paths associated to 1-parameter families of positive functions. Existence of a partial order on the universal cover of (M,ξ) -equivalent to the absence of non-trivial positive loops- is related to squeezing questions* [5, 6] *(i.e., whether certain regions of a contact manifold can be contact isotoped into others).*

It is worth observing that point 2 in proposition 5 implies that one cannot find a 1-parameter group of contact transformations everywhere tangent to ξ. Therefore, if we want to understand the action of $\mathrm{Diff}_0(M)$ on contact structures up to contactomorphism, it is reasonable to work infinitesimally with vector fields tangent to ξ.

Proof of Gray's stability theorem. The proof uses Moser's method along ξ_t. Indeed, let α_t a 1-parameter family of contact forms. Let's assume

$$\phi_t^* \alpha_t = f_t \alpha_0 \tag{10}$$

for an isotopy associated to X_t tangent to ξ_t. If this were the case, then differentiating 10 we get

$$\phi_t^*(i_{X_t} d\alpha_t + \frac{d}{dt}\alpha_t) = \frac{d}{dt} f_t \alpha_0 = \frac{d}{dt} f_t \frac{1}{f_t} \phi_t^* \alpha_t = \phi_t^* h_t \alpha_t \tag{11}$$

Clearly,

$$i_{X_t} d\alpha_t + \frac{d}{dt}\alpha_t = h_t \alpha_t \tag{12}$$

has a unique solution along ξ_t for any h_t. Obviously, for a unique

$$h_t := d/dt \alpha_t(R_t)$$

the previous solution gives an equality of 1-forms. □

Exercise 8. *Show that if two non-co-oriented contact structures are contact homotopic, then they are contact isotopic.*

Hint: Work in the co-orientable double cover and make sure an equivariant construction is available.

3. Contact geometry and symplectic geometry I.

Recall that a symplectic form $\Omega \in \Omega^2(Y)$ is such that

(1) Ω has no kernel.
(2) $d\Omega = 0$.

Definition 10. *Let (M, α) a contact manifold. Then its symplectization is the manifold $M \times (-\infty, \infty)$ with symplectic form $\Omega = d(\mathrm{e}^t \alpha)$, where t is the coordinate of the real line.*

To recover the contact structure we use the more general result.

Definition 11. *Let (M, Ω) be a symplectic manifold. A Liouville vector field X is a vector field such that*

$$L_X \Omega = \Omega$$

Proposition 6. *Let (Y, Ω) be a symplectic manifold, X a Liouville vector field and H a hypersurface such that $H \pitchfork X$. Then $\alpha := i_X \Omega_{|H}$ is a contact form.*

Proof. Since restriction (pullback) commutes with exterior derivative we have

$$d\alpha = d i_X \Omega_{|H} = L_X \Omega_{|H} = \Omega_{|H}$$

So we have to prove that $\Omega_{|\ker\alpha}$ is symplectic. Since Ω is symplectic, we know that $\mathrm{Ann}^\Omega(X)$ is a hyperplane containing X; more precisely

$$\mathrm{Ann}^\Omega(X) = \langle X \rangle \oplus \ker\alpha$$

The kernel of $\Omega_{|\mathrm{Ann}^\Omega(X)}$ is spanned by X, so the symplectic form descends to the quotient

$$(\mathrm{Ann}^\Omega(X)/X, \Omega) \simeq (\ker\alpha, d\alpha)$$

□

Observe that by definition in the symplectization of (M, α) the vector field $\partial/\partial t$ is Liouville. A simple computation shows that for the hypersurface $t = 0$ we have

$$i_{\frac{\partial}{\partial t}} \Omega_{|t=0} = \alpha$$

More generally we conclude

Corollary 4. *For any $f \in C^\infty(M)$ the hypersurface $t = f(x)$ inherits a contact structure. If M is compact all such structures are contact diffeomorphic.*

Proof. By proposition 6 any such hypersurface is contact. Given any such two hypersurfaces H_1, H_2 defined by f_1, f_2, take the convex combination

$$f_t = (1-t)f_1 + tf_2$$

Then each H_t inherits a contact form α_t. Now the restriction of the projection $p_1 \colon H_t \to H_1$ gives a diffeomorphism, producing $\alpha_t' = p_{2*}\alpha_t \in \Omega^1_{\mathrm{cont}}(H_1)$. By Gray's theorem $\ker p_{1*}\alpha_2$ is contact isotopic to $\ker\alpha_1$. □

Remark 8. *One might think that contact topology reduces to symplectic geometry techniques invariant under the Liouville vector field $\partial/\partial t$. This is not true because the symplectization is a non-compact symplectic manifold, for which not many techniques from symplectic topology are available. Therefore, contact geometry must develop its own techniques, often inspired in ideas from symplectic geometry (see for example the proof of Gray's stability theorem).*

Definition 12. *A contact structure on M is called strongly symplectically fillable when it is obtained as in proposition 6 with $M = \partial Y$ and X defined in a neighborhood of ∂X, i.e. when M is the* **strong convex boundary** *of (Y, Ω, X).*

Exercise 9. *Let $S^{2n-1} = \partial B^{2n}(1) \subset \mathbb{R}^{2n}$, with coordinates $x_1, \dots, x_{2n}$. Consider the standard constant symplectic form*

$$\omega_{\mathrm{std}} = \sum_{j=1}^{n} dx_{2j-1} \wedge dx_{2j}$$

The radial vector field

$$X = \sum_{i=j}^{2n} x_j \frac{\partial}{\partial x_j}$$

is Liouville. Since it is transversal to S^{2n-1}, the latter inherits a contact form which is

$$i_X \omega_{\mathrm{std}|TS^{2n-1}} = (\sum_{j=1}^{n} x_{2j-1} dx_{2j} - x_{2j} dx_{2j-1})_{|TS^{2n-1}}$$

The corresponding co-oriented contact structure is the so called standard contact structure ξ_{std} which is also strongly fillable by definition (and any ellipsoid will inherit a contact structure contact isotopic to the one on the sphere).

3.1. **Boothby-Wang examples.** It is well know that there is a 1 to 1 correpondence between isomorphism classes of complex line bundles over M, and $H^2(M;\mathbb{Z})$, the map given by

$$(L \to M) \mapsto c_1(L)$$

The correspondence can be refined as follows: there is no loss of generality in considering isomorphism classes of complex line bundles with hermitian metric $(L, h = \langle \cdot, \cdot \rangle)$ (so we reduce the structural group from $\mathrm{Gl}(\mathbb{C}, 1) = \mathbb{C}^*$ to $U(1) = S^1$). For these, one can consider the sphere bundle

$$S(L) := \{l \in L \mid \langle l, l \rangle = 1\} \subset L,$$

which is a principal S^1-bundle.

Any line bundle admits a **connection** ∇. One can look at a connection in equivalent ways:

(1) A splitting $TL = T^vL \oplus \mathcal{H}$ so that $\lambda_*\mathcal{H} = \mathcal{H}, \lambda \in \mathbb{C}^*$ (in particular it must be tangent to the zero section).

(2) An operator

$$\nabla\colon \Omega^1(M)\otimes\Gamma(L)\to\Gamma(L)$$

subject to the Leibniz rule

$$\nabla fs = dfs + f\nabla s$$

so that one can make sense of differentiating sections of L

Notice that the first description is equivalent to giving

$$A\in\Omega^1(L\backslash\{\underline{0}\},\mathbb{C}=\mathrm{Lie}(\mathbb{C}^*))^{\mathbb{C}^*},$$

and so that along vertical directions describes the action of $\mathbb{C}$.

To go from 1 to 2 we just notice that given $s\in\Gamma(L)$, we can split

$$D_us(x)=D_us(x)^v+D_us(x)^h,\; D_us(x)^v\in T^v_{s(x)}L,\; D_us(x)^h\in\mathcal{H}_{s(x)}$$

In particular, if $\nabla s(x)=0$, then $\mathcal{H}_{s(x)}=Ts(x)$.

If $s\in\Gamma(U)$ is no-where vanishing, then

$$\nabla s/s\in\Omega^1(U;\mathbb{C})$$

Moreover, we can always choose it compatible with the hermitian metric. Using the second point of view it means

$$\mathrm{d}\langle s,t\rangle=\langle\nabla s,t\rangle+\langle s,\nabla t\rangle,\quad s,t\in\Gamma(L) \tag{13}$$

We claim that for a hermitian connection is tangent to $S(L)$: notice that if $h(s,s)=1$ then equation 13 implies

$$\nabla s/s+\overline{\nabla s/s}=0,$$

or equivalently

$$\nabla s/s\in\Omega^1(U;i\mathbb{R})$$

At each x, take $f\in\mathbb{C}^\infty(U)$ so that

$$\nabla s/s(x)+idf(x)=0\,,\;f(x)=0$$

Leibniz's rule imply that $\mathrm{e}^{if}s$ is tangent to $\mathcal{H}(x)$, and this proves the claim.

The curvature of a hermitian connection F_∇ belongs to $\Omega^2(M;i\mathbb{R})$. It can be computed locally using a non-vanishing section as

$$d\nabla s/s\in\Omega^2(U;i\mathbb{R})$$

Indeed, if $t=fs$, f no-where vanishing,

$$\frac{\nabla(fs)}{fs}=\frac{df}{f}+\frac{\nabla s}{s}=d\ln f+\frac{\nabla s}{s},$$

and therefore

$$d\frac{\nabla(fs)}{fs}=d\frac{\nabla(s)}{s}$$

The curvature is a closed form and such that

$$\left[-\frac{1}{2\pi i}F_\nabla\right]=c_1(L)\in H^2(M;\mathbb{Z})\otimes\mathbb{R}$$

The previous procedure can be reversed, so we get

Theorem 3 (Weil)**.** *Any $a\in\Omega^2(M)$ such that $da=0$ and $[a]\in H^2(M;\mathbb{Z})$, determines a hermitian complex line bundle with compatible conection $(L,\langle\cdot,\cdot\rangle,\nabla)$ so that*

$$-\frac{1}{2\pi i}F_\nabla=a,$$

the connection being unique up to the addition of idf, $f\in C^\infty(M)$.

Notice that the connection ∇ restricted to $S(L)$ is given by $-iA_{|S(L)} \in \Omega^1(S(L))$, an S^1-invariant 1-form so that $A(R) = 1$, where R is the generator of the S^1-action (with period 2π).

One has

$$-idA = -ip^*F_\nabla, \tag{14}$$

where p is the projection $S(L) \to M$.

Exercise 10. *Prove equation 14.*

Hint: Trivialize $S(L)$ by a (unitary) section s. Then

$$S(L) = U \times \mathbb{R}/2\pi\mathbb{Z}$$

and $\mathcal{H}$ is the kernel of $i\nabla s/s - dz$, where z is the coordinate on $\mathbb{R}$.

Theorem 4 (Boothby-Wang)**.** *A is a contact form if and only if the curvature is a symplectic form.*

Corollary 5. *Each symplectic manifold has an associated exact contact structure on its* **pre-quantum line***. Two such exact forms differ by the action of the gauge group. The Reeb vector field is the generator of the S^1-action.*

Example 4. *Since $(T^*M, -d\lambda_{\text{liouv}})$ is an exact symplectic manifold, the Chern class of the pre-quantum line bundle is trivial and hence the pre-quantum line bundle is*

$$S^1 \times T^*M$$

We can work on the covering $\mathbb{R} \times T^*M$. There we need to put a connection invariant under the $\mathbb{R}$-action by translations, and whose differential is $p^* - d\lambda_{\text{liouv}}$. Clearly we can choose

$$dz - p^*\lambda_{\text{liouv}}$$

Notice that we recover the exact contact manifold

$$(\mathcal{J}^1M, dz - p^*\lambda_{\text{liouv}})$$

Example 5. *The standard contact structure on S^{2n+1} comes from the Boothby-Wang construction, where the base manifold is $\mathbb{CP}^n$.*

Indeed, consider the projection

$$\mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{CP}^n$$

The tautological complex line bundle $\mathcal{O}(-1)$ has fiber

$$\mathcal{O}(-1)_{[v]} = \mathbb{C}v = \pi^{-1}(v) \cup \{0\}$$

Recall that we can restrict ourselves to $\mathcal{O}(-1)\backslash\{\underline{0}\}$

But notice

$$\mathcal{O}(-1)\backslash\{\underline{0}\} \simeq \backslash\{0\}$$

On $\mathbb{C}^{n+1}$ we use the standard hermitian metric

$$\langle v, w\rangle = v\bar{w}$$

which we restrict to each line through the origin, so we get a hermitian metric on the tautological line bundle. Next the connection is given by the distribution

$$\mathcal{H}_v = \mathbb{C}v^\perp$$

It is clear that $\mathcal{H}$ is invariant under the $\mathbb{C}^*$-action (notice that the S^1-action is by unitary transformations).

We next prove that it is hermitian: For local sections $s, t\colon U \subset \mathbb{C}^n \to \mathbb{C}^{n+1}$ we have

$$\langle ds, t\rangle = \langle \nabla s, t\rangle$$

Since

$$d\langle s, t\rangle = \langle ds, t\rangle + \langle s, dt\rangle$$

the result follows.

Notice that the 1-form defining $\mathcal{H}$ *is given by*

$$A_v = \frac{1}{\langle v, v\rangle}\langle \cdot, v\rangle$$

When restricted to the sphere becomes

$$A_v = \langle \cdot, v\rangle$$

In coordinates $x_1, y_1, \ldots, x_{n+1}, y_{n+1}$ *one easily checks*

$$-iA_{|TS^{2n+1}} = (\sum_{j=1}^{n+1}(x_{2j_1}dx_{2j} - x_{2j}dx_{2j-1}))_{|TS^{2n+1}}$$

Since we know that in the r.h.s. we have a contact form, we also conclude that $-\frac{i}{2}F_\nabla$ *is a symplectic form on* $\mathbb{CP}^{n+1}$*, the so called* **Fubini-Study symplectic form.**

3.2. **More on sphere bundles:** Assume that $(Y, d\alpha)$ is an exact symplectic manifold. Then by Cartan's formula the unique vector field solving the equation

$$i_X d\alpha = \alpha$$

is Liouville.

Let (Y, α) as above be $(T^*M, -d\lambda_{\text{liouv}})$. In dual coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$, the vector field is

$$X = \sum_{j=1}^{n} p_j \frac{\partial}{\partial p_j}$$

By proposition 6 for any hypersurface transverse to X the restriction of α is a contact form. In particular this is the case of the sphere bundle of T^*M w.r.t. any metric. Since any two metrics can be joined by a path, by Gray's stability we give another proof about the contact structures on different sphere bundles being contactomorphic (for compact base).

Exercise 11. *On* $S(T^*M)$*, for a fixed Riemannian metric* g*, we can consider the dual of the geodesic flow. One possible definition is that it is the flow associated to the following vector field* $X \in \mathfrak{X}(T^*M)$*: On a point* (x, p)*, it lifts the vector on* T_xM *dual (w.r.t* g*) to* p*. The lift is given by the dual of Levi-Civita connection on* T^*M*. The dual geodesic flow preserves the dual metric induced on* T^*M*, in particular it is tangent to* $S(T^*M)$*. Show that this flow is by contactomorphisms, and its the contact hamiltonian is*

$$h(x, p) = -|p|^2$$

Hint: For $(x, p) \in S(T^*M) \cong \mathrm{Ct}^{\mathrm{or}}(M)$*, show that the projection* $\pi_* X(x, p)$ *is orthogonal to the contact element (hyperplane on* T_xM*). Parallel transport along a geodesic of* $v \in T_xM$ *can be performed as follows: take a curve representing it and parallel translate the velocity of the geodesic. Then push the curve using the geodesic flow, to get a curve whose derivative is the parallel transport. Deduce from this and the previous fact that* ϕ^X_{t*} *preserves the contact distribution. Therefore* X *integrates into a Hamiltonian isotopy. To compute the contact Hamiltonian, just evaluate* λ_{liouv} *on* X*.*

4. Contact geometry and complex geometry I

Let (Y, J) be a complex manifold and $H \subset Y$ a hypersurface. The hypersurface inherits a CR structure (of hypersurface type), the distribution ξ being $TH \cap JTH$.

We would like to know when ξ is of contact type. Let us suppose that ξ is co-orientable and TM/ξ has been triviallized. That is, we have a global section V that we use to construct the isomorphism

$$\begin{array}{rcl} \Phi\colon TM/\xi & \longrightarrow & \underline{\mathbb{R}} := M \times \mathbb{R} \\ u_x & \longmapsto & (x, \dfrac{u_x}{V_x}) \end{array}$$

In CR geometry the is a way of measuring the "convexity in the complex sense" of ξ.

The **Levi form** is the bilinear form defined

$$\begin{array}{rcl} \mathcal{L}\colon \xi \times \xi & \longrightarrow & TM/\xi \cong \underline{\mathbb{R}} \\ (U, V) & \longmapsto & [U, JV]/\sim \end{array} \tag{15}$$

Exercise 12. *Show that $\mathcal{L}$ is a tensor, and that it is symmetric. For the latter, use the vanishing of the Nihenjuis tensor which is equivalent to*

$$[JU, JV] = [U, V] + J[JU, V] + J[U, JV], \tag{16}$$

Definition 13. *H is called strictly/strongly pseudoconvex (resp. pseudoconcave) if the Levi form is strictly positive (resp. negative).*

The hypersurface H (being its normal bundle orientable) can be defined as the zero set of a function $\rho\colon Y \to \mathbb{R}$ which has no singular points at H.

To any such function we can associate the $(1,1)$ real valued 2-form

$$-dd^c\rho, \tag{17}$$

where $d^c := d \circ J$,

$$d^c\colon C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{J^*} \Omega^1(M)$$

Notice that $d^c\rho_{|H}$ is a (real valued) 1-form whose kernel is ξ.

To explain what a $(1,1)$ form is, just notice that the above complex also works for complex valued functions and forms.

Any (almost) complex structure gives a splitting

$$T^*Y_{\mathbb{C}} = T^{*1,0}Y + T^{*0,1}Y$$

into complex linear and anti-complex linear part. So we get

$$\Omega(Y;\mathbb{C}) = \Omega^{1,0}(Y,\mathbb{C}) \oplus \Omega^{0,1}(Y;\mathbb{C}),$$

and therefore

$$\Omega^r(Y;\mathbb{C}) = \sum_{p+q}^{r} \Omega^{p,q}(Y;\mathbb{C})$$

Using the previous splitting one defines

$$\partial := \pi^{1,0} \circ d,\ \bar\partial := \pi^{0,1} \circ d \tag{18}$$

Equivalently,

$$\partial := \frac{d - idJ}{2},\ \ \bar\partial := \frac{d + idJ}{2},$$

with $d = \partial + \bar\partial$

Since a function f is holomorphic iff its derivative is complex linear, we can write it equivalently as

$$\bar{\partial} f = 0$$

In complex coordinates $z_j = x_j + iy_j$ one defines

$$dz_j := dx_j + idy_j,\, d\bar{z}_j = dx_j - idy_j$$

and checks that $(dz_j)_{z=1,\ldots,n}$ (resp. $(d\bar{z}_j)_{z=1,\ldots,n}$ gives a basis of complex linear (resp. anti-complex linear) 1-forms.

Also notice that

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j,\, \bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),\, \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Then one checks

$$\partial^2 = \bar{\partial}^2 = 0$$

In particular from $d^2 = 0$ we also conclude

$$\partial\bar{\partial} + \bar{\partial}\partial = 0$$

Remark 9. *As we mentioned, the splitting of complex linear forms and the definition of the operators $\partial, \bar{\partial}$ as in equation 18. A foundational result in almost complex geometry is*

Theorem 5 (Newlander-Nirenberg)**.** *Given (Y, J) an almost complex manifold, then the following assertion are equivalent-*

(1) *The almost complex structure is integrable, meaning that about any point there exists complex coordinates.*
(2) $\partial^2 = 0$ *or* $\bar{\partial}^2 = 0$.
(3) *The Nijenhuis tensor (equation 16) is vanishing.*

Observe that $d^c = -i(\bar{\partial} - \partial)$ and therefore another definition of 17 is

$$-dd^c = 2i\partial\bar{\partial} \tag{19}$$

and thus it follows that $-dd^c\rho$ is of type $(1,1)$.
If $\gamma \in \Omega^{1,1}(M;\mathbb{C})$, then

$$\gamma(J\cdot, J\cdot) = \gamma(\cdot,\cdot)$$

Indeed, it is enough to check it for the elements of a basis and

$$dz_j \circ J \wedge d\bar{z}_l \circ J = -i^2 dz_j \wedge d\bar{z}_l$$

So we can define a symmetric and hermitian form by the formulas

$$g(u,v) := -dd^c\rho(u, Jv),\, h =: g - idd^c\rho$$

Exercise 13. *The 1-form $d^c\rho_{|H}$ defines ξ and co-orients it by the declaring $w \in T_xH$ to be positive if $d^c\rho(w) > 0$. Prove that*

$$\mathcal{L}(u,v) = fg(u,v), \tag{20}$$

where f is a strictly positive function

Proposition 7. *(H, ξ) is contact iff the Levi form is non-degenerate. Moreover it is strongly psedoconvex iff J is a compatible almost complex structure for $d(d^c\rho)_{|\xi}$.*

Proof. The signature of the Levi form coincides with that of g, so the hypothesis implies that $d(d^c\rho)_{|\xi}$ is symplectic. □

4.1. **Strong convexity and plurisubharmonic functions.**

Definition 14. *Let $Z \subset \mathbb{R}^m$ a domain. A function $f \in C^\infty(Z)$ is strictly convex if $f_{|[a,b]}$ is strictly convex for any $[a,b] \subset V$.*

Remark 10. *Convexity of a function is not a Riemannian concept, but an affine one. In particular the concept makes sense in affine manifolds (given by charts with transition functions given by a translation followed by an affine transformation).*

Notice as well that convexity of a function is given by the positivity of the degree 2 operator $\frac{d}{dt}^2$ in segments, or globally by the Hessian Hess$f$.

Recall that for a domain $Z \subset \mathbb{R}^n$, we say that ∂Z is strictly convex if for a defining function ρ,

$$\mathrm{Hess} f_{|\partial Z}$$

is strictly positive (this is the same as saying that the second fundamental form of ∂Z w.r.t. the Euclidean metric is strictly positive).

Strict convexity is equivalent to segments with boundary in ∂V having interior in the interior of Z. For domains in affine manifolds, this is true at least for small enough segments (about each point in the boundary).

In complex geometry one may think of holomorphic disks $\mathbb{D} \hookrightarrow Y$ as being substitutes of affine segments.

Even more, up to scaling we have a substitute for $\frac{d}{dt}^2$ on disks. Indeed, take z a complex coordinate, and use the Laplacian Δ w.r.t. the Euclidean metric. A holomorphic change of coordinates induces a conformal change in the metric, and hence just rescales the Laplacian, not affecting its signature.

Exercise 14. *Show that if $x'(x,y), y'(x,y)$ is a holomorphic change of coordinates, then*

$$\Delta f(x'(x,y), y'(x,y)) = \Delta f(x', y')\mathrm{Jac}(x', y')$$

Definition 15. *A function $f \in C^\infty(\mathbb{D})$ is strictly subharmonic if $\Delta f > 0$.*

A function $f \in C^\infty(Y)$ is strictly plurisubharmonic if its restriction to any holomorphic disk is strictly subharmonic.

Definition 16. *Given $f \in C^\infty(Y)$, the complex Hessian $\mathrm{Hess}_{\mathbb{C}} f$ is the symmetric to form associates to $i\bar{\partial}\partial f$.*

Notice that

$$i\bar{\partial}\partial f(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \frac{1}{2}\Delta f$$

Corollary 6. *A function f is strictly plurisubharmonic iff at every point the complex Hessian is strictly definite positive, or in other words the $(1,1)$ form $2i\bar{\partial}\partial f$ is symplectic and J is a c.a.c.s. for it.*

Corollary 7. *A hypersurface H is strictly pseudoconvex if for any defining function its complex Hessian is strictly positive along $TH \cap JTH$.*

Remark 11. *Strict convexity imply that small enough holomorphic disks whose boundary hits H, stay in one side.*

Definition 17. *A domain $Z \subset (Y, J)$ is strictly pseudoconvex if any defining function is strictly pseudoconvex at all points of the boundary along the complex tangencies.*

Lemma 5. *If $Z \subset (Y, J)$ has a defining function ρ for the boundary which is strictly plurisubharmonic (along all directions), then the g-gradient of ρ is Liouville for the symplectic form $-dd^c\rho$*

Exercise 15. *Prove lemma 5.*

So domains whose boundary is defined by a function which is s.p.s.h. in a neighborhood of the boundaryare a (weak) "integrable analog" of strongly convex symplectic domains. On the one hand the symplectic form is only defined near the boundary.

Definition 18. *A complex manifold (Y, J) is Stein iff there exist $\rho\colon Y \to [b, \infty)$ a s.p.s.h. exhaustion function.*

It is of finite type or with conical end if ρ is Morse with only a finite number of critical points.

A Stein domain is any domain of the form $\rho^{-1}([b, r])$, r regular value.

Exercise 16. *Show that there is no loss of generality in assuming ρ to be Morse.*

Clearly, a Stein domain is the "integrable analog" of strongly convex symplectic domain for which the Liouville vector field is globally defined and it is gradientlike (also called **Weinstein manifolds**). See [3] for different notions of convexity.

Example 6. *Strictly pseudoconvex hypersurfaces, and in particular boundaries of Stein domains are examples of contact manifolds.*

Remark 12. *One has the following source of Stein domains; consider (Y, J, g) a closed Kahler manifold such that the Kahler form ω_g is integral. Then the prequantum-line bundle is holomorphic (actually we just need the 2-form in Weil's construction to be $(1, 1)$). This admits equivalent descriptions:*

1. *One can choose local trivializations $s_i\colon U_i \to L_{|U_i}$ so that the transition functions*
$$\varphi_{ij}\colon U_i \cap U_j \to \mathbb{C}^*$$
are holomorphic.
2. *The total space of L admits a complex structure $\hat{J}$ which extend the complex linear structure of the fibers, and makes the projection into a holomorphic map.*

Using the first definition one can define the following operator on sections
$$\bar{\partial}_L t_{|U_i} = \bar{\partial} f s_i,$$
where $t_{|U_i} = f s_i$.

The definition is consistent because the change of trivialization is holomorphic. One declares a section t to be holomorphic if
$$\bar{\partial} t = 0$$

Notice in particular that all the s_i are holomorphic.

Proposition 8. *The two statementsabove are equivalent.*

Proof. To go from the first to the second, use the tangent space to the holomorphic sections to push the complex structure form the base to the total space. This, together with the complex linear structure of the fibers defines an almost complex structure which also makes the derivative of the projection commute with the almost complex structures. It is integrable because if $z_1, \ldots, z_n$ are coordinates on the base, then
$$\pi^* z_1, \ldots, \pi^* z_n, w/s_i(\pi(w))$$
are holomorphic.

Exercise 17. *Prove the above assertion about the coordinates being holomorphic w.r.t. $\hat{J}$.*

To prove the other direction, we just need to find local holomorphic sections. Since $\hat{J}$ is integrable we have local holomorphic coordinates $w_1, \ldots, w_{n+1}$. Pick any w and one coordinate z_1 say whose kernel is transversal to the fiber.

The hypersurface $\{z_1 \equiv 0\}$ is holomorphic, because so is the coordinate. The restriction of the projection to $\{z_1 \equiv 0\}$ is therefore a holomorphic local diffeomorphism. By the inverse function theorem the inverse is also holomorphic, giving thus the desired section. $\square$

The fibers of L are complex linear 1-dimensional spaces. Therefore, we can split

$$\Omega^1(M;L) = \Omega^{1,0}(M;L) \oplus \Omega^{0,1}(M;L)$$

In particular any connection

$$\nabla\colon \Gamma(L) \to \Omega(M) \otimes \Gamma(L)$$

splits into $\nabla^{1,0} + \nabla^{0,1}$.

Notice that the operator

$$\bar{\partial}_L\colon \Gamma(L) \to \Omega^{0,1}(M) \otimes \Gamma(L) \to \Gamma(L)$$

satisfies the Liebniz rule

$$\bar{\partial}_L(fs) = \bar{\partial} f s + f \bar{\partial}_L$$

as the $(0,1)$ of a connection would do.

Definition 19. *A connection ∇ on L is compatible with the (integrable) complex structure $\hat{J}$ if*

$$\nabla^{0,1} = \bar{\partial}_L$$

Exercise 18. *Proof that ∇ is compatible with $\hat{J}$ iff $\mathcal{H}_\nabla$ is $\hat{J}$-complex.*

Another proof of the curvature F_∇ being of type $(1,1)$ is the following: choose s a local holomorphic section and compute

$$F_\nabla = d\frac{\nabla s}{s} = d\frac{\nabla^{0,1} s}{s},$$

so $F^{0,2} = 0$.

$$F_\nabla + \bar{F}_\nabla = 0,$$

which together with the vanishing of the $(0,2)$ part implies the desired result.

It is a very non-trivial result that because ω_g is strictly definite positive, then large enough powers $L^{\otimes k}$ (with the induced complex structure, connection that we still call ∇,...) have many **global holomorphic sections**. *recall that the hermitian holomorphic line bundle carries a connection $\nabla = \partial + \bar{\partial}$, and s is holomorphic iff $\bar{\partial} s = 0$. In particular for $L^{\otimes k}$ one can find $s\colon Y \to L^{\otimes k}$ a holomorphic section transverse to the zero section, so $W = s^{-1}(\mathbf{0})$ is a complex hypersurface. Then $(Y \backslash W, J)$ is Stein.*

Indeed, consider the function

$$f = -\log\langle s, s\rangle \colon Y \backslash W \to \mathbb{R}$$

If we now take the restriction of s to W, we can compute the curvature. Since the curvature is of type $(1,1)$ and s is holomorphic we have

$$F_\nabla = \bar{\partial}\frac{\nabla^{1,0} s}{s}$$

Now

$$\partial f = df^{1,0} = -\frac{d\langle s,s\rangle}{\langle s,s\rangle}^{1,0} = -\frac{\left(\langle \nabla^{1,0}s, s\rangle + \langle s, \nabla^{1,0} s\rangle\right)}{\langle s,s\rangle}^{1,0} = -\frac{\langle \nabla^{1,0}s, s\rangle}{\langle s,s\rangle} = -\frac{\nabla^{1,0}s}{s}$$

As a consequence

$$i\partial\bar{\partial}f = iF_\nabla = 2\pi\omega_g,$$

and this proves the claim.

Now we have the tools to prove that in exercise 5

$$-\frac{i}{2}F_\nabla = \omega_{FS}$$

Indeed in $\mathbb{CP}^n$ *with homogeneous coordinates* $[Z_0 : \cdots : Z_n]$ *consider the open set* U_0 *for which* $Z_0 \neq 0$. *It is parametrized by the chart*

$$\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & U_0 \\ (z_1, \ldots, z_n) & \longmapsto & [1 : z_1 : \cdots : z_n] \end{array}$$

Then we have the local holomorphic section

$$\begin{array}{rcl} s_0 \colon \mathbb{C}^n & \longrightarrow & \mathbb{C}^{n+1} \backslash \{0\} \\ (z_1, \ldots, z_n) & \longmapsto & (1, z_1, \ldots, z_n) \end{array}$$

The covariant derivative of s_0 *is the hermitian projection onto the fibre of the usual derivative*

$$\nabla s_0 = \langle (0, dz_1, \ldots, dz_n), (1, z_1, \ldots, z_n) \rangle \frac{(1, z_1, \ldots, z_n)}{\langle (1, z_1, \ldots, z_n), (1, z_1, \ldots, z_n) \rangle},$$

and therefore

$$\frac{\nabla s_0}{s_0} = \frac{\sum_{j=1}^n \bar{z}_j dz_j}{1 + \sum_{j=1}^n z_j \bar{z}_j} = \partial \log(1 + \sum_{j=1}^n z_j \bar{z}_j)$$

So we conclude

$$\frac{i}{2} F_{\nabla_{|U_0}} = \frac{i}{2} \bar{\partial} \partial \log(1 + \sum_{j=1}^n z_j \bar{z}_j) = \omega_{\mathrm{FS}|U_0}$$

5. (Semi)-local normal forms

Recall that a submanifold $N \hookrightarrow (M^{2n+1}, \xi)$ is isotropic if $TN \subset \xi$.

At the linear level, the tangent space of an isotropic submanifold is isotropic inside ξ_x (for the conformal symplectic structure), therefore it can have at most dimension n.

Example 7.

(1) *Any submanifold* $N \subset M$ *gives rise to a Legendrian inside of* $\mathrm{Ct}(M)$ *by taking the submanifold of hyperplanes containing its tangent subspace.*

(2) *The zero section of* T^*M *is Legendrian inside* $\mathcal{J}^1 M$ *with the canonical contact structure.*

Isotropic and contact submanifolds are important because -up to some extent- they determine the contact structure (even the contact form) in a tubular neighborhood.

Why is this important?

Proposition 9. *Let* $N \subset M$ *be a compact submanifold, and* α, α' *contact forms on* M *such that*

(1) $\alpha_{|N} = \alpha'_{|N}$

(2) $d\alpha_{|N} = d\alpha'_{|N}$

Then there exists U, U' neighborhoods of N and $\phi\colon U \to U'$ diffeomorphism extending the identity on N such that

$$\phi^*\alpha' = f\alpha$$

Proof. Consider the convex combination $\alpha_t := t\alpha + (1-t)\alpha'$. Then the conditions guarantee that we have a family of contact forms in a small tubular neighborhood V of N. Indeed, write $\alpha_t = \alpha' + t(\alpha - \alpha')$. Then

$$\alpha_t \wedge \alpha_t^n = \alpha' \wedge d\alpha'^n + (\alpha - \alpha') \wedge \beta_t + d(\alpha' - \alpha) \wedge \beta_t',\ \beta_t,\ \beta_t' \in \Omega^n(U)$$

Now we can apply the same Moser trick leading to the proof of Gray's stability theorem. Just observe that the solution of equation 12 vanishes at N for all time, so the isotopy fixes the submanifold. □

Remark 13. *Firstly, observe that conditions 1, 2 would follow from $\alpha = \alpha'$ at N at first order. In particular for a given contact form α' we can take a model of normal bundle to linearize α in the normal directions $\alpha_N^{(1)}$, and the proposition implies that α and $\alpha_N^{(1)}$ define contactomorphic structures in suitable tubular neighborhoods.*

Theorem 6 (Darboux theorem for contact forms)**.** *Let (M, α) be a contact manifold. About each $x \in M$ there exist a coordinate chart $\varphi\colon \mathbb{R}^{2n+1} \to U$, with coordinates $x_1, \dots, x_{2n+1}$, so that*

$$\varphi^*\alpha_{\text{std}} = \alpha \tag{21}$$

Proof. Notice that the statement is a normal form for the 1-form, not just for the contact structure, so it is stronger than what one would expect from Moser's technique as it has been used so far.

Take any coordinates $x_1, \dots, x_{2n+1}$ and linearize the 1-form α to obtain $\alpha^{(1)}$. As noticed in remark 13 we can apply Moser's technique to pull back α into $f\alpha^{(1)}$.

But we can do a bit better. We apply Moser's technique to solve this time the equation

$$\phi_t^*\alpha_t = \alpha_0$$

After differentiating we obtain

$$\phi_t^*(i_{X_t} d\alpha_t + d i_{X_t}\alpha_t + \frac{d}{dt}\alpha_t) = 0 \tag{22}$$

So we aim at solving

$$i_{X_t} d\alpha_t + d i_{X_t}\alpha_t + \frac{d}{dt}\alpha_t = 0, \tag{23}$$

where this time $X_t = Z_t + h_t R_t$ is not necessarily tangent to ξ_t. Equation 24 transforms into

$$i_{Z_t} d\alpha_t + dh_t + \frac{d}{dt}\alpha_t = 0, \tag{24}$$

Since R_t is never vanishing in a small neighboorhood of the origin one can solve

$$dh_t(R_t) + \frac{d}{dt}\alpha_t(R_t) = 0,\ h_t(0) = 0,$$

and then find the unique X_t so that equation 24 holds.

We can actually choose the original coordinates so that the Reeb vector field is $\partial/\partial x_{2n+1}$ and $\xi(0)$ is spanned by $\partial/\partial x_1, \dots, \partial/\partial x_{2n}$. We can make a further linear change of coordinates so that $x_1, \dots, x_{2n}$ are Darboux coordinates for $d\alpha(0)_{|\xi}$.

Then

$$\alpha^{(1)} = dx_{2n+1} + \sum_{i=1}^{n} l_i dx_i,$$

and the Darboux assumption implies that $t\alpha^{(1)} + (1-t)\alpha_{\mathrm{std}}$ is a family of contact forms. □

Suppose $N \subset (M,\alpha)$ is isotropic. What information do we need to determine $\alpha_{|N}, d\alpha_{|N}$?.

Being isotropic implies that $TN \subset TN^{d\alpha}$, the latter the symplectic annihilator inside $(\xi, d\alpha)$, so we have the isomorphism

$$TM \simeq \langle R\rangle \oplus \xi/TN^{d\alpha} \oplus TN^{d\alpha}/TN \oplus TN, \tag{25}$$

where we see the quotient bundles as subbundles (so rather than $\simeq$ we have an equality).

The summands $(\xi/TN^{d\alpha} \oplus TN, d\alpha)$ are easy to understand: first, take I complementary to $TN^{d\alpha}$ and isotropic; for example if J is a compatible almost complex structure, then $I := JTN$ does the job. Then we have

$$(I \oplus TN, d\alpha),$$

which is a symplectic vector subbundle of ξ.

On $TN \oplus T^*N$ we have a canonical linear symplectic structure

$$\omega((u,a),(v,b)) = a(v) - b(u)$$

Then one easily shows

Lemma 6. *The map*

$$\begin{array}{rcl}\Phi\colon (I \oplus TN, d\alpha) & \longrightarrow & (T^*N \oplus TN, \omega)\\ (u,v) & \longmapsto & (d\alpha(u,\cdot), v)\end{array}$$

is an isomorphism of symplectic vector bundles.

Definition 20. *The bundle $TN^{d\alpha}/TN$ -which clearly carries a symplectic linear structure inherited from $d\alpha$- is called the symplectic subnormal bundle. The bundle does not depend on the contact form, but the symplectic linear structure conformally depends on α.*

Notice that if we identify $TN^{d\alpha}/TN$ with $TN^{d\alpha}\cap I^{d\alpha}$, then we have the splitting

$$(\xi, d\alpha) = (I \oplus TN, d\alpha) \oplus (TN^{d\alpha} \cap I^{d\alpha}, d\alpha)$$

as symplectic vector bundles.

Theorem 7. *Let $N \hookrightarrow (M,\alpha)$, $N' \hookrightarrow (M',\alpha')$ isotropic submanifolds. Suppose that there exist a diffeomorphism $\phi\colon N \to N'$ and a lift to the conormal bundles*

$$\varphi\colon (TN^{d\alpha}/TN, d\alpha) \to (TN'^{d\alpha'}/TN', d\alpha')$$

which is an isomorphism of symplectic vector bundles. Then there exist tubular neighborhoods U, U' of the isotropic submanifolds an a diffeomorphism

$$\Psi\colon U \to U'$$

such that

(1) $\Psi_{|N} = \phi$
(2) $\Psi_*\xi = \xi'$

Proof. Define $\Psi\colon TM_{|N} \to TM'_{|N'}$ by

$$\Psi(R) = R',\ \Psi_{|L\oplus TN} = \Phi'^{-1} \circ (\psi^* \oplus \psi_*) \circ \Phi,\ \Psi_{|TN^{d\alpha}/TN} = \varphi$$

Then $\Psi^*\alpha' = \alpha$, $\Psi^* d\alpha' = d\alpha$ in the points of N.

□

Corollary 8. *A diffeomorphism of Legendrian manifolds extends to a contactomorphism of tubular neighborhoods.*

Remark 14. *In principle theorem 7 matches contact structures in tubular neighborhoods, but not contact forms unlike Darboux' theorem. We will see in section 6 that we can also achieve equality at the level of contact forms.*

6. Symplectic cobordisms and surgery

Let M, M' be oriented manifolds.

Definition 21. *An oriented cobordism from M to M' is an oriented m+1 manifold Y so that*

$$\partial Y = -M \coprod M,$$

where we use the outward normal first rule to orient the boundary. We also denote $\partial Y_- := M$, $\partial Y_+ := M'$.

Clearly, oriented cobordisms can be composed/concatenated.

Any (oriented) cobordism can be split into a sequence of very simple cobordisms, called "handle attaching".

Definition 22. *An n-dimensional k-handle is*

$$h^k := D^k \times D^{n-k}$$

$$\partial h^k = \partial D^k \times D^{n-k} \coprod D^k \times \partial D^{n-k},$$

where we also denote

$$\partial h^k_- := \partial D^k \times D^{n-k}, \, \partial h^k_+ := D^k \times \partial D^{n-k}$$

k handles in n-dimensional space are associated to the quadratic Morse functions of index k

$$f_k(x_1, \ldots, x_n) = x_1^2 + \cdots + x_{n-k}^2 - x_{n-k+1}^2 - \cdots - x_n^2$$

Indeed, one should understand

$$h^k = W^s(f_k) \times W^u(f_k),$$

and the gradient vector field ∇f_k enters through ∂h^k_- and leaves through ∂h^k_+.

The attaching sphere is $\partial D^k \times \{0\} = \partial W^s(f_k) \subset \partial h^k_-$.

Let $\phi \colon \partial h^k_- \to \partial Y_+$ be a diffeomorphism onto its image

Definition 23. *The result of attaching h_k to Y via ϕ is the manifold*

$$Y \cup_\phi h_k := Y \coprod h_k / x \sim \phi(x)$$

once "corners have been rounded".

Recall that the new manifold $Y \cup_\phi h_k$ -up to diffeomorphism- is entirely determined by the isotopy class of ϕ. Notice that

$$\partial h^k_- = S^{k-1} \times D^{n-k},$$

the trivial disk bundle over the attaching sphere S^{k-1}. The attaching map induces a unique up to isotopy bundle map

$$S^{k-1} \times D^{n-k} \to \nu(\phi(S^{k-1})),$$

which in particular admits a representative which is a linear isomorphism. In short, the attaching map ϕ is determined by a homotopy class of isomorphisms

$$S^{k-1} \times \mathbb{R}^{n-k} \to \nu(\phi(S^{k-1})),$$

also called a **framing** of $\phi(S^{k-1})$.

Definition 24. *A (directed) symplectic cobordism is a (compact for us) symplectic manifold (Y, Ω) together with Liouville vector fields X_-, X_+, so that X_- (resp. X_+) is defined near ∂Y_- (resp. ∂Y_+), transversal to it, and inwards (resp. outwards) pointing.*

In other words it is a symplectic manifold whose boundary splits into an (strongly) concave part ∂Y_- and an (strongly) convex part ∂Y_+.

If given a (directed) symplectic cobordism we are able to attach a handle (to ∂Y_+ say) so that we produce a new symplectic cobordism Y', the contact manifold ∂Y_+ will have changed into $\partial Y'_+$. Notice for example that for a given contact manifold (M, α), $M \times [0, a]$ inside the symplectization is a symplectic cobordism so that the negative boundary coincides with (M, α).

In [17] (see also [2]) Weinstein described a way of attaching certain "symplectic handles" to a symplectic cobordism to get a new one. It is based on the following two points:

(1) Given symplectic cobordisms Y, W, find conditions (as flexible as possible) for a gluing map $\phi\colon \partial W_- \to \partial Y_+$ so that $Y \cup_\phi W$ is symplectic (extending both symplectic structures).
(2) Arrange a (directed) symplectic cobordism structure on a 2n-dimensional k-handle h^k, so that after symplectically attaching it to ∂Y_+, one can still induce an outwards pointing Liouville vector field in the new boundary
$$\partial Y \backslash \phi(\partial h^k_-) \cup \partial h^k_+$$

The answer to the first point is very much related to the study of neighborhoods of isotropic submanifolds as done in the previous section.

Definition 25 (Weinstein,[17])**.** *Given a symplectic manifold (Y, Ω) a isotropic setup is a quintuple (Y, Ω, H, X, N) were H is a hypersurface, X a Liouville vector field transversal to H (perhaps locally defined) and $N \subset M$ an isotropic submanifold (either isotropic for Ω in Y or isotropic for $\alpha = i_X\Omega_{|H}$ in H).*

Theorem 8 (Weinstein,[17])**.** *Let (Y, Ω, H, X, N) and $(Y', \Omega', H', X', N')$ be two isotropic setups. Let $\phi\colon N \to N'$ a diffeomorphism and $\Phi\colon TN^{d\alpha}/TN \to TN'^{d\alpha'}/TN'$ a symplectic vector bundle isomorphism lifting it. Then there exists V, V' neighborhoods of N, N' in Y, Y' and an isomorphism of isotropic setups*
$$\Psi\colon (Y \cap V, \Omega, H, X, N) \to (Y' \cap V', \Omega', H', X', N')$$
which restricts to the given data.

We refer to [17] for the proof, which is based again on Moser's method.

Remark 15. *Given isotropic submanifolds and diffeomorphism as in the statement of 7, we can symplectize both contact manifolds to obtain a couple of isotropic setups together with the bundle map between the symplectic subnormal bundles. Then theorem 7 implies*
$$\Psi^*\alpha' = \alpha,$$
because the equality holds for both the Liouville vector fields and the symplectic forms.

Recall that according to example 7 the zero section of $(\mathcal{J}^1 L, dz - \lambda)$ is Legendrian.

Corollary 9. *Let $L \subset (M, \alpha)$ Legendrian, then a neighborhood of it is isomorphic to a neighborhood of the zero section of $\mathcal{J}^1 L$ as exact contact manifolds.*

Point 2 is based on choosing the right model for a given k-handle. Let 2n+2 be the dimension of Y, and let $0 \leq k \leq n$.

In $\mathbb{R}^{2n+2}$ with coordinates $x_1, \dots, x_n, p_1, \dots, p_n$ and the standard symplectic structure

$$\Omega_{\text{std}} = \sum_{i=1}^{n} dx_i \wedge dp_i$$

All subspaces $x_1, \dots, x_{n-k}, p_1, \dots, p_n = 0$ are isotropic.

The vector field

$$X_k = \sum_{i=1}^{n-k} \left(\frac{1}{2} p_i \frac{\partial}{\partial p_i} + \frac{1}{2} x_i \frac{\partial}{\partial x_i} \right) + \sum_{i=n-k+1}^{n} \left(2p_i \frac{\partial}{\partial p_i} + x_i \frac{\partial}{\partial x_i} \right) \tag{26}$$

is Liouville. Notice that it is the gradient of the Morse function

$$f'_k = \sum_{i=1}^{n-k} \left(\frac{1}{4} p_i^2 + \frac{1}{4} x_i^2 \right) + \sum_{i=n-k+1}^{n} \left(p_i^2 - \frac{1}{2} x_i^2 \right) \tag{27}$$

having an index k critical point at the origin.

Let $\alpha_k := i_{X_k} \Omega_{\text{std}}$

The unstable manifold W^k_- for f'_k is one of the aforementioned isotropic vector subspaces vector subspace

$$x_1, \dots, x_{n-k}, p_1, \dots, p_n = 0$$

The hypersurfaces $\partial h^k_- := \{f'_k = -1\}$, $\partial h^k_+ := \{f'_k = 1\}$ inherit a contact structure, since the gradient vector field is Liouville contact for α_k and the attaching sphere is isotropic. Weinstein gives a model for the k-handle $h^{k,\text{std}}$ so that

$$\partial h^{k,\text{std}} = \partial h^k_- \cup \partial h^{k,\text{std}}_+,$$

and $\partial h^{k,\text{std}}_+$ is everywhere transverse to X_k, the Liouville vector field pointing outwards.

As a simple consequence Weinstein gives a rather clean proof of the following result of Meckert [13].

Corollary 10. *The connected sum of to exact contact manifolds (M, α), (M', α') carries an exact contact structure, unique up to isotopy.*

Proof. Consider the contact manifold $(M, \alpha) \cup (M, \alpha')$, and then its symplectization. Points $p \in M, p' \in M'$ together with a choice of Darboux basis at $(\xi_p, d\alpha_p)$ and $(\xi_{p'}, d\alpha'_{p'})$ allow the gluing of a symplectic 1-handle. Notice that since the symplectic linear group is connected, up to isotopy there is a unique symplectic framing of the symplectic subnormal bundle of $p \cup p'$, therefore no choices are involved (because the handle is essentially unique). Besides, any two pair of points can be joined by an isotopy though coisotropic submanifolds (points!), and therefore (see remark 5) through contact transformations. □

Remark 16. *More generally in [2] a theory for attaching "Stein handles" is developed. As a result a topological characterization of Stein manifolds of dimension $2n > 4$ is given. More precisely, for any (Y, J) almost complex manifold with a Morse exhaustion function ρ with critical points of index $k \leq n$, there exist a homotopic integrable almost complex structure $\tilde{J}$ for which f is ρ is strictly plurisubharmonic.*

7. Contact geometry and complex and symplectic geometry II: open book decompositions

Definition 26. *The canonical open book decomposition $\mathcal{B}_0$ of $\mathbb{C}$ is given by the partition of $\mathbb{C}$ origin (the binding), and the open half lines through the origin (leaves).*

Definition 27. *Given a manifold M an open book decomposition is given by a smooth function $s\colon M \to \mathbb{C}$ transversal to $\mathcal{B}_0$ (i.e. to all submanifolds in the decomposition). The binding and leaves are the pullbacks of the binding and leaves of $\mathcal{B}_0$.*

Two open book decompositions are equivalent if there is a diffeomorphism preserving bindings and leaves.

An open book decomposition, once we fix a leaf F, gives rise to a return map $\phi\colon F \to F$ which is the identity near ∂F (away from the binding, the open book decomposition is a fiber bundle over S^1, and hence a mapping torus). The leaf and isotopy class of ϕ determine the open book decomposition.

We follow mostly [14] in the following exposition. Let $F\colon \mathbb{C}^n \to \mathbb{C}$ a holomorphic function with an isolated complex singulatiy at the origin.

Let $S_\epsilon \subset \mathbb{C}^n$ denote the sphere of radius ϵ. This is a hypersurface level of the s.p.s.h. function

$$\begin{aligned} \rho\colon \mathbb{C}^n &\longrightarrow \mathbb{R} \\ z &\longmapsto z\bar{z} \end{aligned} \tag{28}$$

Since it is a polynomial function it has a finite number of critical values points when restricted to $F^{-1}(0)\backslash\{0\}$ (this is because the critical set is an algebraic variety, and hence has a finite number of connected components). Therefore if ϵ is small enough the singular fiber $Y := F^{-1}(0)$ is tranversal to S_ϵ (notice this says Whitney B condition holds in complex coordinates). Since ρ restricted to $Y\backslash\{0\}$ is s.p.s.h.,

$$M_\epsilon := F^{-1}(0) \cap S_\epsilon$$

-called the link of the singularity defined by F- carry a contact structure, the **Milnor contact structure**. By Gray's theorem it does not depend on the radius ϵ.

Now take any holomorphic function $G\colon \mathbb{C}^n \to \mathbb{C}$

(1) G vanishes at 0.
(2) $G^{-1}(0) \cap Y$ has an isolated singularity at the origin.

The closed subset in $\mathbb{CP}^{n-1}$ of limit tangent hyperplanes is not the whole projective space [15]. Any function G vanishing at the origin, regular at it and with tangent hyperplane not in the limit set satisfies the previous requirements.

Theorem 9 ([14, 1, 16])**.** *For every ϵ small enough the function $G\colon M_\epsilon \to \mathbb{C}$ defines an open book decomposition, the* **Milnor open book** *associated to G.*

In Milnor's original construction [14] the embedded singular variety is $\mathbb{C}^n$ and M_ϵ is S_ϵ. The function giving rise to a singular hypersurface is F, and hence according to theorem 9 the link of the singularity is the binding of an open book decomposition of S_ϵ. In particular for complex dimension 2 the binding is a knot, and the open book decomposition implies that it is a **fibered knot**.

Definition 28. *Let (M,ξ) be a contact manifold. A submanifold $N \hookrightarrow M$ is contact if $TN \pitchfork \xi$, so $TN \cap \xi$ is a hyperplane distribution of N, and $TN \cap \xi$ is contact.*

Definition 29 (Giroux)**.** *Given (M,ξ) a co-oriented contact manifold, an open book decomposition carries the contact structure if there exist α positive such that*

- *The binding is a contact submanifold (with the right orientation).*
- *The leaves are symplectic for $d\alpha$ (with the right orientation).*

If F denotes the leaf, the monodromy of such open book belongs to $\mathrm{Symp(F, d\alpha)}$. The contact structure -up to symplectic isotopy- only depends on $(F, d\alpha)$ and the symplectic monodromy, so a new link between contact geometry and symplectic geometry is stablished in the presence of such an open book.

Theorem 10 ([14, 1, 16]). *Any Milnor open book carries the Milnor contact structure of any complex isolated singulary (the exact form comming from $2i\bar{\partial}\partial\rho$).*

There is a more general result along the same lines.

Theorem 11 ([10]). *For any (M, ξ) a co-oriented contact manifold, there exists an open book carrying it.*

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