A characterization of computable analysis on unbounded domains using differential equations

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Abstract

The functions of *computable analysis* are defined by enhancing normal Turing machines to deal with real number inputs. We consider characterizations of these functions using function algebras, known as *real recursive functions*. Since there are numerous incompatible models of computation over the reals, it is interesting to find that the two very different models we consider can be set up to yield exactly the same functions. Bournez and Hainry [6] used a function algebra to characterize computable analysis, restricted to the twice continuously differentiable functions with compact domains. In our paper [11], we found a different (and apparently more natural) function algebra that also yields computable analysis, with the same restriction. In this paper we improve our result in [11], finding three function algebras characterizing computable analysis, removing the restriction to twice continuously differentiable functions and allowing unbounded domains. One of these function algebras is built upon the widely studied *real primitive recursive functions*. Furthermore, the proof of this paper uses our *method of approximation* from [12], whose applicability is further evidenced by this paper.

Key words: computable analysis, real recursive functions, function algebras, analog computation, differential equations

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1. Introduction

When is a function over the reals *computable*? The answer to this question when working over the naturals has a generally agreed upon answer (e.g. Turing-computability, recursive, etc.), but over the reals, there are a number of incompatible computational models. We can make an informal categorization of these models: 1) Models that evolve in discrete time steps, and 2) Models that evolve in continuous-time. In the first category we have models like computable analysis [23] [29], Grzegorczyk's algebras of functionals [20], BSS-machines [2] [1], real random access machines [27] [8], and a recursive characterization of computable real functions [7]. In the second category, we have models like Shannon's circuit model (the General Purpose Analog Computer) [28] [18], continuous neural networks [24], and Moore's real recursive functions [25] (for an up-to-date review of continuous-time models see [3]). Discrete-time models typically use some abstract machine, like a Turing machine, where there is a clear notion of the "next state" within a computation. Continuous-time models typically use differential equations to describe a computation which can be viewed as proceeding continuously in real time, with no clear notion of "next state." The dissimilarity of the models makes it interesting to investigate connections between them, as a number of recent papers have done. It is known that several approaches to computability over the real numbers coincide: In particular, computable analysis, computable functionals [20], and continuous domains [17] all yield the same class of functions. Here, we are most interested in comparing computable analysis with continuous-time models. Bournez et. al. [4] characterize computable analysis with General Purpose Analog Computers. Bournez and Hainry [5] [6] partially characterize computable analysis with real recursive functions. We continue in this direction, providing various characterizations of computable analysis with real recursive functions. Finding different models for the same set of functions could be useful from a technical point of view, allowing one to prove facts about one model by using one of its characterizations. Furthermore, understanding when models of computation over the reals agree and disagree should be vital for a deeper perspective on what we mean by computing over the reals.

Computable analysis seeks to give a realistic account of how a digital computer calculates with real numbers: A function is computable if from approximations to the input, we can approximate the output (using a standard discrete-time Turing machine). Moore's *real recursive functions* is a generic name we use for models based on function algebras over the reals, i.e. a specific set of *basic functions* are closed under a specific set of operations, some involving differential equations (for background see Moore's original paper [25], along with the clarifying papers [15] and [22]). Moore's motivation was to develop an analog version of the normal recursive functions over the naturals, replacing the recursion operation with a differential equation operation. The main result of this paper (theorem 2.17) characterizes the functions of computable analysis via real recursive functions, using three different function algebras.

Bournez and Hainry [6] proposed a class of real recursive functions that partially captured computable analysis. In particular, their function algebra characterizes the twice continuously differentiable ($\mathcal{C}^2$) functions with compact domain. Our function algebras remove the restriction to $\mathcal{C}^2$ and compact domains. Their function algebra contains a set of basic functions and is closed under the following operations: Composition, linear integration, a limit operation and a root-finding operation. One of our characterizations will be similar, and another one will replace the root-finding operation with an operation that finds the inverse of a function. The third, and most interesting function algebra, does not have a root-finding operation or inverse operation, but instead strengthens the operation of integration, removing the linearity restriction. This third characterization (called $\mathbf{ODE}_k^\star(\mathsf{LIM})$) of computable analysis appears to be about as natural as one could hope for (using real recursive functions); its underlying functions (before a kind of limit operation is applied) are merely a few basic functions, along with functions that can be built from these, using differential equations. In fact, this function algebra is very similar to the *real primitive recursive functions*, which have been studied by a number of authors (e.g. [15], [22], [6]; note that there are slight differences between their definitions and only [22] actually uses the name "real primitive recursive functions").

In addition to providing new characterizations which apply to all of the functions of computable analysis, our proofs use our *method of approximation* (developed in [12]). To capture computable analysis with an analog model, earlier approaches have proceeded by fixing a particular characterization of computable analysis, and then exploiting its particular properties in order to simulate its operation in the analog model (e.g. Turing machines are used in [4], and computable functionals are used by Bournez and Hainry [6]). While we of course begin with some model of computable analysis (we choose oracle Turing machines), we convert the problem into a question about function algebras, and are no longer explicitly concerned with computable analysis. In particular, we introduce the notion of one class of functions approximating another one, and reduce the work to proving a series of approximations. Our approach offers a number of advantages. Due to the transitivity of the approximation relation, we can break up the proof into a number of natural steps. The approximation context works naturally with the inductive structure of the function algebras. And finally, our approach seems to be more general, facilitating reasoning about our problem and other problems of this kind. The significance of the method of approximation is discussed in more detail in the conclusion (section 5).

Section 2 introduces the terminology and discusses the main result. Section 3 outlines our proof, breaking it up into a *minor step* and a *major step.* The minor step follows directly from our work in [11], and thus we simply summarize the ideas for this step. The major step is set up in section 3 (page 10), leaving the technical details of this step for section 4 (page 20). In section 5 (page 41) we reflect on the significance of our approach to this problem, comparing it to other approaches, and also consider strengthening our result by simplifying our function algebra. *There is an index at the end of the paper.*

2. Formulating the Main Result

We now provide the basic definitions and state the result, leaving the proof for the next section.

Definition 2.1. *For us, unless stated otherwise, a* ***function*** *will always have domain $\mathcal{D} \subseteq \mathbb{R}^k$ and codomain $\mathbb{R}$. To indicate that a function f is defined on* all *of $\mathcal{D}$, with codomain $\mathcal{E}$, we write $f : \mathcal{D} \to \mathcal{E}$. By "*dom*$f$" we mean the domain of f. If $f : D^k \to \mathcal{E}$ and $X \subseteq D$, we write $f_{|X}$ for the restriction of f to the domain X^k.*

Typical domains in this paper are: The naturals $\mathbb{N} = \{0, 1, 2, \ldots\}$, the integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, the rationals $\mathbb{Q}$, and the reals $\mathbb{R}$.

Definition 2.2. *For $\bar{x} = (x_1, \ldots, x_k)$, and $X \subseteq \mathbb{R}$, we write $\bar{x} \in X$ to mean that each x_i is in X; for a unary function f, the vector $(f(x_1), \ldots, f(x_k))$ is abbreviated by $f(\bar{x})$.*

One of the models we will consider is *computable analysis*, also known as *type-2 computability* (see [23] or [29] for details). A function $f : \mathbb{R} \to \mathbb{R}$ is computable in the sense of computable analysis if there is an oracle Turing machine, which on input n (which we call the *accuracy input*), using an oracle for $x \in \mathbb{R}$, outputs a rational within $1/n$ of $f(x)$. The oracle is used as follows: If the machine writes a number m on the oracle tape, it receives a rational within $1/m$ of x. The following definition generalizes this discussion to functions with domain $\mathbb{R}^k$.

Definition 2.3. *We say a function $f : \mathbb{R}^k \to \mathbb{R}$ is in $\mathbf{C}_{\mathbb{R}}$ iff it is computable in the sense of computable analysis.*

It is common in computable analysis to restrict the domain to bounded intervals; however the functions of $\mathbf{C}_{\mathbb{R}}$ are defined on unbounded domains.

We now turn our attention to function algebras. We use the term **operation** to refer to an operator that maps a finite number of functions to a single function. Some operations are *partial*, meaning that they are undefined given certain functions as arguments. The next few pages of technical definitions will be followed by some (hopefully) helpful examples.

Definition 2.4. *Suppose $\mathbf{B}$ is a set of functions (called* ***basic*** *functions), and $\mathcal{O}$ is a set of operations. Then $\mathsf{FA}[\mathbf{B}; \mathcal{O}]$ is called a* ***function algebra****, and it denotes the smallest set of functions containing $\mathbf{B}$ and closed under the operations in $\mathcal{O}$. For ease of readability, we display the elements of $\mathbf{B}$ and $\mathcal{O}$ as comma separated lists.*

Some of the most important operations will be defined using differential equations. From a vector valued function $\bar{g} = (g_1, \ldots, g_k)$ we can form an **initial value problem (IVP)** in parameters $\bar{a} = (a_1, \ldots, a_k)$, given by the following system of equations:

$$(\bar{h})' = \bar{g}(\bar{a}, \bar{h}) \quad , \quad \bar{h}(0, \bar{a}) = \bar{a}.$$

We can also write the same system of equations more explicitly as follows:

$$\begin{array}{rclcrcl} \frac{\partial}{\partial x} h_1(x,\bar{a}) & = & g_1(\bar{a}, h_1, \ldots, h_k) & & h_1(0,\bar{a}) & = & a_1 \\ & \vdots & & , & & \vdots & \\ \frac{\partial}{\partial x} h_k(x,\bar{a}) & = & g_k(\bar{a}, h_1, \ldots, h_k) & & h_k(0,\bar{a}) & = & a_k \end{array} \tag{1}$$

We understand the vector $\bar{a}$ to be parameter variables, as opposed to just fixed reals; thus we write the solution $\bar{h} = (h_1, \ldots, h_k)$ with the parameter variables $\bar{a}$ as arguments (i.e. exactly the situation described in [21], p. 93).

In general, one IVP can have many distinct solutions; when we use an IVP to define an operation below, we want to avoid this case. Roughly, we will say an IVP is *well-posed* if it has a unique solution, though more precisely, we mean the following.

Definition 2.5. *Consider the IVP (1).*

- *$\bar{h}(x, \bar{a})$ is a **maximal solution** with domain D if $\bar{h}$ is a solution on some open set $D \subseteq \mathbb{R}^{k+1}$, and for any open set E (such that $D \subsetneq E$), there is no solution to the IVP on E.*
- *The IVP is **well-posed** if there is a unique maximal solution.*

We can now define the operations we will be using (note that in the operations there is an implicit choice of which arguments of the function we choose to use; any choice is allowed). Strictly speaking, all the following operations are partial. In the operations, conditions are described that need to be satisfied in order for the operation to output a function. If any condition is not satisfied by the input functions, then the operation is undefined for that input.

Definition 2.6. ***(Operations for real functions)***

1. *The operation ODE takes as input, functions $g_1, \ldots, g_k$. It is defined if the IVP given by the g_i is well-posed. Otherwise the operation is undefined. If the operation is defined, the IVP has a solution $\bar{h} = (h_1, \ldots, h_k)$ with domain D as described in definition 2.5. The operation outputs h_1 and* dom $h_1 = D$.
2. *LI is the same as ODE, with the additional condition that the $\bar{g}$ are linear in the $\bar{h}$.*
3. *Let comp be the* composition operation*. The operation comp takes as input, functions $f, g_1, \ldots, g_k$. It is defined if the functions f and g_i have appropriate arities; otherwise it is undefined. When defined, it is as follows. Given*

$$f(y_1, \ldots, y_k),\ g_1(x_1, \ldots, x_m),\ \ldots\ ,\ g_k(x_1, \ldots, x_m),$$

it returns the simultaneous composition:

$$h(x_1, \ldots, x_m) = f(g_1(x_1, \ldots, x_m), \ldots, g_k(x_1, \ldots, x_m)),$$

which is defined on the maximal well-defined domain, i.e.

$$\text{dom } h = \{\bar{x} \in \mathbb{R}^m : \bar{x} \in \text{dom } \bar{g} \text{ and } \bar{g}(\bar{x}) \in \text{dom} f\}.$$

4. *The operation* Inverse *receives a function* $f(x, \bar{a}) : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ *as input. The operation* Inverse *is defined on* f *if:*
   (a) *For all real* $\bar{a}$*,* $f(x, \bar{a})$ *is a bijection (of* $\mathbb{R}$*) in* x*, and*
   (b) *For all real* $\bar{a}$ *and* x*,* $\frac{\partial}{\partial x} f(x, \bar{a})$ *exists and is positive.*

   Otherwise it is undefined. If Inverse *is defined on* f *then it returns* $f^{-1} : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$*, the inverse in* x*, i.e.* $f(f^{-1}(x, \bar{a}), \bar{a}) = x = f^{-1}(f(x, \bar{a}), \bar{a})$ *(*f^{-1} *may be referred to as* Inverse(f, x)*; the "*x*" indicates the variable of* f *to which* Inverse *is applied).*
5. *The root-finding operation* UMU *("*unique μ*") receives a function* $f(x, \bar{a}) : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ *as input. The operation* UMU *is defined on* f *if:*
   (a) *For all real* $\bar{a}$*,* $f(x, \bar{a})$ *is increasing in* x *(not necessarily strictly), and*
   (b) *For all real* $\bar{a}$*, there is a unique* x *such that* $f(x, \bar{a}) = 0$ *(and at that* x*,* $\frac{\partial}{\partial x} f$ *exists and is positive).*

   Otherwise it is undefined. If UMU *is defined on* f*, it returns the function* $g : \mathbb{R}^k \to \mathbb{R}$ *such that for* $\bar{a} \in \mathbb{R}^k$*,* $g(\bar{a})$ *is the unique* x *such that* $f(x, \bar{a}) = 0$ *(*g *may be referred to as* UMU(f, x)*; the "*x*" indicates the variable of* f *to which* UMU *is applied).*

Our most important operation, ODE (also called **differential recursion** by other authors) is defined *without* analysis style conditions such as requiring the g_i be continuous, Lipschitz, etc. In doing so, our definition is similar to [22, Def. 3.5], rather than to that of [15]. However, as we shall see in lemma 2.15, whenever we actually use the ODE operation, all the functions involved will be *smooth* (i.e. $\mathcal{C}^n$ for some $n \geq 1$, as defined below). Thus, whether our ODE operation is defined as in [22] or with the requirement of being locally Lipschitz as in [15], our results remain the same. Furthermore, the two classes of functions we define with ODE ($\mathbf{ODE}_k$ and $\mathbf{ODE}_k^\star$; see definition 2.10) are both closed under composition, so we can use standard manipulations on differential equations to derive the following more flexible looking operation:

> On input $\bar{f}$ and $\bar{g}$, the derived operation operates just like ODE, except that the IVP it solves is the following:
>
> $$(\bar{h})' = \bar{g}(x, \bar{a}, \bar{h}) \ , \ \bar{h}(0, \bar{a}) = \bar{f}(\bar{a})$$
>
> (i.e. in the earlier system (1), we can replace each initial condition "$h_i(0, \bar{a}) = a_i$" by "$h_i(0, \bar{a}) = f_i(\bar{a})$" and allow the system to be non-autonomous, i.e. allow x as an input to $\bar{g}$).

Definition 2.7. *We say that* f *is* $\mathcal{C}^n$ *if* f *has continuous partial derivatives of all order* $k \leq n$*, on its domain.*

The following well known result (see [21], chapter V, theorem 3.1 and corollary 4.1, and chapter II, theorem 3.1) will be central to prove lemma 2.15. In particular, this proposition shows that if the g_i are $\mathcal{C}^n$ then the corresponding IVP is well-posed.

Proposition 2.8. *Consider the IVP*

$$(\bar{h})' = \bar{g}(\bar{a}, \bar{h}) \quad , \quad \bar{h}(0) = \bar{a},$$

and suppose $n \geq 1$. If $\bar{g}$ is $\mathcal{C}^n$ and has an open domain, then its unique maximal solution $\bar{h}(x, \bar{a})$ has an open domain and is $\mathcal{C}^n$.

Among our basic functions, one of the most significant will be a function which indicates if a number is to the left or right of zero. Such a function is the *Heaviside step function*: $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x \geq 0$. However, instead we will use a function of this sort, with some smoothness constraint. For integer $k \geq 1$, we let

$$\theta_k(x) = \begin{cases} 0, & \text{if } x < 0; \\ x^k, & \text{if } x \geq 0. \end{cases}$$

Thus, we think of the function θ_k as a $\mathcal{C}^{k-1}$ substitute for the Heaviside function. Besides θ_k we will also have basic functions like the constant function "0" and the projection functions $\mathbf{P}$ (e.g. $\mathbf{P}$ contains $P_{(2,1)}(x, y) = x,\ P_{(3,2)}(x, y, z) = y$, etc.). We will use the same names for these functions in the context of various domains ($\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$).

The **rank** of a function with respect to an operation counts the number of nested applications of the operation in the construction of the function.

Definition 2.9. *Given a function algebra $\mathbf{F} = \mathsf{FA}[\mathbf{B}; \mathsf{op}_1, \ldots, \mathsf{op}_n, \mathsf{OP}]$, we define the* ***rank of the construction*** *of the a function f in $\mathbf{F}$ with respect to the operation OP:*

1. *If f is in $\mathbf{B}$ (i.e. f is a basic function) then $\mathsf{rank}(f) = 0$.*
2. *If f is $\mathsf{op}_i(g_1, g_2, \ldots)$, then $\mathsf{rank}(f) = \mathsf{max}\{\mathsf{rank}(g_1), \mathsf{rank}(g_2), \ldots\}$.*
3. *If f is $\mathsf{OP}(g_1, g_2, \ldots)$ then*
   $\mathsf{rank}(f) = 1 + \mathsf{max}\{\mathsf{rank}(g_1), \mathsf{rank}(g_2), \ldots\}$.

We say that f is of ***rank*** *c if it has a construction of rank less than or equal to c.*

Note that by definition, a function of rank c is also of rank n for $n \geq c$. We now define the relevant function algebras over the reals.

Definition 2.10. ***(Function algebras over the reals)*** *Let $k \geq 1$ be an integer.*

- *Let $\mathbf{RMU}_k$ be $\mathsf{FA}[0, 1, \theta_k, \mathbf{P}; \mathsf{comp}, \mathsf{LI}, \mathsf{UMU}]$.*

- *Let $\mathbf{RMU}_k^{(c)}$ be the functions of $\mathbf{RMU}_k$ that have rank c with respect to the operation UMU.*

- *Let $\pi\mathbf{RMU}_k^{(c)}$ be the function algebra $\mathbf{RMU}_k^{(c)}$, with the constant functions -1 and π added as basic functions.*

- *Let* $\mathbf{INV}_k$ *be* $\mathit{FA}[0,1,-1,\theta_k,\mathbf{P};\mathsf{comp},\mathsf{LI},\mathsf{Inverse}]$.
- *Let* $\mathbf{INV}_k^{(c)}$ *be the functions of* $\mathbf{INV}_k$ *that have rank c with respect to the operation* Inverse.
- *Let* $\mathbf{ODE}_k$ *be* $\mathit{FA}[0,1,-1,\theta_k,\mathbf{P};\mathsf{comp},\mathsf{ODE}]$.
- *We define* $\mathbf{ODE}_k^{\star}$ *as follows: A function* f *is in* $\mathbf{ODE}_k^{\star}$ *iff* f *is in* $\mathbf{ODE}_k$ *and* f *has domain* $\mathbb{R}^n$ *for some* n.

We now discuss some examples.

Example 2.11. *Consider the initial value problem:*

$$\frac{\partial}{\partial x}h(x,y) = 1 \quad , \quad h(0,y) = y$$

The unique function satisfying this differential equation is $h(x,y) = x+y$. *Since the above function algebras all contain the constant function* 1*, and are closed under linear differential equations, they all contain the addition function. (As an exercise, do multiplication).*

The following example shows us a function in $\mathbf{ODE}_k$ that is not in the other function algebras.

Example 2.12. *The* non-linear *initial value problem*

$$h' = h^2 \quad , \quad h(0) = 1,$$

defines the function $h(y) = \frac{1}{1-y}$ *and so* $\frac{1}{x} = h(1-x)$ *is in* $\mathbf{ODE}_k$*, for any* $k \geq 1$ *(following our conventions, the domain of* $\frac{1}{x}$ *is* $(0,+\infty)$*, not containing any non-positive numbers). Note that the linear differential equations of* LI *do not suffice to define this function.*

Since the function $\frac{1}{x}$ is not total (over $\mathbb{R}$), it is not in $\mathbf{ODE}_k^{\star}$; for this reason we often work with $\mathbf{ODE}_k$ in the proofs, even though in the end we care about $\mathbf{ODE}_k^{\star}$. The next example uses the power of the basic function θ_k.

Example 2.13. *Campagnolo et. al. [9, lemma 4.7] defined a kind of step function,* step $: \mathbb{R} \to \mathbb{R}$*, which is increasing, continuous, and satisfies the following property:*

> *For any* $u \in \mathbb{Z}$, step$(x) = u$ *for* $x \in [u, u+1/2]$.

This construction can be carried out in $\mathbf{RMU}_k^{(c)}$*, for* $k \geq 1, c \geq 0$.

Example 2.14. *The constant functions* -1 *and* π *can each be constructed in* $\mathbf{RMU}_k^{(1)}$*. To get* -1 *we just find the root of* $x+1$*. For* π*, see the proof of lemma 5.1 in [6] (in case the reader checks the reference, note that despite initial appearances, a single application of* UMU *suffices to define* π*).*

Other examples of functions in all the function algebras are $\sin x$, e^x, and the constant functions q (for any rational q).

We list some basic properties of these function algebras in the next lemma.

Lemma 2.15. ***(Properties of Function Algebras)***

1. $\pi\mathbf{RMU}_k^{(c)} \subseteq \mathbf{RMU}_k^{(c+1)}$.
2. $\mathbf{F}_k \subseteq \mathbf{F}_{k-1}$*, where* $\mathbf{F}_k$ *is any of the function algebras in definition 2.10 containing* θ_k *as a basic function, and* $\mathbf{F}_{k-1}$ *is the same function algebra with the basic function* θ_{k-1} *instead.*
3. *For every function algebra of definition 2.10, with the exclusion of the function algebra* $\mathbf{ODE}_k$*, if* f *is one of its functions, then the domain of* f *is* $\mathbb{R}^n$ *for some* n*.*
4. *Every function in* $\mathbf{ODE}_k$ *has an open domain and is* $\mathcal{C}^{k-1}$*, for* $k \geq 2$*.*

Proof

Point 1: Follows from example 2.14

Point 2: Follows, since $\theta_k(x) = x \cdot \theta_{k-1}(x)$, and all the algebras are closed under multiplication.

Point 3: By $\mathbb{R}^\times$ we mean $\mathbb{R}^n$ for some $n \geq 1$. The conditions on the input to Inverse and UMU in fact require the input function to have domain $\mathbb{R}^\times$; the output function also has domain $\mathbb{R}^\times$. Since the basic functions have domain $\mathbb{R}^\times$, and the operations (comp, Inverse, UMU, and LI) all preserve that property whenever they are defined, all the functions in $\mathbf{RMU}_k$, $\mathbf{RMU}_k^{(c)}$, $\pi\mathbf{RMU}_k^{(c)}$, $\mathbf{INV}_k$, and $\mathbf{INV}_k^{(c)}$ have domain $\mathbb{R}^\times$; by definition, the functions of $\mathbf{ODE}_k^\star$ have domain $\mathbb{R}^\times$.

Point 4: The basic functions, and in particular θ_k, are all $\mathcal{C}^{k-1}$ and have open domains. To conclude with this point, we just need to check that all the operations preserve these properties. Suppose the function f and the vector-valued function $\mathbf{g} = (g_1, \ldots, g_n)$ are $\mathcal{C}^{k-1}$ and have open domains. It is known that the composition $f(g_1, \ldots, g_n)$ is $\mathcal{C}^{k-1}$. Suppose the domains of f and $\mathbf{g}$ are the open sets F and G, respectively. Then the domain of $f(g_1, \ldots, g_n)$ is $G \cap \mathbf{g}^{-1}(F)$, which is open, since the continuity of $\mathbf{g}$ implies $\mathbf{g}^{-1}(F)$ is open. For the ODE operation, suppose that $g_1, \ldots, g_k$ are in the algebra. By inductive hypothesis they are $\mathcal{C}^{k-1}$ and have open domains. Therefore, by lemma 2.8 (using the fact that $k-1 \geq 1$), the unique solution output by ODE is $\mathcal{C}^{k-1}$ and has an open domain.

■

To make the connection to computable analysis, we consider a limit operation which allows us to take the limit of a function as some argument goes to infinity, provided the function *converges rapidly* to its limit, i.e. this is exactly the kind of limit that is implicit in the definitions of computable analysis.

Definition 2.16. ***(Limits)***

- *The operation* LIM *takes as input a function* $f^*(\bar{x}, t) : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$*. It is defined on* f^* *if:*

  1. *For all real* $\bar{x}$*, the limit* $f(\bar{x}) = \lim_{t \to \infty} f^*(\bar{x}, t)$ *exists, and*
  2. *For all real* $t > 0$ *and all real* $\bar{x}$*,* $|f(\bar{x}) - f^*(\bar{x}, t)| \leq 1/t$*.*

  Otherwise it is undefined. If LIM *is defined on* f^* *then it returns* $f(\bar{x}) : \mathbb{R}^k \to \mathbb{R}$ *(the function* f *may be referred to as* $\mathsf{LIM}(f^*)$*).*

  We will see in definition 3.7, that f^* *is an "approximation" of* f*.*

- *Given a class of functions* $\mathbf{F}$*, we let* $\mathbf{F}(\mathsf{LIM})$ *denote the closure of* $\mathbf{F}$ *under the operation* LIM*.*

It is easy to check that $\mathbf{C}_{\mathbb{R}}$ is closed under LIM, i.e. $\mathbf{C}_{\mathbb{R}} = \mathbf{C}_{\mathbb{R}}(\mathsf{LIM})$. For all the classes $\mathbf{F}$ considered in this paper, we in fact only need to apply LIM once to any function, i.e. $\{\mathsf{LIM}(h) \mid h \text{ is in } \mathbf{F}\} = \mathbf{F}(\mathsf{LIM})$.

We now state the main theorem, obtaining three characterizations of the total computable functions via three function algebras.

Theorem 2.17. *For* $k \geq 2$*,* $\mathbf{C}_{\mathbb{R}} = \mathbf{ODE}^{\star}_k(\mathsf{LIM}) = \mathbf{RMU}_k(\mathsf{LIM}) = \mathbf{INV}_k(\mathsf{LIM})$.

We consider the characterization by $\mathbf{ODE}^{\star}_k(\mathsf{LIM})$ the most natural and interesting. In characterizing $\mathbf{C}_{\mathbb{R}}$ by $\mathbf{ODE}^{\star}_k(\mathsf{LIM})$, we have characterized computable analysis by a function algebra which differs from the real primitive recursive functions in two essential ways: The presence of the limit operation and the presence of the extra basic function θ_k. The theorem improves the previous results of this kind. In our paper [11] we partially characterized $\mathbf{C}_{\mathbb{R}}$ by $\mathbf{ODE}^{\star}_k(\mathsf{LIM})$; namely, we only characterized those function of $\mathbf{C}_{\mathbb{R}}$ which are $\mathcal{C}^2$ and have a compact domain. With the same restriction on $\mathbf{C}_{\mathbb{R}}$, Bournez and Hainry [6] partially characterized $\mathbf{C}_{\mathbb{R}}$ by $\mathbf{RMU}_k(\mathsf{LIM})$, where the operation LI is replaced by a slightly unnatural variant; however, we should note that their result allows the limit operation to be interleaved with the other operations of the function algebra. In addition to providing a full characterization, we introduce the new characterization by $\mathbf{INV}_k(\mathsf{LIM})$, and our proof uses our method of approximation.

3. The Proof

Since our proof boils down to proving a number of inclusions, we summarize these inclusions below, making reference to the lemmas which immediately imply them. The first step (the *main step*) will be discussed in part 3.2. The second step (the *minor step*), discussed in part 3.1, summarizes a series of inclusions that follow immediately from the referenced lemmas. The third step simply puts together the first two steps in order to prove theorem 2.17.

1. (**The Main Step**) Lemma 3.6 will show that for any $k \geq 1$, we have:

$$\mathbf{C}_{\mathbb{R}} \subseteq \pi\mathbf{RMU}_k^{(1)}(\mathsf{LIM}).$$

2. (**The Minor Step**) The lemmas 2.15, 3.1, 3.2, 3.4, and 3.5 imply the following two chains of inclusions:
   - For $k \geq 5$:
     $\pi\mathbf{RMU}_k^{(1)} \subseteq \mathbf{RMU}_k^{(2)} \subseteq \mathbf{INV}_k^{(2)} \subseteq \mathbf{ODE}_{k-2}^{\star} \subseteq \mathbf{ODE}_{k-3}^{\star} \subseteq \mathbf{C}_{\mathbb{R}}$
   - For $k \geq 2$:
     $\pi\mathbf{RMU}_k^{(1)} \subseteq \mathbf{RMU}_k^{(2)} \subseteq \mathbf{RMU}_k \subseteq \mathbf{INV}_k \subseteq \mathbf{C}_{\mathbb{R}}$
3. The theorem 2.17 follows immediately: We arrive at a cycle of inclusions by closing the classes of step 2 under limits, combining the resulting inclusions with the inclusion of step 1, and using the fact that $\mathbf{C}_{\mathbb{R}}$ is closed under limits.

Our proof in fact also characterizes $\mathbf{C}_{\mathbb{R}}$ by $\pi\mathbf{RMU}_k^{(1)}(\mathsf{LIM})$, $\mathbf{RMU}_k^{(2)}(\mathsf{LIM})$ and $\mathbf{INV}_k^{(2)}(\mathsf{LIM})$ (all for $k \geq 2$), though we only included the cleaner characterizations in theorem 2.17.

3.1. The Minor Step

The inclusions for the minor step are discussed in this section. The lemmas of this part are taken right from our previous paper [11], though we have (hopefully) improved the notation in this paper. We use **RMU** for BH, **INV** for $\mathcal{L}$, and **ODE** for G (in the case of BH, it had the extra basic function "-1", which we can remove as per example 2.14). We will discuss the intuitions behind some of the proofs (for the detailed proofs, see the indicated citations of [11]).

The next easy lemma shows that the operation $\mathsf{Inverse}$ can simulate the root-finding operation UMU. From a function $f(x, \bar{a})$, to use $\mathsf{Inverse}$ to find the x_0 such that $f(x_0, \bar{a}) = 0$, we simply invert f (or rather, a function with the same root as f) in the argument x, to obtain some $\tilde{f}(x, \bar{a})$, and then we output $\tilde{f}(0, \bar{a})$ (which equals x_0).

Lemma 3.1. *[11, Lemma 3.2] For $k \geq 2$ and $c \geq 0$, $\mathbf{RMU}_k^{(c)} \subseteq \mathbf{INV}_k^{(c)}$.*

An immediate consequence of the lemma is that $\mathbf{RMU}_k \subseteq \mathbf{INV}_k$, for $k \geq 2$.

The next lemma shows that the operation $\mathsf{Inverse}$ can be simulated by the function algebra $\mathbf{ODE}_k$. While the actual proof gets somewhat involved (see [11] for the detailed proof), the intuition is quite simple. Supposing we want to invert the function $f(x)$ in $\mathbf{ODE}_k$, we recall that the Inverse Function Theorem tells us that

$$\frac{\partial}{\partial x} f^{-1}(x) = \frac{1}{\frac{\partial}{\partial x} f(f^{-1}(x))}.$$

Since the function $\frac{1}{x}$ in $\mathbf{ODE}_k$, we can use $\frac{\partial}{\partial x} f$ to set up the previous differential equation, and thus f^{-1} is in $\mathbf{ODE}_k$. However, having f in $\mathbf{ODE}_k$ does not imply $\frac{\partial}{\partial x} f$ is in $\mathbf{ODE}_k$; yet it suffices to work in the larger class $\mathbf{ODE}_{k-c}$ (for some c).

Lemma 3.2. *[11, Lemma 3.5]* $\mathbf{INV}_k^{(c)} \subseteq \mathbf{ODE}_{k-c}^\star$*, for* $c \geq 0$ *and* $k \geq c+3$.

For our result, it will be fundamental to know when the solution to a differential equation is computable (in the sense of computable analysis). By a classic result of Pour-El and Richards [26], the solution may *not* be computable, even if the differential equation is defined using computable functions and initial conditions. However, under a uniqueness condition, Collins and Graça [13] [14] show that the solution is computable.

Proposition 3.3. *[14, Theorem 21] Consider the initial value problem*

$$(\bar{h})' = \bar{g}(\bar{h}) \quad , \quad \bar{h}(0) = \bar{a},$$

where $\bar{g}$ *is continuous on an open subset of* $\mathbb{R}^n$*, and such that for each fixed initial condition* $\bar{a}$*, there is a* unique solution *on a maximum interval. Then the operator that takes* $\bar{g}$ *and* $\bar{a}$ *to its unique solution, is computable.*

Or, in our terminology, if $\bar{g}$ *is computable, the solution* $\bar{h}$ *(as a function of its main variable and its parameter variables) is computable.*

Note that in our statement of proposition 3.3, we slightly modified the statement appearing in [14]; they had a slight ambiguity which we clarify by writing that "for each fixed initial condition $\bar{a}$, there is a unique solution on a maximum interval." By reading the proof of their theorem (and communicating with an author of [14]), we see that this is their intended meaning, and what they in fact prove. Using the last proposition, we reprove (a slightly strengthened form of) lemma 3.11 of [11] in a much simpler manner; in [11], we used the weaker result of Graça et. al. [19], while now we use proposition 3.3.

Lemma 3.4. *(Almost lemma 3.11 from [11]) For* $k \geq 2$, $\mathbf{ODE}_k^\star \subseteq \mathbf{C}_\mathbb{R}$.

Proof

> We proceed by induction on the structure of $\mathbf{ODE}_k$ (k is fixed), showing that these functions are computable on their domain; the result then holds since $\mathbf{ODE}_k^\star$ is a subset of $\mathbf{ODE}_k$. The basic functions of $\mathbf{ODE}_k$ are all computable. It is well-known that the composition of two (real) computable functions is computable [29]. For the operation ODE, suppose $g_1, \dots, g_k$ are in $\mathbf{ODE}_k$, and are used to set up the differential equation (1). Since $k \geq 2$, each g_i is $\mathcal{C}^1$ and has an open domain, by lemma 2.15 (part 4), and thus the IVP has a unique solution. By inductive hypothesis, the g_i are computable, thus all the requirements of proposition 3.3 are satisfied, so the result of the operation ODE is computable on its domain.
>
> ■

To show that the functions of $\mathbf{INV}_k$ are computable (in the sense of computable analysis) we proceed by induction. The basic functions are computable.

The composition operation preserves computability. The operation LI also preserves computability, using proposition 3.3 in the same say we did in the proof of lemma 3.4 (note that the related function algebra of Bournez and Hainry [6] used a slightly unnatural variant of LI because proposition 3.3 was proved after their work). The inverse operation is known to preserve computability: See [29] (p.180) for the case of a function with a bounded domain; our situation with an unbounded domain is similar (in both cases we can use a binary search algorithm). Thus we have proved the following lemma.

Lemma 3.5. $\mathbf{INV}_k \subseteq \mathbf{C}_{\mathbb{R}}$*, for* $k \geq 2$.

3.2. The Main Step: The Setup

Now we discuss the main step, whose goal is to prove the following lemma.

Lemma 3.6. ***(Main Lemma)*** $\mathbf{C}_{\mathbb{R}} \subseteq \pi\mathbf{RMU}_k^{(1)}(\mathit{LIM})$*, for* $k \geq 1$.

This lemma will follow from a series of lemmas, in which we will use the notion of one class of functions approximating another class of functions. We discuss the notion of approximation (a simplified version of what we developed in [12]) and then introduce a number of intermediary classes of functions which are used to facilitate the proof. At the end of this section we outline the proof, leaving the technical lemmas for section 4.

Definition 3.7. ***(Approximation)***

- *Consider functions* f *and* f^**. We write* $f(\bar{x}) \preceq f^*(\bar{x}, t)$*, if*

$$|f(\bar{x}) - f^*(\bar{x}, t)| < \frac{1}{t},$$

  for all $\bar{x}$ *in the domain of* f*, and all* $t > 0$ *(the variable "t" is called the* ***approximation parameter****); we emphasize that if some* $\bar{x}$ *is in the domain of* f*, then* $(\bar{x}, t)$ *is in the domain of* f^* *for all* $t > 0$*.*

- *For classes of functions* $\mathbf{A}$ *and* $\mathbf{B}$*, we write* $\mathbf{A} \preceq \mathbf{B}$ *if for any* f *in* $\mathbf{A}$ *there is some* f^* *in* $\mathbf{B}$ *such that* $f \preceq f^*$*.*

We reserve the variable t, and sometimes t_1 and t_2, for the approximation parameters.

Remark *From the definition of* LIM *we have the following relations:*

1. *If* $f \preceq f^*$ *then* $f = \mathit{LIM}(f^*)$*;*
2. *If* $\mathbf{A} \preceq \mathbf{B}$ *then* $\mathbf{A} \subseteq \mathbf{B}(\mathit{LIM})$*.*

The approximation relation is transitive, assuming a *niceness* condition.

Definition 3.8. ***(Nice classes)*** *A class of functions* $\mathbf{F}$ *is called* ***nice*** *if it satisfies the following properties:*

1. *It contains the addition function, i.e. for $f(x, y) = x + y$, f is in* $\mathbf{F}$.
2. *It contains the unary identity function, i.e. for $f(x) = x$, f is in* $\mathbf{F}$.
3. *It is closed under composition.*

Lemma 3.9. ***(Transitivity)*** *Suppose* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are classes of functions and suppose* $\mathbf{C}$ *is nice. Then* $\mathbf{A} \preceq \mathbf{B}$ *and* $\mathbf{B} \preceq \mathbf{C}$ *implies* $\mathbf{A} \preceq \mathbf{C}$.

Proof

> Let $f(\bar{x})$ be in $\mathbf{A}$. Thus we have $g(\bar{x}, t_1)$ in $\mathbf{B}$ such that $|f(\bar{x}) - g(\bar{x}, t_1)| < 1/t_1$, and $h(\bar{x}, t_1, t_2)$ in $\mathbf{C}$ such that $|g(\bar{x}, t_1) - h(\bar{x}, t_1, t_2)| < 1/t_2$. Thus $\tilde{h}(\bar{x}, t) = h(\bar{x}, 2t, 2t)$ is in $\mathbf{C}$ since $\mathbf{C}$ is nice; furthermore, $f \preceq \tilde{h}$, by the triangle inequality.
>
> ■

To obtain our result over the reals we will make significant use of the classical computable functions over the naturals.

Definition 3.10. *Let* $\mathbf{C}_{\mathbb{N}}$ *be the functions* $f : \mathbb{N}^k \to \mathbb{N}$, *such that* f *is computable.*

There are numerous characterizations of $\mathbf{C}_{\mathbb{N}}$ with function algebras; we pick one that will be useful to us, defined using the following operations.

Definition 3.11. ***(Bounded sums and products)*** *Let* $f : \mathbb{N} \times \mathbb{N}^k \to \mathbb{N}$ *be a function.*

- *The* ***bounded summation*** *operation* ($\sum$) *takes* f *as input and returns* $g : \mathbb{N} \times \mathbb{N}^k \to \mathbb{N}$, *where* $g(y, \bar{a}) = \sum_{x=0}^{y} f(x, \bar{a})$.
- *The* ***bounded product*** *operation* ($\prod$) *takes* f *as input and returns* $g : \mathbb{N} \times \mathbb{N}^k \to \mathbb{N}$, *where* $g(y, \bar{a}) = \prod_{x=0}^{y} f(x, \bar{a})$.

We define a *search* operation similar to the classic μ operation (i.e. given a function $f(x, \bar{a})$, $\mu(f)$ is the function $g(\bar{a}) =$ the first x_0 such that $f(x_0, \bar{a}) = 0$). Our operation will be limited, but in the end it will be just as powerful as the full μ operation (note, that it searches for a *one* instead of a zero).

Definition 3.12. *The operation* MU *receives a function* $f : \mathbb{N} \times \mathbb{N}^k \to \mathbb{N}$ *as input.* MU *is defined on* f *if the following two conditions are satisfied:*

1. *For each* $\bar{a}$, *there is a unique* $x_{\bar{a}} \geq 1$ *(called the* ***one*** *of* f*) such that*

$$f(x, \bar{a}) = \begin{cases} 0, & \text{if } x < x_{\bar{a}}; \\ 1, & \text{if } x = x_{\bar{a}}; \\ 2, & \text{if } x > x_{\bar{a}}. \end{cases}$$

$a=4$	0	0	0	0	0	0	0
$a=3$	0	0	0	0	0	0	1
$a=2$	0	0	0	0	0	1	2
$a=1$	0	0	1	2	2	2	2
$a=0$	0	0	1	2	2	2	2
	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$	$x=6$

Figure 1: Example of $f(x, a)$ that satisfies the requirements of definition 3.12

2. *The function* $g(\bar{a}) = x_{\bar{a}}$ *is increasing in all arguments.*

Otherwise MU *is undefined on* f. *If* MU *is defined, then its output is* g.

Figure 1 is an example of a function $f(x, a)$ on which MU is defined (i.e. we have just indicated some values of $f(x, a)$, but we could extend f). For any fixed a, $f(x, a)$ satisfies the first requirement of MU. For any fixed x, $f(x, a)$ is decreasing in a, which is exactly what is needed to guarantee the second requirement of MU.

We now use the preceding operations to define a modification of the standard function algebra for the computable functions over $\mathbb{N}$.

Definition 3.13. *(**The function algebra** **NMU**)*

- *Let* $x \dot{-} y = \begin{cases} x - y, & \text{if } x \geq y; \\ 0, & \text{otherwise.} \end{cases}$ *(called **cut-off subtraction**)*
- *Let* **NMU** *be* $\mathsf{FA}[0, 1, +, \dot{-}, \mathbf{P}; \mathsf{comp}, \sum, \prod, \mathsf{MU}]$.
- *Let* $\mathbf{NMU}^{(c)}$ *be the functions of* **NMU** *that have rank* c *with respect to the operation* MU.

We chose the basic functions of **NMU** so that without MU, we get the elementary computable functions, i.e. $\mathsf{FA}[0, 1, +, \dot{-}, \mathbf{P}; \mathsf{comp}, \sum, \prod]$ is exactly the elementary computable functions. The next lemma follows from standard characterizations of the computable functions, since we can simulate the standard $\mu-$operation with our restricted MU by pre-processing a given function with elementary computable functions (similar to [6, proposition 2.2]). The standard normal form theorem (where Kleene's predicate can be taken as elementary, [16, corollary 12-4.5]) allows to get away with a single application of μ or MU.

Lemma 3.14. $\mathbf{C}_{\mathbb{N}} = \mathbf{NMU}^{(1)}$

We will define various classes of functions over $\mathbb{Q}$, used as a bridge between the classes on the naturals and the reals. In the end we will not care about these classes; they are simply intermediary classes defined not with the intention of looking pretty, but with the goal of making the proofs run smoothly.

Definition 3.15. *Let* $\mathbf{C}_{\mathbb{Q}}$ *be* $\{f_{|\mathbb{Q}} \mid f \text{ is in } \mathbf{C}_{\mathbb{R}}\}$.

Thus, $\mathbf{C}_{\mathbb{Q}}$ is just the functions of $\mathbf{C}_{\mathbb{R}}$ with their domains (but not their ranges) restricted to $\mathbb{Q}$. For other classes over $\mathbb{Q}$, the following definitions of a kind of denominator, numerator, and sign function will be convenient.

Definition 3.16. *We define three functions from $\mathbb{Q}$ to $\mathbb{Z}$. For a rational $(\text{-}1)^s\,\frac{p}{q}$ presented in lowest terms, where p and q are positive integers, let:*

$$\mathsf{D}\left((\text{-}1)^s\,\frac{p}{q}\right) = (\text{-}1)^s q \quad (\mathsf{D}(0) = 0)$$

$$\mathsf{N}\left((\text{-}1)^s\,\frac{p}{q}\right) = (\text{-}1)^s p \quad (\mathsf{N}(0) = 0)$$

$$\mathsf{sign}(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ 1, & \text{if } x > 0. \end{cases}$$

Note that both D and N implicitly reduce their argument to lowest terms and carry the sign (the latter property is simply to facilitate some technical development); e.g. $\mathsf{D}(-\frac{2}{6}) = -3$. We define another notion of computability over $\mathbb{Q}$ using Turing machines, but unlike $\mathbf{C}_{\mathbb{Q}}$ the machine gets the rational input exactly, coded using naturals.

Definition 3.17. *A function $f : \mathbb{Q} \to \mathbb{Q}$ is in $\mathbf{dC}_{\mathbb{Q}}$ if there is a Turing machine over $\mathbb{N}$ that computes it in the following sense:*

> *On input $x \in \mathbb{Q}$ the machine is given the triple $\langle\, |\mathsf{N}(x)|, |\mathsf{D}(x)|, \mathsf{sign}(x)\,\rangle$, and it computes the triple $\langle\, |\mathsf{N}(f(x))|, |\mathsf{D}(f(x))|, \mathsf{sign}(f(x))\,\rangle$.*

For a function $f : \mathbb{Q}^k \to \mathbb{Q}$, the definition is similar (on input $x_1, \ldots, x_k \in \mathbb{Q}$ the machine is given a corresponding triple for each x_i).

Note that $\mathbf{C}_{\mathbb{Q}}$ contains only continuous functions, while $\mathbf{dC}_{\mathbb{Q}}$ contains discontinuous functions (where we say a function with domain $\mathbb{Q}^k$ is continuous if it can be extended to a continuous function with domain $\mathbb{R}^k$); e.g. given the exact rational as a code we can easily decide if it is larger than 0 or not, thus the discontinuous function sign is in $\mathbf{dC}_{\mathbb{Q}}$. In general, if a class of functions (over $\mathbb{R}$ or $\mathbb{Q}$) contains only continuous functions we call it a **continuous class** and otherwise we call it a **discontinuous class**; we will put the symbol "**d**" in front of discontinuous classes, as was done with $\mathbf{dC}_{\mathbb{Q}}$. In fact it will typically be important for us that a class of functions is not just continuous, but that it has *modulus functions*, i.e. its continuity is exhibited by a function (definition 3.19 differs from our definition in [12]; to simplify the discussion, this paper uses the reciprocal of what we used earlier).

Definition 3.18. ***(Norm-increasing)*** *A function $f(x, \bar{a})$ is* ***norm-increasing in x*** *if f is* positive everywhere *and for any fixed $\bar{a}$, $f(x, \bar{a})$ increases as $|x|$ increases. The function f is simply called norm-increasing if it is norm-increasing in all its arguments.*

It is easy to check that if f and g are norm-increasing then so are $f + g$, $f \cdot g$, and $f \circ g$. We let $|x_1, \ldots, x_k|$ abbreviate $|x_1| + \ldots + |x_k|$; thus $|\bar{b} - \bar{a}|$ abbreviates $|b_1 - a_1| + \ldots + |b_n - a_n|$.

Definition 3.19. ***(Modulus)*** *Suppose $f(\overline{x}) : D^k \to D$ and $m(\overline{x}, z) : D^{k+1} \to D$ are functions, where D is either $\mathbb{Q}$ or $\mathbb{R}$. We say m is a* ***modulus*** *for f if m is* norm-increasing *and for all $\bar{x}$, $\bar{y}$, $z \in D$, $z > 0$,*

$$|\bar{x} - \bar{y}| \leq \frac{1}{m(\bar{x}, z)} \text{ implies } |f(\bar{x}) - f(\bar{y})| \leq \frac{1}{z}.$$

Note that a function (over $\mathbb{Q}$ or $\mathbb{R}$) that has a modulus is continuous. Also, if m is a modulus for f, then so is a larger norm-increasing function.

Throughout the paper, we will use the important technical idea of **linearizing** a function. Suppose $f(x, \bar{a})$ is a function, and for a fixed $\bar{a}$, as x varies, $f(x, \bar{a})$ is shown in figure 3(a) (see page 32) as the function whose graph is the dashed line. If we just consider the values of $f(x, \bar{a})$ when x is an integer, connecting these values by straight lines yields what we call $\mathsf{Lin}(f, x)$, *linearizing f with respect to the argument x*, shown as the function $\hat{f}$ of figure 3(a), whose graph is a solid line. By $\lfloor x \rfloor$ we mean the greatest integer less than or equal to x, and by $\lceil x \rceil$ we mean the smallest integer greater than or equal to x.

Definition 3.20. ***(Linearization)*** *Suppose $f(x_1, \ldots, x_k) : D \times E \to \mathbb{R}$, where $\mathbb{Z} \subseteq D \subseteq \mathbb{R}$, and $E \subseteq \mathbb{R}^{k-1}$. The* ***linearization to $\mathbb{Q}$ (respectively, to $\mathbb{R}$) of*** *$f(x_1, \ldots, x_k)$* ***with respect to*** *x_1, denoted $\mathsf{Lin}(f, x_1)$, is the function h with domain $\mathbb{Q} \times E$ (respectively, $\mathbb{R} \times E$), defined by*

$$\begin{aligned} h(x_1, \ldots, x_k) = {} & f(\lfloor x_1 \rfloor, x_2, \ldots, x_k) \cdot (\lfloor x_1 \rfloor + 1 - x_1) \\ & + f(\lceil x_1 \rceil, x_2, \ldots, x_k) \cdot (x_1 - \lfloor x_1 \rfloor). \end{aligned}$$

We define $\mathsf{Lin}(f, x_i)$ similarly, but with respect to the variable x_i. By $\mathsf{Lin}(f)$ we mean the ***full linearization*** *$\mathsf{Lin}(\ldots \mathsf{Lin}(\mathsf{Lin}(f, x_1), x_2) \ldots x_k)$.*

Whether we are linearizing to $\mathbb{Q}$ or to $\mathbb{R}$ will generally be clear from context, and so go unmentioned. Also note that while we defined $\mathsf{Lin}(f)$ by linearizing f with respect to x_1, then x_2, and so on, in fact the order does not matter. The next lemma holds because the full linearization operation ignores the values of the function off of the integers, and is well-behaved in-between.

Lemma 3.21. *Suppose $f(\bar{x})$ is a function, and $\hat{f}(\bar{x}) = \mathsf{Lin}(f)$, the linearization to $\mathbb{Q}$ or to $\mathbb{R}$. The following holds.*

1. *For $\overline{x} \in \mathbb{Z}$, $\hat{f}(\overline{x}) = f(\overline{x})$.*

2. $\hat{f}$ *is continuous.*
3. *For any* $\bar{x}$ *(in* $\mathbb{Q}$ *or* $\mathbb{R}$*), let*
$$\mathcal{X} = \{f(\lfloor x_1 \rfloor, \ldots, \lfloor x_k \rfloor), \ldots, f(\lceil x_1 \rceil, \ldots, \lceil x_k \rceil)\},$$
where we range over all 2^k *combinations of* $\lfloor\cdot\rfloor$ *and* $\lceil\cdot\rceil$*. The following holds:*
$$\mathsf{min}(\mathcal{X}) \leq \hat{f}(\bar{x}) \leq \mathsf{max}(\mathcal{X})$$

We will want to *linearize operations*, converting operations over $\mathbb{N}$ to operations over $\mathbb{Q}$.

Definition 3.22. *Suppose* OP *is an operation which takes a function* $f : \mathbb{N}^k \to \mathbb{N}$ *and returns a function* $g : \mathbb{N}^m \to \mathbb{N}$*. By* $\mathsf{OP}_{\mathbb{Q}}$*, we mean the following operation:*

1. $\mathsf{OP}_{\mathbb{Q}}$ *takes as input* $f : \mathbb{Q}^k \to \mathbb{Q}$ *such that* $f_{|\mathbb{N}} : \mathbb{N}^k \to \mathbb{N}$*.*
2. $\mathsf{OP}_{\mathbb{Q}}$ *then applies* OP *to* $f_{|\mathbb{N}}$ *to get some* $g : \mathbb{N}^m \to \mathbb{N}$*.*
3. *Let* $\tilde{g}$ *extend* g *to the integers so that* $\tilde{g}$ *is zero if any argument is negative.*
4. $\mathsf{OP}_{\mathbb{Q}}$ *outputs* $\mathsf{Lin}(\tilde{g})$*.*

We illustrate the previous definition by considering an example with MU (recall definition 3.12) and the derived operation $\mathsf{MU}_{\mathbb{Q}}$. Consider some function $f : \mathbb{Q}^2 \to \mathbb{Q}$ which is an extension of the function over the naturals illustrated in figure 1 (see page 15); thus f satisfies the condition of step 1 of the above definition. We apply $\mathsf{MU}_{\mathbb{Q}}$ to f and argument x. Step 2 defines a function g over the naturals such that $g(0) = 2$, $g(1) = 2$, $g(2) = 5$, $g(3) = 6$, and so on. Step 3 extends g to $\tilde{g}$, a function with domain $\mathbb{Z}$, whose value is zero on negative integers. Finally step 4 connects the following points by straight line segments: $\{\ldots, (-1, \tilde{g}(-1)),\ (0, \tilde{g}(0)),\ (1, \tilde{g}(1)),\ (2, \tilde{g}(2)), \ldots\} = \{\ldots, (-1, 0), (0, 2), (1, 2), (2, 5), \ldots\}$. The operation $\sum_{\mathbb{Q}}$ is more intuitive: To sum up to a rational y, we compute two sums, the sum up to $\lfloor y \rfloor$ and the sum up to $\lceil y \rceil$, and we return the weighted average of the two sums according to where y is between $\lfloor y \rfloor$ and $\lceil y \rceil$.

We will now define two function algebras over $\mathbb{Q}$, one continuous and the other discontinuous.

Definition 3.23. ***(The continuous function algebra* QMU*)***

- *Let* $\mathsf{div} : \mathbb{Q} \to \mathbb{Q}$ *such that* $\mathsf{div}(x) = \mathsf{Lin}(f)$*, where* $f(x) = \begin{cases} 1/x, & \text{if } x \geq 1; \\ 1, & \text{otherwise.} \end{cases}$
- *Let* **QMU** *be* $\mathsf{FA}[0, 1, -1, +, *, \mathsf{div}, \theta_1, \mathbf{P}; \mathsf{comp}, \sum_{\mathbb{Q}}, \prod_{\mathbb{Q}}, \mathsf{MU}_{\mathbb{Q}}, \mathsf{Lin}]$*.*
- *Let* $\mathbf{QMU}^{(c)}$ *be the functions of* **QMU** *that have rank* c *with respect to the operation* $\mathsf{MU}_{\mathbb{Q}}$*.*

The function div allows us to construct rationals within the function algebra. While other functions could do the job, div will have some technical advantages.

The next function algebra is our *discontinuous* function algebra over $\mathbb{Q}$, differing from **QMU** in two ways: It does not have the operation Lin and it does have the extra basic function D, though we restrict how D can be used in composition (this restriction is used in the proof of lemma 4.3).

Definition 3.24. ***(The discontinuous function algebra dQMU)***

- *Let* $\mathbf{dQMU}$ *be* $FA[0, 1, -1, +, *, div, \theta_1, D, \mathbf{P}; comp, \sum_{\mathbb{Q}}, \prod_{\mathbb{Q}}, MU_{\mathbb{Q}}]$, except that in **dQMU** we restrict the comp operation as follows*:*

  On input $f(y_1, \ldots, y_k), g_1(x_1, \ldots, x_m), \ldots, g_k(x_1, \ldots, x_m)$*, we can only form the composition* $f(g_1(x_1, \ldots, x_m), \ldots, g_k(x_1, \ldots, x_m))$ *if* f *is* **D−free***, meaning that the function D never appears in its construction.*

- *Let* $\mathbf{dQMU}^{(c)}$ *be the functions of* **dQMU** *that have rank c with respect to the operation* $MU_{\mathbb{Q}}$*.*

The main difference between **dQMU** and **QMU** is that **dQMU** can break up rational inputs into their integer components (using the function D) while **QMU** cannot. To make up for this weakness, **QMU** has the extra operation Lin, which is important for proving that **QMU** can approximate the functions of **dQMU** which have an appropriate modulus (lemma 4.5).

We have now defined all the classes of functions that we will need, modulo the restricting of the two discontinuous classes to certain subsets of their continuous functions. We will restrict $\mathbf{dC}_{\mathbb{Q}}$ and $\mathbf{dQMU}^{(1)}$ to their functions that happen to have a modulus in $\mathbf{QMU}^{(1)}$ (other classes, such as $\mathbf{C}_{\mathbb{N}}$, which have the same "growth rate" as $\mathbf{QMU}^{(1)}$ could have been used in its place, though our choice seems most convenient).

Definition 3.25. ***(Continuous restrictions of*** $\mathbf{dC}_{\mathbb{Q}}$ ***and dQMU)***

- *Let* $\widetilde{\mathbf{dC}_{\mathbb{Q}}}$ *be the functions in* $\mathbf{dC}_{\mathbb{Q}}$ *that have a modulus function in* $\mathbf{QMU}^{(1)}$*.*

- *Let* $\widetilde{\mathbf{dQMU}}^{(1)}$ *be the functions in* $\mathbf{dQMU}^{(1)}$ *that have a modulus function in* $\mathbf{QMU}^{(1)}$*.*

Outline of the proof of the main lemma 3.6.

We wish to prove that $\mathbf{C}_{\mathbb{R}} \subseteq \pi\mathbf{RMU}_k^{(1)}(\mathsf{LIM})$ for $k \geq 1$. Consider the sequence of approximations in figure 2 (note that containment is an approximation with zero error). Once we have proved these four approximations, we apply transitivity (lemma 3.9) to obtain:

$$\mathbf{C}_{\mathbb{Q}} \preceq \pi\mathbf{RMU}_k^{(1)}.$$

$$\mathbf{C}_{\mathbb{Q}} \overset{\text{Lem. 4.6}}{\preceq} \widetilde{\mathbf{dC}_{\mathbb{Q}}} \overset{\text{Lem. 4.2}}{\subseteq} \widetilde{\mathbf{dQMU}}^{(1)} \overset{\text{Lem. 4.5}}{\preceq} \mathbf{QMU}^{(1)} \overset{\text{Lem. 4.12}}{\preceq} \pi\mathbf{RMU}_k^{(1)}$$

Figure 2: Approximations to prove $\mathbf{C}_{\mathbb{Q}} \preceq \pi\mathbf{RMU}_k^{(1)}$.

To apply transitivity we have a niceness condition, which is obvious in all the required cases, namely $\mathbf{QMU}^{(1)}$ and $\pi\mathbf{RMU}_k^{(1)}$. Note that we need not worry about the niceness of $\widetilde{\mathbf{dQMU}}^{(1)}$ since the inclusion in figure 2 allows us to conclude $\mathbf{C}_{\mathbb{Q}} \preceq \widetilde{\mathbf{dQMU}}^{(1)}$ without relying on lemma 3.9. Now consider some $f_{\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}$ in $\mathbf{C}_{\mathbb{R}}$. By definition of $\mathbf{C}_{\mathbb{Q}}$, $f_{\mathbb{R}}$ is the extension of some function $f_{\mathbb{Q}} : \mathbb{Q}^n \to \mathbb{R}$ in $\mathbf{C}_{\mathbb{Q}}$. We have shown that $f_{\mathbb{Q}}$ is approximated by some continuous function, say f^*, with domain $\mathbb{R}^{n+1}$, in $\pi\mathbf{RMU}_k^{(1)}$. Since all the functions in $\mathbf{C}_{\mathbb{R}}$ are continuous, we can easily verify that $f_{\mathbb{R}}$ is also approximated by f^*, which means that $\mathbf{C}_{\mathbb{R}} \preceq \pi\mathbf{RMU}_k^{(1)}$. Recalling the remark after definition 3.7, we see that the main lemma 3.6 follows by closing $\pi\mathbf{RMU}_k^{(1)}$ under LIM.

■

The next section is devoted to proving the four approximations in figure 2.

4. The Main Step: Technical Aspects

We prove the sequence of approximations in figure 2. The final approximation, $\mathbf{QMU}^{(1)} \preceq \pi\mathbf{RMU}_k^{(1)}$, is the technical heart of the argument. The other three approximations are similar to approximations appearing in our paper [12]; where the proofs are similar we will make reference to the appropriate parts of [12]. First we discuss the three easier approximations, and then we discuss the final one.

4.1. The First Three Approximations

The next lemma holds because the classes **QMU** and **dQMU** contain extensions of the basic functions of **NMU** (note that $x \dot{-} y = \theta_1(x-y)$) and their operations extend appropriately those of **NMU**.

Lemma 4.1. *Every function of* $\mathbf{NMU}^{(1)}$ *can be extended to a function in* $\mathbf{QMU}^{(1)}$ *and to a D−free function in* $\mathbf{dQMU}^{(1)}$*.*

A typical use of the last lemma will be to start with a function which is clearly in $\mathbf{C}_{\mathbb{N}}$ (and thus in $\mathbf{NMU}^{(1)}$, by lemma 3.14), extend it to a function in one of the rational classes, and then perform some basic manipulations to this extension so that it works properly for negative integers and rationals (we will typically just refer to the last lemma without discussing the basic manipulations done in the rational classes). We note some useful functions in $\mathbf{dQMU}^{(1)}$ (recall definition 3.16):

- sign is in $\mathbf{dQMU}^{(1)}$ (because $\mathsf{sign}(x) = \mathsf{sign}_{|\mathbb{Z}}(\mathsf{D}(x))$, and using lemma 4.1, $\mathsf{sign}_{|\mathbb{Z}}$ has an extension in $\mathbf{dQMU}^{(1)}$).
- $|\mathsf{N}(x)|$ is in $\mathbf{dQMU}^{(1)}$ (because $|\mathsf{N}(x)| = x * \mathsf{D}(x)$).

The following lemma proves the inclusion from figure 2.

Lemma 4.2. $\widetilde{\mathbf{dC}_{\mathbb{Q}}} \subseteq \widetilde{\mathbf{dQMU}}^{(1)}$

Proof

We show that $\mathbf{dC}_{\mathbb{Q}} \subseteq \mathbf{dQMU}^{(1)}$ and the lemma follows immediately. Suppose $f(x)$ is in $\mathbf{dC}_{\mathbb{Q}}$, so there are f_1, f_2, f_3 in $\mathbf{C}_{\mathbb{N}}$, (and so in $\mathbf{NMU}^{(1)}$ by lemma 3.14) such that

$$f(x) = \frac{f_1(|\mathsf{N}(x)|, |\mathsf{D}(x)|, \mathsf{sign}(x))}{f_2(|\mathsf{N}(x)|, |\mathsf{D}(x)|, \mathsf{sign}(x))}(-1)^{f_3(|\mathsf{N}(x)|, |\mathsf{D}(x)|, \mathsf{sign}(x))}$$

By lemma 4.1, f_1, f_2, and f_3 have extensions in $\mathbf{dQMU}^{(1)}$, say g_1, g_2, and g_3 respectively, which are D-free. The class $\mathbf{dQMU}^{(1)}$ also contains a function $S(x)$ such that $S(0) = 1$ and $S(1) = -1$. Thus $f(x)$ is in $\mathbf{dQMU}^{(1)}$ because we can write it as a composition meeting the requirements of $\mathbf{dQMU}^{(1)}$:

$$\begin{aligned} & g_1(|\mathsf{N}(x)|, |\mathsf{D}(x)|, \mathsf{sign}(x)) \\ & \quad * \mathsf{div}(g_2(|\mathsf{N}(x)|, |\mathsf{D}(x)|, \mathsf{sign}(x))) \\ & \quad\quad * S(g_3(|\mathsf{N}(x)|, |\mathsf{D}(x)|, \mathsf{sign}(x))) \end{aligned}$$

■

The next lemma (a simplification of [12, lemma 17]) essentially shows that we can assume that any function in $\mathbf{dQMU}^{(1)}$ only applies D directly to arguments, and not to more complicated constructions.

Lemma 4.3. *For any function $h(\bar{x})$ in $\mathbf{dQMU}^{(1)}$, there is a function $\tilde{h}(\bar{x}, \bar{y})$ in $\mathbf{QMU}^{(1)}$ such that $h(\bar{x}) = \tilde{h}(\bar{x}, \mathsf{D}(\bar{x}))$.*

Proof

We proceed by induction on $\mathbf{dQMU}^{(1)}$. The basic functions in $\mathbf{dQMU}^{(1)}$ trivially satisfy the claim since all of them except D are in $\mathbf{QMU}^{(1)}$ and D itself fits the desired form.

For composition, suppose $h(x) = f(g(x))$ (composition with more functions works similarly), where f is $\mathsf{D}-$free. Thus f is in $\mathbf{QMU}^{(1)}$, and by inductive hypothesis, we have $\tilde{g}(x,y)$ in $\mathbf{QMU}^{(1)}$ such that $g(x) = \tilde{g}(x, \mathsf{D}(x))$. Let $\tilde{h}(x,y) = f(\tilde{g}(x,y))$, a function in $\mathbf{QMU}^{(1)}$ such that $h(x) = \tilde{h}(x, \mathsf{D}(x))$ as claimed.

For the three other operations, the inductive steps are easy, due to their definition via linearization. For example, consider bounded sums. Consider the function $\sum_{\mathbb{Q}}(f(x,\bar{a}))$, where inductively we can write $f(x,\bar{a}) = \tilde{f}(x,\bar{a},D(x),D(\bar{a}))$, for some $\tilde{f}$ in $\mathbf{QMU}^{(1)}$. In $\mathbf{QMU}^{(1)}$, we can easily define (using lemma 4.1) a function $s(u)$, such that for $u \in \mathbb{N}$, $s(u) = \begin{cases} 0, & \text{if } u = 0; \\ 1, & \text{if } u \geq 1 \end{cases}$ (we do not care what s does off of the naturals); i.e. on the naturals $s(u) = \mathsf{D}(u)$. So $f(x,\bar{a}) = \tilde{f}(x,\bar{a},s(x),s(\bar{a}))$ for $x,\bar{a} \in \mathbb{N}$. Recall that by definition (recall definition 3.22) $\sum_{\mathbb{Q}}$ ignores the values of the input function off of the naturals, thus $\sum_{\mathbb{Q}}(f(x,\bar{a})) = \sum_{\mathbb{Q}}(\tilde{f}(x,\bar{a},s(x),s(\bar{a})))$ is in $\mathbf{QMU}^{(1)}$. The other operations follow similarly, since they also ignore values off of the naturals.

■

We write $\lceil u \rfloor$ in a statement to indicate that the statement holds if each occurrence of $\lceil u \rfloor$ is replaced by either $\lfloor u \rfloor$ or $\lceil u \rceil$; we allow one occurrence of $\lceil u \rfloor$ to be replaced by $\lfloor u \rfloor$ and another occurrence in the same statement to be replaced by $\lceil u \rceil$ (this notation will be used only in lemmas 4.4, 4.5, and 4.6). Notice that given x, for large M, $\frac{\lceil Mx \rfloor}{\lceil M \rfloor}$ is approximately x, thus a small calculation proves the next lemma.

Lemma 4.4. *Given $r(x,\bar{a})$ in $\mathbf{QMU}^{(1)}$ (respectively, in $\mathbf{C}_{\mathbb{Q}}$) there is $\tilde{r}(x,\bar{a})$ in $\mathbf{QMU}^{(1)}$ (respectively, in $\mathbf{C}_{\mathbb{Q}}$) such that*

$$\left| x - \frac{\lceil \tilde{r}(x,\bar{a}) \cdot x \rfloor}{\lceil \tilde{r}(x,\bar{a}) \rfloor} \right| \leq \frac{1}{r(x,\bar{a})}, \text{ for all } x, \bar{a} \in \mathbb{Q}.$$

We now prove an approximation from figure 2, showing that the class of continuous functions $\mathbf{QMU}^{(1)}$ can approximate functions from the discontinuous class $\mathbf{dQMU}^{(1)}$ which have a $\mathbf{QMU}^{(1)}$ modulus. The core of the proof depends on the fact that $\mathbf{QMU}^{(1)}$ has Lin as one of its operations (in fact, this proof is the reason for including the operation in the definition of $\mathbf{QMU}^{(1)}$); the proof is similar to part of the proof of lemma 18 of [12].

Lemma 4.5. $\widetilde{d\mathbf{QMU}}^{(1)} \preceq \mathbf{QMU}^{(1)}$

Proof

Let $f(x)$ be in $\mathbf{dQMU}^{(1)}$, assuming just one variable for ease of readability, and suppose $m(x,t)$ is a $\mathbf{QMU}^{(1)}$ modulus function for f. Our goal is now to find some $f^*(x,t)$ in $\mathbf{QMU}^{(1)}$, such that $f \preceq f^*$.

It will be more convenient to write the rational x as a/b, where $a, b \in \mathbb{Z}$. By lemma 4.3, $f(a/b)$ can be written as $g(a/b, \mathsf{D}(a/b))$, where g is in $\mathbf{QMU}^{(1)}$. We define a function $F : \mathbb{Z}^2 \to \mathbb{Q}$ by $F(a,b) = g(a/b, \mathsf{D}(a/b))$. The functions a/b and $\mathsf{D}(a/b)$ (as functions of type $\mathbb{Z}^2 \to \mathbb{Q}$) have extensions in $\mathbf{QMU}^{(1)}$, using lemma 4.1 and functions (like div) in $\mathbf{QMU}^{(1)}$ (for one concrete approach see the functions dv and bottom in the proof of lemma 18 of [12]). Thus F has an extension in $\mathbf{QMU}^{(1)}$ (which we will also call F), and $\hat{F} = \mathsf{Lin}(F)$ is in $\mathbf{QMU}^{(1)}$. To relate $\hat{F}$ and f, consider the rational grid $\mathbb{Q} \times \mathbb{Q}$, where $\hat{F}$ and f have been evaluated at all the points, i.e. for all $(u,v) \in \mathbb{Q} \times \mathbb{Q}$ we consider $\hat{F}(u,v)$ and $f(u/v)$. Note that on the integer sub-grid $\mathbb{Z} \times \mathbb{Z}$, $\hat{F}$ and f agree, i.e. for $(u,v) \in \mathbb{Z} \times \mathbb{Z}$, $\hat{F}(u,v) = f(u/v)$. The idea of the proof is as follows. Given a rational x, we find large $u, v \in \mathbb{Q}$ such that $x = u/v$. Since u and v are large, the fraction u/v is approximately the same as any of the four fractions $\lfloor u \rfloor / \lfloor v \rfloor, \lceil u \rceil / \lfloor v \rfloor, \lfloor u \rfloor / \lceil v \rceil, \lceil u \rceil / \lceil v \rceil$. Consequently $\hat{F}(u,v)$ and $f(u/v)$ will both be close to the four values $\hat{F}(\lfloor u \rfloor, \lfloor v \rfloor), \hat{F}(\lceil u \rceil, \lfloor v \rfloor), \hat{F}(\lfloor u \rfloor, \lceil v \rceil), \hat{F}(\lceil u \rceil, \lceil v \rceil)$ and so as desired, $\hat{F}(u,v)$ will be approximately equal to $f(x)$. We now carry out the details of the proof.

Applying lemma 4.4 to m, we get some $\tilde{m}(x,t)$ in $\mathbf{QMU}^{(1)}$ such that:

$$(\bigstar) \qquad \left| x - \frac{\lceil x \cdot \tilde{m}(x,t) \rfloor}{\lceil \tilde{m}(x,t) \rfloor} \right| \leq \frac{1}{m(x,t)}.$$

We define $f^*(x,t) = \hat{F}(x \cdot \tilde{m}(x,t), \tilde{m}(x,t))$ in $\mathbf{QMU}^{(1)}$. We just need to show that $|f(x) - f^*(x,t)| \leq 1/t$. For any $x, t \in \mathbb{Q}$, let X be the following set of 4 numbers:

$$\begin{aligned}
\{ \;\; & \hat{F}(\lfloor x \cdot \tilde{m}(x,t) \rfloor, \lfloor \tilde{m}(x,t) \rfloor), \\
& \hat{F}(\lfloor x \cdot \tilde{m}(x,t) \rfloor, \lceil \tilde{m}(x,t) \rceil), \\
& \hat{F}(\lceil x \cdot \tilde{m}(x,t) \rceil, \lfloor \tilde{m}(x,t) \rfloor), \\
& \hat{F}(\lceil x \cdot \tilde{m}(x,t) \rceil, \lceil \tilde{m}(x,t) \rceil) \;\; \}.
\end{aligned}$$

By lemma 3.21, we know that for any $x, t \in \mathbb{Q}$: $\mathsf{min}(X) \leq f^*(x,t) \leq \mathsf{max}(X)$. Thus it suffices to show that $|f(x) - \hat{F}(\lceil x \cdot \tilde{m}(x,t) \rfloor, \lceil \tilde{m}(x,t) \rfloor)| \leq 1/t$ (for example, if $f(x) \leq \mathsf{min}(X)$, we would then have $|\mathsf{max}(X) - f(x)| \leq 1/t$, and $f^*(x,t)$ is even closer to $f(x)$ than $\mathsf{max}(X)$). By definition,

$$\begin{aligned}
\hat{F}(\lceil x \cdot \tilde{m}(x,t) \rfloor, \lceil \tilde{m}(x,t) \rfloor) &= g\left(\frac{\lceil x \cdot \tilde{m}(x,t) \rfloor}{\lceil \tilde{m}(x,t) \rfloor}, \mathsf{D}\left(\frac{\lceil x \cdot \tilde{m}(x,t) \rfloor}{\lceil \tilde{m}(x,t) \rfloor} \right) \right) \\
&= f\left(\frac{\lceil x \cdot \tilde{m}(x,t) \rfloor}{\lceil \tilde{m}(x,t) \rfloor} \right).
\end{aligned}$$

By property ★ and the definition of a modulus, $|f(x) - f\left(\frac{\lceil x \cdot \tilde{m}(x,t) \rfloor}{\lceil \tilde{m}(x,t) \rfloor}\right)| \leq 1/t$, as desired.

■

In the next lemma, we want to show that the continuous class $\mathbf{C}_{\mathbb{Q}}$ can be approximated by the functions of the discontinuous class $\mathbf{dC}_{\mathbb{Q}}$ which happen to have a $\mathbf{QMU}^{(1)}$ modulus; the proof is very similar to part of the proof of lemma 16 of [12].

Lemma 4.6. $\mathbf{C}_{\mathbb{Q}} \preceq \widetilde{\mathbf{dC}_{\mathbb{Q}}}$

Proof

Let $f(x)$ be in $\mathbf{C}_{\mathbb{Q}}$ (assume one argument for ease of exposition), and we need some $f^*(x,t)$ in $\mathbf{dC}_{\mathbb{Q}}$ such that $f \preceq f^*$, and f^* has a $\mathbf{QMU}^{(1)}$ modulus. Let M be the Turing machine that computes f in the sense of computable analysis. Thus M has an oracle tape which gives approximations of x, and an input tape for the accuracy input (recall definition 2.3 and the immediately preceding discussion). The function $f^*(x,t)$ will be defined by a Turing machine which takes $x, t \in \mathbb{Q}$ as input, where each rational is given exactly as a triple of natural numbers, though to keep it simple, we ignore the sign and consider rationals as pairs, i.e. the positive rational p/q is given by the pair (p,q). To obtain the condition $f \preceq f^*$ alone would be straightforward. We could define f^* in terms of M, by inputting the desired accuracy, $\lceil t \rceil$, to the machine M, and use the exact x as the oracle to M. This is roughly how f^* will be defined, however with such a definition, for fixed t, $f^*(x,t)$ could fluctuate arbitrarily between $f(x) - \frac{1}{t}$ and $f(x) + \frac{1}{t}$, and thus not even be continuous; i.e. though the final function f defined from M is continuous, the "approximations" defined from M may not be. Guaranteeing the modulus condition will then require some care and is the reason we will use (2) to define f^*.

Since f is computable in the sense of computable analysis, it has a computable modulus, e.g. a modulus function $m(x,z)$ in $\mathbf{C}_{\mathbb{Q}}$. By lemma 4.4, there is some $\tilde{m}(x,z)$ in $\mathbf{C}_{\mathbb{Q}}$ such that $\left|x - \frac{\lceil x \cdot \tilde{m}(x,z) \rfloor}{\lceil \tilde{m}(x,z) \rfloor}\right| \leq \frac{1}{m(x,z)}$.

Now we will define a function $h(n,p,q) : \mathbb{Z}^3 \to \mathbb{Q}$:

> Run M with accuracy input n, using p/q as its oracle. When we say to use p/q as the oracle, we mean that whenever some oracle query is made, enough bits of the binary expansion of p/q are given. Define $h(n,p,q)$ to be the output of this run of M.

We can define h to be zero for non-integer rationals so that the linearization $\hat{h} = \mathsf{Lin}(h)$ is a function $\hat{h} : \mathbb{Q}^3 \to \mathbb{Q}$. We define

$$f^*(x,t) = \hat{h}(2t+1, x \cdot \tilde{m}(x, 2t+1), \tilde{m}(x, 2t+1)), \qquad (2)$$

which is contained in both $\mathbf{C}_\mathbb{Q}$ and $\mathbf{dC}_\mathbb{Q}$. Since f^* is in $\mathbf{C}_\mathbb{Q}$ it has a modulus in $\mathbf{C}_\mathbb{Q}$. By lemmas 3.14 and 4.1, we can conclude that the modulus is in $\mathbf{QMU}^{(1)}$ (since the growth rate of $\mathbf{C}_\mathbb{N}$ is the same as $\mathbf{C}_\mathbb{Q}$, and for modulus functions, growth rate is all that matters). It is left to check that $|f(x) - f^*(x,t)| \leq 1/t$. By lemma 3.21 (as in the proof of lemma 4.5), it suffices to show

$|f(x) - \hat{h}(\lceil 2t+1 \rfloor, \lceil x \cdot \tilde{m}(x, 2t+1) \rfloor, \lceil \tilde{m}(x, 2t+1) \rfloor)| \leq 1/t$. By the definition of M, the result of a run with accuracy input $\lceil 2t+1 \rfloor \geq 2t$ and oracle $\dfrac{\lceil x \cdot \tilde{m}(x, 2t+1) \rfloor}{\lceil \tilde{m}(x, 2t+1) \rfloor}$ must be within $1/(2t)$ of $f\left(\dfrac{\lceil x \cdot \tilde{m}(x, 2t+1) \rfloor}{\lceil \tilde{m}(x, 2t+1) \rfloor}\right)$, and so

$$\left| f\left(\frac{\lceil x \cdot \tilde{m}(x, 2t+1) \rfloor}{\lceil \tilde{m}(x, 2t+1) \rfloor}\right) - \hat{h}(\lceil 2t+1 \rfloor, \lceil x \cdot \tilde{m}(x, 2t+1) \rfloor, \lceil \tilde{m}(x, 2t+1) \rfloor) \right| \leq \frac{1}{2t}.$$

Because $\left| x - \dfrac{\lceil x \cdot \tilde{m}(x, 2t+1) \rfloor}{\lceil \tilde{m}(x, 2t+1) \rfloor} \right| \leq \dfrac{1}{m(x, 2t+1)}$, the definition of the modulus yields

$$\left| f(x) - f\left(\frac{\lceil x \cdot \tilde{m}(x, 2t+1) \rfloor}{\lceil \tilde{m}(x, 2t+1) \rfloor}\right) \right| \leq \frac{1}{2t}.$$

By the triangle inequality we are done.

■

4.2. The Final Approximation: $\mathbf{QMU}^{(1)} \preceq \pi\mathbf{RMU}_k^{(1)}$

We now begin the core technical work needed to prove the final approximation of figure 2, $\mathbf{QMU}^{(1)} \preceq \pi\mathbf{RMU}_k^{(1)}$. For the technical development we will use a restriction of $\pi\mathbf{RMU}_k^{(c)}$ to those functions that we can bound appropriately.

Definition 4.7. *Let* $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ *be the functions* $f(\bar{x})$ *in* $\pi\mathbf{RMU}_k^{(c)}$ *such that there is a norm-increasing function* $b(\bar{x})$ *in* $\pi\mathbf{RMU}_k^{(c)}$ *such that* $|f(\bar{x})| \leq b(\bar{x})$*.*

By definition, $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)} \subseteq \pi\mathbf{RMU}_k^{(c)}$. Our approach and the proofs to come, make essential use of the class $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$, however, we claim (but do not prove here) that for all but possibly a few integers c and k, the classes $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ and $\pi\mathbf{RMU}_k^{(c)}$ are the same.

Note that $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ contains the functions 0,1, -1, π, θ_k, and $\mathbf{P}$. The property of being norm-increasing is preserved by composition. From norm-increasing bounds on f and g we can get a norm increasing bound on $h =$

$\mathsf{LI}(f,g)$. Thus by induction on the structure of $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ we can prove the following lemma.

Lemma 4.8. *$\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ contains the basic functions 0,1, -1, π, θ_k, and $\mathbf{P}$, and is closed under* comp *and* LI.

Thus we can work with $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ as if it were $\pi\mathbf{RMU}_k^{(c)}$, except when we want to use the UMU operation (in which case we need to make sure we can appropriately bound the output of UMU).

We now begin a somewhat involved technical discussion in the next two lemmas (4.9 and 4.11) and corollary 4.10. The reader who would rather avoid this technical part enjoys our sympathy, however natural alternatives to using linearization seem to further complicate matters. We will show how to deal with linearization (approximately) in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$. It may be helpful to keep in mind that we will use linearization for two distinct purposes. On the one hand, we will need to be able to "approximate" the operation Lin, which is an operation of the function algebra $\mathbf{QMU}^{(1)}$ (the point of lemma 4.9 and corollary 4.10). On the other hand, the approximating function we construct has a number of nice properties that we make explicit (in lemma 4.11), and use in conjunction with the UMU operation in the final step of lemma 4.12. The next lemma says that $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ can approximately do linearization in one variable, and constructs a kind of "partial modulus."

Lemma 4.9. *Suppose $f(x,\bar{a})$ is in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$. There are $\mathbf{L}(x,\bar{a},t)$ and $m(x,\bar{a},z)$ in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$, such that (for $\hat{f}(x,\bar{a})$ denoting* Lin(f,x)*):*

1. $\hat{f}(x,\bar{a}) \preceq \mathbf{L}(x,\bar{a},t)$ *and* $\mathbf{L}(0,\bar{a},t) = f(0,\bar{a})$,
2. $|x-x'| \le \dfrac{1}{m(x,\bar{a},z)}$ *implies* $|\hat{f}(x,\bar{a}) - \hat{f}(x',\bar{a})| \le \dfrac{1}{z}$,

Proof

The notation is a little involved: We initially use the notation L when dealing with $x \ge 0$; we then define $\mathbf{L}$ which extends the construction to all real x. We will define and discuss the function L and then carry out the error analysis and the construction of $\mathbf{L}$ to prove part 1. The proof of part 2 will follow from the constructions in part 1. Figure 3 (on page 32) summarizes the proof.

Part 1

Proof setup.

To define the function $L(x,\bar{a},t)$ in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$, we will first define $\tilde{L}(x,\bar{a},u)$, and then show that there is a function $\alpha(x,\bar{a},t)$ in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ such that for the definition

$$L(x,\bar{a},t) = \tilde{L}(x,\bar{a},\alpha(x,\bar{a},t)),$$

we have $|\hat{f}(x,\bar{a}) - L(x,\bar{a},t)| \leq \dfrac{1}{t}$ for all $t > 0$ and all $x \geq 0$. We will define other functions with arguments x, $\bar{a}$ and u; throughout, we will assume $u > 0$, and when we draw such functions we will assume $\bar{a}$ and u are fixed, while x varies.

Working backwards, we define $\tilde{L}$, providing intuition via figure 3 on page 32. Consider the (typically discontinuous) step function $\Delta(x,\bar{a}) = f(\lfloor x \rfloor + 1, \bar{a}) - f(\lfloor x \rfloor, \bar{a})$, which gives the slope of $\hat{f}$ on the interval $(\lfloor x \rfloor, \lfloor x \rfloor + 1)$; the functions f, $\hat{f}$, and Δ are pictured in figure 3(a). We will approximate the discontinuous function $\Delta(x,\bar{a})$ with a *continuous slope function* $S(x,\bar{a},u)$, where the approximation will only be bad near integers; functions Δ and S are pictured in figure 3(b). $\tilde{L}$ will then be defined by making its slope $S(x,\bar{a},u)$, i.e.

$$\tilde{L}(x,\bar{a},u) = f(0,\bar{a}) + \int_0^x S(y,\bar{a},u)\,dy. \qquad (3)$$

To construct S we will use LI to define a function W that approximates $f(\lfloor x \rfloor + 1, \bar{a})$ on most of the interval $[\lfloor x \rfloor, \lfloor x \rfloor + 1]$. Let $F(x,\bar{a}) = f(\mathsf{step}(x),\bar{a})$, and note that $F(x,\bar{a}) = f(\lfloor x \rfloor, \bar{a})$ for $x \in [\lfloor x \rfloor, \lfloor x \rfloor + \frac{1}{2}]$; recall the function step from example 2.13. We define W in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ with the following linear differential equation (W is drawn along with f and F in figure 3(c)).

$$\frac{\partial}{\partial x} W(x,\bar{a},u) = (F(x+1,\bar{a}) - W(x,\bar{a},u))\,\theta_k(\sin 2\pi x)\,u \qquad (4)$$

and initial condition $W(0,\bar{a},u) = f(0,\bar{a})$. Once one believes that figure 3(c) accurately portrays W (technical discussion follows), it makes sense to define

$$S(x,\bar{a},u) = W(x,\bar{a},u) - W(x-1,\bar{a},u).$$

We have now worked backwards to a complete definition of $\tilde{L}$; in the error analysis we will define the function $\alpha(x,\bar{a},t)$, and thus complete the definition of $L(x,\bar{a},t)$.

Definition of $\tilde{L}$ and α: technical details.
Now we find useful expressions for W and S. Let $K(x) = \int_{\lfloor x \rfloor}^x \sin^k(2\pi y)dy$, which is strictly increasing for $x \in [\lfloor x \rfloor, \lfloor x \rfloor + 1/2]$. As an approximation for $\Delta(n,\bar{a})$ we let

$$\tilde{\Delta}_n = f(n+1,\bar{a}) - W(n,\bar{a},u), \quad \text{for integer } n \geq 0;$$

note that $\tilde{\Delta}_n$ *does* depend on $\bar{a}$ and u, though we drop these arguments to simplify the notation (throughout this proof, the letter "n" refers to elements of $\mathbb{N}$). We can derive the following expressions for W and S, for $x \in [n, n+1)$:

$$W(x,\bar{a},u) = f(n+1,\bar{a}) - \tilde{\Delta}_n\,\Theta(x,u) \tag{5}$$

$$S(x,\bar{a},u) = f(n+1,\bar{a}) - f(n,\bar{a}) + (\tilde{\Delta}_{n-1} - \tilde{\Delta}_n)\,\Theta(x,u) \tag{6}$$

where

$\underline{\Theta}(u) = \exp\{-u\,K(\frac{1}{2})\}$, and

$\Theta(x,u) = \begin{cases} \exp\{-uK(x)\}, & \text{for } n \le x \le n+\frac{1}{2}; \\ \underline{\Theta}(u), & \text{for } n+\frac{1}{2} < x < n+1. \end{cases}$

The expression for S follows immediately from the expression for W, so we now discuss the expression for W. For $x \ge 0$, either $\tilde{\Delta}_n = 0$ and $W(x,\bar{a},u) = f(n+1,\bar{a})$ for $x \in [n, n+\frac{1}{2}]$, or $\tilde{\Delta}_n \ne 0$ and the differential equation (4) can be explicitly solved by separating variables. In either case, its solution for $x \in [n, n+1/2]$ is the above expression. For $x \in [n+\frac{1}{2}, n+1]$, $\frac{\partial}{\partial x} W = 0$ and thus $W(x,\bar{a},u) = W(n+\frac{1}{2},\bar{a},u)$.

Now we carry out the error analysis. By definition of $\tilde{L}$, the (signed) error of approximation of $\hat{f}(x,\bar{a})$ by $\tilde{L}$ is

$$E(x,\bar{a},u) = \tilde{L}(x,\bar{a},u) - \hat{f}(x,\bar{a}) = \int_0^x S(y,\bar{a},u) - \Delta(y,\bar{a})\,dy. \tag{7}$$

If we let $\varepsilon(u) = \int_0^{\frac{1}{2}} \exp\{-u\,K(x)\}\,dx + \frac{1}{2}\exp\{-u\,K(\frac{1}{2})\}$,

$$\int_n^{n+1} S(x,\bar{a},u) - \Delta(x,\bar{a})\,dx = (\tilde{\Delta}_{n-1} - \tilde{\Delta}_n)\,\varepsilon(u),$$

and the total approximation error at $x = n$ is (using a telescoping sum):

$$\begin{aligned} E(n,\bar{a},u) &= \textstyle\sum_{i=0}^{n-1} \int_i^{i+1} S(x,\bar{a},u) - \Delta(x,\bar{a})\,dx \\ &= (\tilde{\Delta}_{-1} - \tilde{\Delta}_{n-1})\,\varepsilon(u) = -\tilde{\Delta}_{n-1}\,\varepsilon(u). \end{aligned} \tag{8}$$

For non-integer $x \in (n, n+1)$, (6) shows that $S(x,\bar{a},u)$ is always above or below $\Delta(x,\bar{a})$, therefore $E(x,\bar{a},u)$ is between $E(n,\bar{a},u)$ and $E(n+1,\bar{a},u)$, and so

$$|E(x,\bar{a},u)| \le \max\{|E(\lfloor x\rfloor,\bar{a},u)|, |E(\lfloor x\rfloor+1,\bar{a},u)|\} = \max\{|\tilde{\Delta}_{\lfloor x\rfloor-1}|, |\tilde{\Delta}_{\lfloor x\rfloor}|\}\,\varepsilon(u).$$

To proceed, we need to find a norm-increasing bound on $\tilde{\Delta}_n$ that is not dependent on u. Recall that $\tilde{\Delta}_n = f(n+1,\bar{a}) - W(n,\bar{a},u)$.

By assumption f (and therefore F) has a norm-increasing bound in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$, so we only need to consider W. Note that $W(\lfloor x \rfloor, \bar{a}, u)$ is in between the largest and smallest of the following three values: $F(\lfloor x \rfloor - 1, \bar{a})$, $F(\lfloor x \rfloor, \bar{a})$ and $F(\lfloor x \rfloor + 1, \bar{a})$; this follows from equation 4, as illustrated in figure 3(c). Therefore, $F(\lfloor x \rfloor, \bar{a})$ is between $F(x, \bar{a})$ and $F(x - \frac{1}{2}, \bar{a})$ for all x. All those bounds only involve the function F and can be assumed norm-increasing. They can be combined with the bound on f to find some function β in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$such that $\max\{|\tilde{\Delta}_{\lfloor x \rfloor - 1}|, |\tilde{\Delta}_{\lfloor x \rfloor}|\} \leq \beta(x, \bar{a})$. Hence, $|E(x, \bar{a}, u)| \leq \beta(x, \bar{a})\, \varepsilon(u)$.

Finally, we check how $\varepsilon(u)$ decreases with u and we use this to define $\alpha(x, \bar{a}, t)$ in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ (recall that α is the missing function we use to define L from $\tilde{L}$) such that

$$|E(x, \bar{a}, \alpha(x, \bar{a}, t))| \leq \beta(x, \bar{a})\, \varepsilon(\alpha(x, \bar{a}, t)) \leq \frac{1}{t}.$$

To do this, we need to bound $\varepsilon(u)$. There is some κ that depends on k such that $K(x) \geq (x - \lfloor x \rfloor)^\kappa$ for $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + \frac{1}{2}$. Thus, if $0 \leq x \leq \frac{1}{2}$ we bound the first term of $\varepsilon(u)$ as follows:

$$\begin{aligned}
\int_0^{\frac{1}{2}} \exp\{-u\, K(x)\}\, dx &\leq \int_0^{\frac{1}{2}} \exp\{-u\, x^\kappa\}\, dx \\
&\leq u^{-1/\kappa} \int_0^{+\infty} \exp\{-x^k\} dx \\
&\leq u^{-1/\kappa}(1 + \int_1^{+\infty} \exp\{-x\}\, dx) \\
&\leq 2u^{-1/\kappa}.
\end{aligned}$$

The second term of $\varepsilon(u)$ is $\frac{1}{2}\exp\{-u\, K(\frac{1}{2})\}$ and is bounded by $u^{-\frac{1}{\kappa}}$ for u large enough. Therefore, $\beta(x, \bar{a})\, \varepsilon(\alpha(x, \bar{a}, t)) \leq 3\, \beta(x, \bar{a})\, (\alpha(x, \bar{a}, t))^{-\frac{1}{\kappa}}$. Thus, choosing a large enough $\alpha(x, \bar{a}, t) \geq (3\, t\, \beta(x, \bar{a}))^\kappa$ we obtain the desired $\frac{1}{t}$ bound, on $|E(x, \bar{a}, \alpha(x, \bar{a}, t))|$.

Definition of $\mathbf{L}$.

To conclude this part of the proof we address the case of $x < 0$. We denote by $L^-(x, \bar{a}, t)$ the function that is defined from $f(-x, \bar{a})$ instead of $f(x, \bar{a})$ as in the construction above; e.g. $L^-(1, \bar{a}, t)$ gives an approximate value of $f(-1, \bar{a})$. Notice that $L(0, \bar{a}, t) = L^-(0, \bar{a}, t) = f(0, \bar{a})$ for any t. To obtain an approximation for $\hat{f}$ over $\mathbb{R}$ we just have to define an appropriate convex combination of L and L^-. We will define

$$\mathbf{L}(x, \bar{a}, t) = (1 - \lambda(x, \bar{a}, t))L^-(-x, \bar{a}, 3t) + \lambda(x, \bar{a}, t)L(x, \bar{a}, 3t), \quad (9)$$

for a function λ (defined below) which resembles the Heaviside function θ, but continuously and quickly switches from 0 to 1 on the interval $[0, \delta]$, for a function $\delta = \delta(\bar{a}, t)$ which will be discussed. Therefore, $\mathbf{L}$ will be precisely L, for $x \geq \delta$, and L^-, for $x \leq 0$; hence the previous error analysis still holds off of $[0, \delta]$. Next, we consider the error $|\hat{f}(x, \bar{a}) - \mathbf{L}(x, \bar{a}, t)|$ on $[0, \delta]$.

First, we will want a function B in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ such that $|\Delta(x, \bar{a})| \leq B(x, \bar{a})$. From the constructions above it is clear that u can be chosen large enough (depending on F and k) so that $S(n, \bar{a}, u) - 1 \leq \Delta(n, \bar{a}) \leq S(n, \bar{a}, u) + 1$ (see Figure 3(b)). Given that S is monotonic on $[n, n + \frac{1}{2}]$ and constant on $[n + \frac{1}{2}, n + 1]$, $\Delta(x, \bar{a})$ can be bounded as follows:

$$\min\{S(x, \bar{a}, u), S(x + \frac{1}{2}, \bar{a}, u)\} - 1 \leq \Delta(x, \bar{a}) \leq \max\{S(x, \bar{a}, u), S(x + \frac{1}{2}, \bar{a}, u)\} + 1,$$

and thus we can find such a B. To conclude part 1 of the proof we only need $B(0, \bar{a})$; however, we will need $B(x, \bar{a})$ in part 2 of the proof. Let $\delta(\bar{a}, t) = \dfrac{1}{B(0, \bar{a})(3t + 1)}$, and define

$$\lambda(x, \bar{a}, t) = \begin{cases} 0, & x \leq 0; \\ 1, & x \geq \delta(\bar{a}, t), \end{cases} \tag{10}$$

and for $0 \leq x \leq \delta(\bar{a}, t)$, $\lambda(x, \bar{a}, t)$ is increasing in x (but we do not care about its exact definition). Note that such a function λ is in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ (even though we do not have the exact division operation needed for δ, we can define a non-zero decreasing function that converges to zero faster than $1/x$, such as e^{-x}). The error, $|\hat{f}(x, \bar{a}) - \mathbf{L}(x, \bar{a}, t)|$, for $x \leq 0$ or $x \geq \delta(\bar{a}, t)$ has been done, so we consider the case of $0 < x < \delta(\bar{a}, t)$ (some technical discussion follows to justify the second equality):

$$\begin{aligned}
|\hat{f}(x, \bar{a}) - \mathbf{L}(x, \bar{a}, t)| &= |\hat{f}(x, \bar{a}) - \{(1 - \lambda(x, \bar{a}, t))L^-(-x, \bar{a}, 3t) + \lambda(x, \bar{a}, t)L(x, \bar{a}, 3t)\}| \\
&= |\hat{f}(x, \bar{a}) - (1 - \lambda(x, \bar{a}, t))f(0, \bar{a}) - \lambda(x, \bar{a}, t)L(x, \bar{a}, 3t)| \\
&\leq |\hat{f}(x, \bar{a}) - \hat{f}(0, \bar{a})| + \lambda(x, \bar{a}, t)|L(x, \bar{a}, 3t) - \hat{f}(0, \bar{a})| \\
&\leq |\hat{f}(x, \bar{a}) - \hat{f}(0, \bar{a})| + \lambda(x, \bar{a}, t)\{|L(x, \bar{a}, 3t) - \hat{f}(x, \bar{a})| + |\hat{f}(x, \bar{a}) - \hat{f}(0, \bar{a})|\} \\
&\leq \Delta(0, \bar{a})\delta(\bar{a}, t) + 1 \cdot (|L(x, \bar{a}, 3t) - \hat{f}(x, \bar{a})| + \Delta(0, \bar{a})\delta(\bar{a}, t)) \\
&\leq \frac{1}{3t} + \frac{1}{3t} + \frac{1}{3t} \\
&\leq \frac{1}{t}
\end{aligned}$$

To justify the second equality we point out why $L^-(-x, \bar{a}, 3t) = f(0, \bar{a})$, for $0 < x < \delta(\bar{a}, t)$. Assume that $\delta(\bar{a}, t) \leq \frac{1}{2}$, which is guaranteed if $B(0, \bar{a}) \geq 2$. We make the following *claim*:

For any f, any u and $x \in [-\frac{1}{2}, 0]$, $\tilde{L}(x, \bar{a}, u) = f(0, \bar{a})$.

We apply this to $f(-x,\bar{a})$ and $u = 3t$ to conclude that $L^-(x,\bar{a},3t) = f(0,\bar{a})$, and thus $L^-(-x,\bar{a},3t) = f(0,\bar{a})$ for $x \in [0,\frac{1}{2}]$. To prove the claim, it suffices (by the definition of $\tilde{L}$) to show that $S(x,\bar{a},u) = 0$ for $x \in [-\frac{1}{2},0]$, and thus (by definition of S) it suffices to show that $W(x,\bar{a},u) = f(0,\bar{a})$ for $x \in [-\frac{3}{2},0]$. To prove the last point, we recall that $W(0,\bar{a},u) = f(0,\bar{a})$ and show that $\frac{\partial}{\partial x}W = 0$ for $x \in [-\frac{3}{2},0]$. To get the derivative to be zero, we consider the definition of W in equation (4), and it suffices to note that: For $x \in [-\frac{3}{2},-1]$ or $x \in [-\frac{1}{2},0]$, $\theta_k(\sin 2\pi x) = 0$ and for $x \in [-1,-\frac{1}{2}]$, by definition, $F(x+1,\bar{a}) = f(0,\bar{a})$.

Part 2.

For $|x - x'| \leq 1$ we have

$$|\hat{f}(x,\bar{a}) - \hat{f}(x',\bar{a})| \leq |x - x'| \max\{|\Delta(x,\bar{a})|, |\Delta(x',\bar{a})|\},$$

since Δ gives the slope of $\hat{f}$. In part 1, we saw that there is a function B in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ such that $|\Delta(x,\bar{a})| \leq B(x,\bar{a})$. We use it to bound $\max\{|\Delta(x,\bar{a})|, |\Delta(x',\bar{a})|\}$ by some $M(x,\bar{a})$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$; then we can simply take $m(x,\bar{a},z) = M(x,\bar{a})z$.

■

The basic point of the next corollary is that from an approximation to a function, we can construct an approximation to its full linearization, and get a genuine modulus for the linearization. Basically, repeated application of the last lemma yields the corollary.

Corollary 4.10. *If $f(\bar{x}) \preceq f^*(\bar{x},t)$ and f^* is in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$, then there are functions $\mathbf{L}$ and m in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ such that $\mathsf{Lin}(f) \preceq \mathbf{L}$ and m is a modulus for $\mathsf{Lin}(f)$.*

Proof

Suppose $f(x_1,\ldots,x_n) \preceq f^*(x_1,\ldots,x_n,t)$, where f^* is in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$. Let $\hat{f}_0 = f$, and let $\hat{f}_{k+1} = \mathsf{Lin}(\hat{f}_k, x_{k+1})$.

We proceed by induction on k from 0 up to n, showing:

There are L_k and norm-increasing m_k in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ such that for $x_1,\ldots,x_k \in \mathbb{R}$, and $x_{k+1},\ldots,x_n \in \mathbb{Z}$:

1. $|\hat{f}_k(x_1,\ldots,x_n) - L_k(x_1,\ldots,x_n,t)| \leq \dfrac{1}{t}$, and
2. $|x_1 - x_1'| + \ldots + |x_k - x_k'| \leq \dfrac{1}{m_k(x_1,\ldots,x_n,z)}$ implies

$$|\hat{f}_k(x_1,\ldots,x_n) - \hat{f}_k(x_1',\ldots,x_k',x_{k+1},\ldots,x_n)| \leq \frac{1}{z}.$$

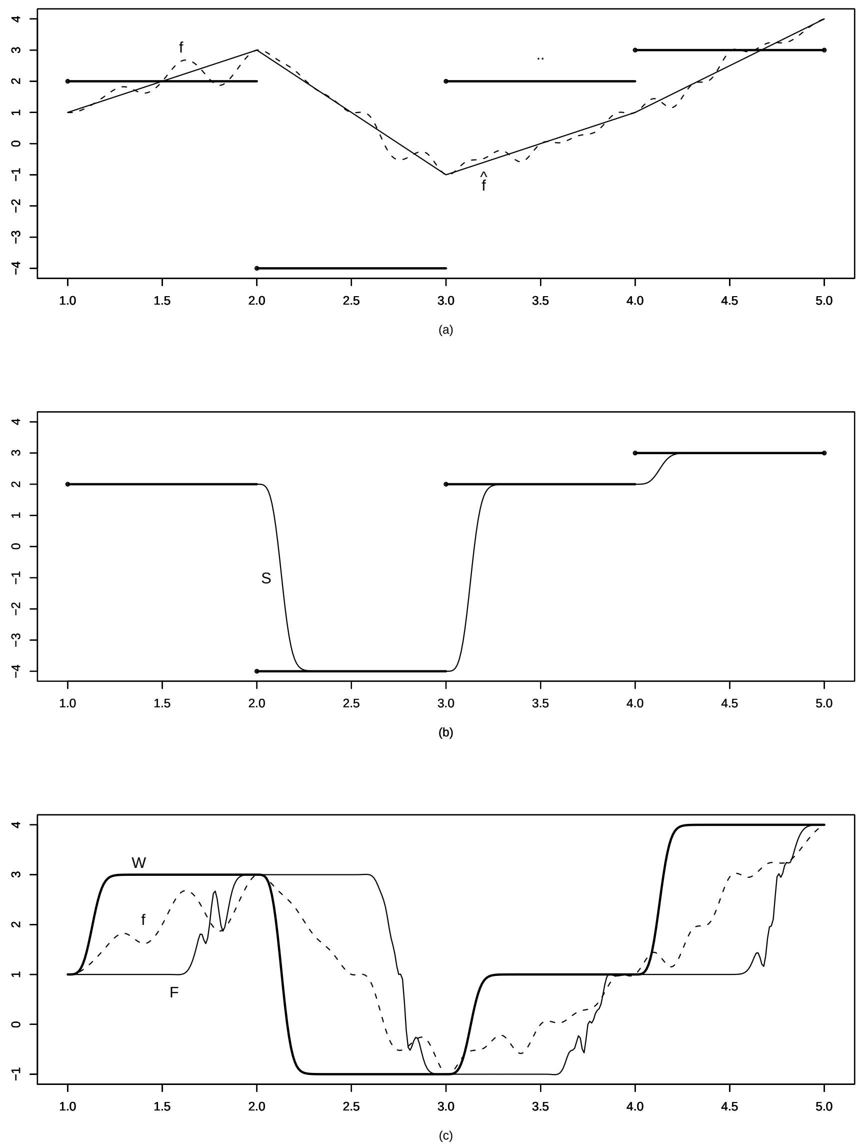

Figure 3: **Constructions in lemma 4.9 for fixed $\bar{a}$ and u:**
(a) The dotted line represents $f(x)$ and the thin solid line represents $\hat{f}(x) = \mathsf{Lin}(f, x)$, which we aim to approximate; the slope of $\hat{f}(x)$ is given by $\Delta(x)$, which is represented by the thick, discontinuous solid line.
(b) We approximate $\Delta(x)$ by $S(x, u)$, represented by the continuous line, where $S(x, u) = W(x, u) - W(x - 1, u)$.
(c) This figure illustrates the construction of W from f: The dotted line represents $f(x)$ as before, the thin solid line represents $F(x)$, which is constant on the intervals $[n, n + 1/2]$, and the thick solid line represents $W(x, u)$ as defined from F with the linear differential equation (4); the figure shows that $W(x, u)$ is close to $F(n)$, and thus to $f(n)$, for most of $x \in [n-1, n]$. Note that the graph of W was in fact obtained by numerically integrating (4) with $u = 10$.

When $k = n$, we have proved the corollary. For $k = 0$, we take $L_0 = f^*$, and we need no m_0 (we may take it to be identically zero so the inductive step works). We now complete the proof by discussing the inductive step from k to $k+1$. Inductively, we have appropriate L_k and m_k. Let $\hat{L}_k = \mathsf{Lin}(L_k, x_{k+1})$ and note that from the inductive hypothesis and the definition of Lin, we have:

(♦) For $x_1, \ldots, x_{k+1} \in \mathbb{R}$, $x_{k+2}, \ldots, x_n \in \mathbb{Z}$, $|\hat{f}_{k+1}(x_1, \ldots, x_n) - \hat{L}_k(x_1, \ldots, x_n, t)| \leq 1/t$.

From lemma 4.9 we have $\mathbf{L}$ and m in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ such that:

- $\hat{L}_k \preceq \mathbf{L}$
- $|x_{k+1} - x'_{k+1}| \leq \dfrac{1}{m(x_1, \ldots, x_n, t, z)}$ implies

$$|\hat{L}_k(x_1, \ldots, x_n, t) - \hat{L}_k(x_1, \ldots, x_k, x'_{k+1}, x_{k+2}, \ldots, x_n, t)| \leq \frac{1}{z}$$

Let $L_{k+1}(x_1, \ldots, x_n, t) = \mathbf{L}(x_1, \ldots, x_n, 2t, 2t)$. For $x_1, \ldots, x_{k+1} \in \mathbb{R}$, and $x_{k+2}, \ldots, x_n \in \mathbb{Z}$: $|\hat{f}_{k+1}(x_1, \ldots, x_n) - \hat{L}_k(x_1, \ldots, x_n, 2t)| \leq \dfrac{1}{2t}$ by ♦, and $|\hat{L}_k(x_1, \ldots, x_n, 2t) - L_{k+1}(x_1, \ldots, x_n, t)| \leq \dfrac{1}{2t}$ by the definition of $\mathbf{L}$, thus by the triangle inequality, L_{k+1} appropriately approximates $\hat{f}_{k+1}$.

Now we consider the modulus claim. By definition, $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ contains a norm-increasing bound on $|m|$; we use the same name "m" for this bound. Let b be some norm-increasing function in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ such that $|x| + 1 \leq b(x)$. Let $m_{k+1}(x_1, \ldots, x_n, z) =$

$$1 + m_k(x_1, \ldots, x_k, b(x_{k+1}), x_{k+2}, \ldots, x_n, 5z) + m(b(x_1), \ldots, b(x_k), x_{k+1}, \ldots, x_n, 5z, 5z).$$

It is a norm-increasing function in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$. Now we check the main property for modulus functions. Suppose we have some $x_1, \ldots, x_{k+1}, x'_1, \ldots, x'_{k+1} \in \mathbb{R}$, $x_{k+2}, \ldots, x_n \in \mathbb{Z}$ satisfying:

$$|x_1 - x'_1| + \ldots + |x_{k+1} - x'_{k+1}| \leq \frac{1}{m_{k+1}(x_1, \ldots, x_n, z)}.$$

Consider:

$$(\bigstar)\ |\hat{f}_{k+1}(x_1, \ldots, x_n) - \hat{f}_{k+1}(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n)|.$$

Letting $r = \lfloor x_{k+1} \rfloor$, by definition of Lin, (★) equals:

$$|(r+1-x_{k+1})(\hat{f}_k(x_1, \ldots, x_k, r, x_{k+2}, \ldots, x_n) - \hat{f}_k(x'_1, \ldots, x'_k, r, x_{k+2}, \ldots, x_n)) + (x_{k+1}-r)(\hat{f}_k(x_1, \ldots, x_k, r+1, x_{k+2}, \ldots, x_n) - \hat{f}_k(x'_1, \ldots, x'_k, r+1, x_{k+2}, \ldots, x_n))|.$$

Since $|r|, |r+1| \le b(x_{k+1})$ and
$m_{k+1}(x_1, \ldots, x_n, z) \ \ge \ m_k(x_1, \ldots x_k, b(x_{k+1}), x_{k+2}, \ldots, x_n, 5z)$, we use the inductive hypothesis and the fact that m_k is norm-increasing, to bound (★) by $\frac{2}{5z}$. Now consider

$|\hat{f}_{k+1}(x_1, \ldots, x_n) - \hat{f}_{k+1}(x'_1, \ldots, x'_{k+1}, x_{k+2}, \ldots, x_n)| \le$

$$\begin{aligned}
&|\hat{f}_{k+1}(x_1, \ldots, x_n) - \hat{f}_{k+1}(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n)| \\
&+ |\hat{f}_{k+1}(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n) - \hat{L}_k(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n, 5z)| \\
&+ |\hat{L}_k(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n, 5z) - \hat{L}_k(x'_1, \ldots, x'_k, x'_{k+1}, x_{k+2}, \ldots, x_n, 5z)| \\
&+ |\hat{L}_k(x'_1, \ldots, x'_k, x'_{k+1}, x_{k+2}, \ldots, x_n, 5z) - \hat{f}_{k+1}(x'_1, \ldots, x'_k, x'_{k+1}, x_{k+2}, \ldots, x_n)|.
\end{aligned}$$

In the above sum of four terms, the first is bounded by $\frac{2}{5z}$, since it is (★); we observe that the other three terms are each bounded by $\frac{1}{5z}$, thus providing the desired $\frac{1}{z}$ bound. The second and fourth terms are bounded by $\frac{1}{5z}$ because of ⧫. For the third term, we note that $|x'_i| \le b(x_i)$ since $m_{k+1}(x_1, \ldots, x_n, z) \ge 1$, thus using the fact that m is norm-increasing we get:

$$\begin{aligned}
m(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n, 5z, 5z) &\le m(b(x_1), \ldots, b(x_k), x_{k+1}, \ldots, x_n, 5z, 5z) \\
&= m_{k+1}(x_1, \ldots, x_n, z)
\end{aligned}$$

By the properties of m we bound the third term by $\frac{1}{5z}$.

■

The next lemma shows that if a function has the right shape for the MU operation (definition 3.12) over the naturals, then the approximate linearization has the analogous behavior over the reals.

Lemma 4.11. *Suppose $f(y,z)$ is in $\pi\overrightarrow{\mathbf{RMU}}^{(c)}_k$, and $f_{|\mathbb{Z}}(y,z)$ satisfies: a) the conditions of MU when $y, z \ge 0$, b) $f_{|\mathbb{Z}}(y,z) = 0$ when $y < 0$, and c) $f_{|\mathbb{Z}}(y,z) = f_{|\mathbb{Z}}(y,0)$ when $z < 0$.*

Then there is a function $\mathbf{L}(y,z,t)$ in $\pi\overrightarrow{\mathbf{RMU}}^{(c)}_k$ such that:

1. *Lin$(f) \preceq \mathbf{L}$,*
2. *For any z and t, $\mathbf{L}(y,z,t)$ increases with y,*
3. *For any z and t, there is exactly one y^* such that $\mathbf{L}(y^*,z,t) = 1$, and for this unique y^*, $\frac{\partial}{\partial y}\mathbf{L}(y^*,z,t) > 0$,*
4. *For any y and t, $\mathbf{L}(y,z,t)$ decreases with z.*

Proof

The basic approach of the proof is to *use the proof* of lemma 4.9, first with respect to y (i.e. y plays the role of x), applied to the function $f(y,z)$, to arrive at the function $\mathbf{L}_1(y,z,t_1)$. Then we use the proof of lemma 4.9 again, but now with respect to z (i.e. z plays the role of x), and applied to the function $\mathbf{L}_1(y,z,t_1)$, to arrive at the function $\mathbf{L}_2(y,z,t_1,t_2)$, so that our desired function will be $\mathbf{L}(y,z,t) = \mathbf{L}_2(y,z,2t,2t)$.

When we use the proof of lemma 4.9 to get $\mathbf{L}_1$ we write L_1 in place of L, and W_1 in place of W; likewise we write L_2 in place of L, and W_2 in place of W, when getting $\mathbf{L}_2$. By assumption, for a integer z, there is a natural y_z such that $f(y_z,z) = 1$; note that $y_z = y_0$ for negative integer z.

Parts 1, 2 and 3. Recall that in lemma 4.9, the construction worked by first constructing the function $\tilde{L}(y,z,u)$, then letting $L(y,z,t) = \tilde{L}(y,z,\alpha(y,z,t))$, for a certain function α that depends on y,z, and t. In this proof, we make a key modification at this point (both for $\mathbf{L}_1$ and $\mathbf{L}_2$), choosing an α which depends only on t. Recall that we chose $\alpha(y,z,t)$ such that $\beta(y,z)\varepsilon(\alpha(y,z,t)) \leq 1/t$. In the current situation, due to the particular shape of f, we can bound $\beta(y,z)$ by 3 (independent of y,z): Recall that β was defined on page 29 to be a bound on $\max\{|\tilde{\Delta}_{n-1}|, |\tilde{\Delta}_n|\}$, approximations to the slope of f (with respect to y or z), which for the exact slope would be at most 2, and for sufficiently large u is less than 3 (since we substitute $\alpha(t)$ for u, we just assume that we have taken $\alpha(t)$ to be sufficiently large). Thus in the construction of $\mathbf{L}_1$, based on f, and $\mathbf{L}_2$, based on $\mathbf{L}_1$ (where $\mathbf{L}_1$ inherits the rough shape of f), we can take α to be a function just of t. The same argument applies to the definition of λ and δ in (10) to conclude we can take δ as a function of just t, λ (in the construction of $\mathbf{L}_1$) as a function of just y and t, and λ (in $\mathbf{L}_2$) as a function of just z and t.

Construction for $\mathbf{L}_1$.

We begin by considering $\mathbf{L}_1$, showing that for z *restricted to being an integer*, $\frac{\partial}{\partial y}\mathbf{L}_1(y,z,t_1) \geq 0$, and for $y > y_z - 1$, $\frac{\partial}{\partial y}\mathbf{L}_1(y,z,t_1) > 0$ (note that since $\mathbf{L}_1$ approximates f, the positive derivative after $y_z - 1$ is enough to guarantee that $\mathbf{L}_1$ has a unique *one*, at which it has a positive derivative, i.e. part 3 of the lemma). Since $\mathbf{L}_1$ is defined as in (9), we will first prove the claim for L_1 on $y \geq 0$; the argument extends to L_1^- on $y \leq 0$. Finally, we will consider the interval $[0,\delta]$, where $\mathbf{L}_1$ is a combination of L_1 and L_1^-.

Since $L_1(y,z,t_1) = \tilde{L}_1(y,z,\alpha(t_1))$, we analyze $\tilde{L}_1(y,z,u)$. We rewrite (6) to get

$$\frac{\partial}{\partial y}\tilde{L}_1(y,z,u) = [f(n+1,z)-f(n,z)]\,(1-\Theta(y,u))+[W_1(n,z,u)-W_1(n-1,z,u)]\,\Theta(y,u), \tag{11}$$

for $y \in [n, n+1)$. Note that $0 < \underline{\Theta}(u) \leq \Theta(y,u) \leq 1$; hence the above expression is a convex combination of $f(n+1,z) - f(n,z)$, which is non-negative by hypothesis, and $W_1(n,z,u) - W_1(n-1,z,u)$, which we claim is also non-negative. To show this, consider integer $k \geq 0$ and $y \in [k, k+1)$, letting $y \to k+1$ in (5) to derive the following recurrence:

$$W_1(k+1,z,u) = [1 - \underline{\Theta}(u)]\, f(k+1,z) + \underline{\Theta}(u)\, W_1(k,z,u). \quad (12)$$

This is now a convex combination of $f(k+1,z)$ and $W_1(k,z,u)$.

We now proceed by induction on k, up to n, showing $W_1(k+1,z,u) \geq W_1(k,z,u)$. For $k = 0$, this is true, since $W_1(0,z,u) = f(0,z)$, and we can use (12) to conclude that $W_1(1,z,u) \geq f(0,z) = W_1(0,z,u)$. Furthermore, if $W_1(k,z,u) \geq W_1(k-1,z,u)$ then by (12) and the assumption that $f(k+1,z) \geq f(k,z)$ we conclude that $W_1(k+1,z,u) \geq [1 - \underline{\Theta}(u)]\, f(k,z) + \underline{\Theta}(u)\, W_1(k-1,z,u) = W_1(k,z,u)$. Therefore, $W_1(n,z,u) \geq W_1(n-1,z,u)$, so the difference $W_1(n,z,u) - W_1(n-1,z,u) \geq 0$, as desired. Thus $\frac{\partial}{\partial y}\tilde{L}_1(y,z,u) \geq 0$ for all $y \geq 0$.

To show $\frac{\partial}{\partial y}\tilde{L}_1 > 0$ for $y > y_z - 1$, we consider two cases: $y_z - 1 < y < y_z$ and $y \geq y_z$. We use equation (11) in both cases, showing that either its first term or its second term is positive (we just showed that both terms of (11) are non-negative). In the case $y_z - 1 < y < y_z$, $\Theta(y,u) < 1$, and $f(y_z,z) - f(y_z - 1,z) = 1$, so the first term is positive. In the case $y \geq y_z$, $\Theta(y,u) > 0$ (it always is) and by induction on integer $n \geq y_z$, $W_1(n,z,u) > W_1(n-1,z,u)$, so the second term is positive (the induction is similar to the previous induction, again using (12)).

To conclude, we consider $\mathbf{L}_1$ over $\mathbb{R}$. Recalling (9), for $y \geq \delta(t_1)$, $\mathbf{L}_1$ is L_1, and for $y \leq 0$, $\mathbf{L}_1(y,z,t_1) = L_1^-(-y,z,t_1)$; the former case was just analyzed, and the latter case can be handled similarly (in fact, the shape of f makes $L_1^-(-y,z,t_1) = 0$ for all $y \leq 0$). We just need to show that $\mathbf{L}_1(y,z,t_1)$ is also increasing for $0 \leq y \leq \delta(t_1)$. Without loss of generality we suppose that $\delta(t) \leq \frac{1}{2}$, so the expression of $\mathbf{L}_1$ simplifies to

$$\mathbf{L}_1(y,z,t_1) = (1 - \lambda(y,t_1)) f(0,z) + \lambda(y,t_1) L_1(y,z,3t_1), \quad (13)$$

as seen at the end of part 1 of lemma 4.9. Since $f(0,z) = L_1(0,z,3t_1)$ and $L_1(y,z,3t_1)$ is increasing in y, $\mathbf{L}_1$ is increasing on $[0,\delta]$.

Construction for $\mathbf{L}_2$.

Now we consider $\mathbf{L}_2(y,z,t_1,t_2)$, the construction of lemma 4.9 applied to $\mathbf{L}_1$, linearizing with respect to z; so now z plays the role of x, and $\bar{a}$ is y together with t_1. Let $\hat{\mathbf{L}}_1 = \mathsf{Lin}(\mathbf{L}_1, z)$; the construction we are now carrying out will approximate $\hat{\mathbf{L}}_1$. We show that the earlier properties of $\mathbf{L}_1$ that held for integer z, also hold for all real

z; in particular, *for all* z, $\frac{\partial}{\partial y}\mathbf{L}_2 \geq 0$, and there is some y_z such that the *one* of $\mathbf{L}_2$ occurs after $y_z - 1$ and $\frac{\partial}{\partial y}\mathbf{L}_2 > 0$ for $y > y_z - 1$.

We start by considering the function $L_2(y, z, t_1, t_2)$, for $z \geq 0$; at the end we briefly discuss $\mathbf{L}_2$. Consider $z \in [n, n+1)$ for some integer $n \geq 0$, and focus on $\tilde{L}_2(y, z, t_1, u)$, where $L_2(y, z, t_1, t_2) = \tilde{L}_2(y, z, t_1, \alpha(t_2))$. We derive from (7) and (8) two convenient expressions when $z \in [n, n+1)$:

$$\begin{aligned}\tilde{L}_2(y, z, t_1, u) &= \hat{\mathbf{L}}_1(y, z, t_1) - [\varepsilon(u) - \sigma(z, u)]\, \tilde{\Delta}_{n-1} - \sigma(z, u)\, \tilde{\Delta}_n \\ &= A_1(z, u)\, \mathbf{L}_1(y, n, t_1) + A_2(z, u)\, \mathbf{L}_1(y, n+1, t_1) + \\ &+ [\varepsilon(u) - \sigma(z, u)]\, W_2(y, n-1, t_1, u) + \sigma(z, u)\, W_2(y, n, t_1, u)\end{aligned} \tag{14}$$

where $\sigma(z, u) = \int_n^z \Theta(r, u) dr$, $A_1(z, u) = n + 1 - z - \varepsilon(u) + \sigma(z, u)$, and $A_2(z, u) = z - n - \sigma(z, u)$.

Now we differentiate (14) with respect to y to write

$$\begin{aligned}\tfrac{\partial}{\partial y}\tilde{L}_2(y, z, t_1, u) &= A_1(z, u)\, \tfrac{\partial}{\partial y}\mathbf{L}_1(y, n, t_1) + A_2(z, u)\, \tfrac{\partial}{\partial y}\mathbf{L}_1(y, n+1, t_1) + \\ &+ [\varepsilon(u) - \sigma(z, u)]\, \tfrac{\partial}{\partial y}W_2(y, n-1, t_1, u) + \sigma(z, u)\, \tfrac{\partial}{\partial y}W_2(y, n, t_1, u).\end{aligned} \tag{15}$$

Is is easy to check the following facts (for $z \in [n, n+1]$): $\sigma(z, u)$ is strictly increasing; $\sigma(n, u) = 0$; $\sigma(n+1, u) = \varepsilon(u)$; $z - n \geq \sigma(z, u)$; and $n + 1 - z \geq \varepsilon(u) - \sigma(z, u)$. Hence $A_1, A_2, \varepsilon(u) - \sigma(z, u), \sigma(z, u) \geq 0$ for $z \in [n, n+1]$. We have shown that $\frac{\partial}{\partial y}\mathbf{L}_1(y, n+1, t_1), \frac{\partial}{\partial y}\mathbf{L}_1(y, n, t_1) \geq 0$, so to prove that $\tilde{L}_2$ has the desired properties, it suffices to show that for integer $k = 0, \ldots, n$, $\frac{\partial}{\partial y}W_2(y, k, t_1, u) \geq 0$ and furthermore, if $y > y_k - 1$ then $\frac{\partial}{\partial y}W_2(y, k, t_1, u) > 0$. We prove these facts by induction on k up to n, similar to the inductive proofs used in the construction of $\mathbf{L}_1$. The induction proceeds like before, using the following recurrence (which can be derived from equation (12)):

$$\frac{\partial}{\partial y}W_2(y, k, t_1, u) = [1 - \underline{\Theta}(u)]\, \frac{\partial}{\partial y}\mathbf{L}_1(y, k, t_1) + \underline{\Theta}(u)\, \frac{\partial}{\partial y}W_2(y, k-1, t_1, u).$$

Recalling once again (9), for $z \geq \delta(t_2)$, $\mathbf{L}_2(y, z, t_1, t_2) = L_2(y, z, t_1, t_2)$, and for $z \leq 0$, $\mathbf{L}_2(y, z, t_1, t_2) = L_2^-(y, -z, t_1, t_2)$. Using the same arguments as above, it is clear that $\mathbf{L}_2(y, z, t_1, t_2)$, satisfies $\frac{\partial}{\partial y}\mathbf{L}_2 \geq 0$ and $\frac{\partial}{\partial y}\mathbf{L}_2 > 0$ for $y > y_z - 1$. To show that this property holds for $\mathbf{L}_2$ when $0 < z < \delta(t_2)$, as before we write

$$\mathbf{L}_2(y, z, t_1, t_2) = [1 - \lambda(z, t_2)]\, \mathbf{L}_1(y, 0, t_1) + \lambda(z, t_2)\, L_2(y, \delta(t_2), t_1, 3t_2).$$

Differentiating with respect to y yields the result, since λ does not depend on y, and both $\frac{\partial}{\partial y}\mathbf{L}_1$ and $\frac{\partial}{\partial y}L_2$ have the right properties.

In the end we take $\mathbf{L}(y,z,t) = \mathbf{L}_2(y,z,2t,2t)$, which has the desired properties.

Part 4. Analogous to (11), for $z \in [n, n+1)$,

$$\frac{\partial}{\partial z}\tilde{L}_2(y,z,t_1,u) = [\mathbf{L}_1(y,n+1,t_1) - \mathbf{L}_1(y,n,t_1)]\,(1-\Theta(z,u)) + [W_2(y,n,t_1,u) - W_2(y,n-1,t_1,u)]\,\Theta(z,u), \tag{16}$$

which we now claim is less than or equal to 0. The argument is similar to the one after (11), except that this time the relevant terms in (16) are non-positive. First, $\mathbf{L}_1(y,n+1,t_1) - \mathbf{L}_1(y,n,t_1) \leq 0$ because of the shape of f (recall figure 1), i.e. since for all $n \in \mathbb{N}$, $\mathbf{L}_1(y,n+1,t_1)$ is just $\mathbf{L}_1(y,n,t_1)$ shifted to the right (or not shifted at all) in the y direction. To prove that $W_2(y,n,t_1,u) - W_2(y,n-1,t_1,u) \leq 0$ we use the fact that $\mathbf{L}_1(y,n+1,t_1) - \mathbf{L}_1(y,n,t_1) \leq 0$ and we proceed inductively along the same lines of the argument that follows (12). Note that for integer $n < 0$, the construction reveals that $W_2(y,n,t,u) = W_2(y,n-1,t,u)$ so the difference is 0. Finally, we adjust the argument surrounding (13) to $\mathbf{L}_2$, to conclude that $\mathbf{L}_2$ is decreasing with respect to $z \in \mathbb{R}$ (in equation (12), and the discussion following it, W_2 is used in place of W_1 and $\mathbf{L}_1$ in place of f).

■

Remark *In the last lemma, we can replace the single parameter z by a tuple, since we can repeat the construction of $\mathbf{L}_2$ of lemma 4.11 for each member of the tuple (similar to corollary 4.10).*

Now we prove the final and most important approximation of figure 2, making significant use of the last corollary and lemma. Proceeding in a straightforward way we ran into difficulties, and thus we generalize the inductive hypothesis, so that we simultaneously build approximating functions and modulus functions.

Lemma 4.12. $\mathbf{QMU}^{(1)} \preceq \pi\mathbf{RMU}_k^{(1)}$ *for any* $k \geq 1$.

Proof

Since $\overrightarrow{\pi\mathbf{RMU}}_k^{(1)} \subseteq \pi\mathbf{RMU}_k^{(1)}$, it suffices to show $\mathbf{QMU}^{(1)} \preceq \overrightarrow{\pi\mathbf{RMU}}_k^{(1)}$. We proceed inductively on the construction of the function algebra $\mathbf{QMU}^{(c)}$, showing the following stronger claim (for both $c = 0$ and $c = 1$):

For any $f(\bar{x})$ in $\mathbf{QMU}^{(c)}$ the following two properties hold,

1. (*Approximation property*) There is $f^*(\bar{x}, t)$ in $\overrightarrow{\pi\mathbf{RMU}}_k^{(c)}$ such that $f \preceq f^*$, and

2. (*Modulus property*) There is $m(\bar{x}, z)$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ which is a modulus for f.

We will use a few functions in the proof which are similar to the function step (recall example 2.13), and can be defined using it, or defined in a similar manner, all in $\pi\overrightarrow{\mathbf{RMU}}_k^{(0)}$. We can define a continuous *integer rounding* function $\mathsf{rd} : \mathbb{R} \to \mathbb{R}$ which has the following property: *For any $n \in \mathbb{N}$, $rd(x) = n$ for all $x \in [n - 1/4, n + 1/4]$*, and for $x \leq 0$, $\mathsf{rd}(x) = 0$. We can define a continuous *sign function* $\mathsf{sg} : \mathbb{R} \to \mathbb{R}$ such that $\mathsf{sg}(x) = 0$ for $x \leq 0$, and $\mathsf{sg}(x) = 1$ for $x \geq 1$; its values for $0 < x < 1$ do not matter.

Now we discuss the steps of the induction, showing the approximation and modulus property in each case; technically we also proceed inductively on the rank c, though in our case we just do it for $c = 0$ and then for $c = 1$.

- **Basic Functions.**
  The modulus of all the basic functions are simple to deal with, so we just consider the approximation property. Except for θ_1 and div, $\pi\overrightarrow{\mathbf{RMU}}_k^{(0)}$ contains expansions to $\mathbb{R}$ of the basic functions of $\mathbf{QMU}^{(c)}$, and so approximates them exactly. For θ_1 and div, we find functions in $\pi\overrightarrow{\mathbf{RMU}}_k^{(0)}$ which approximate these functions on $\mathbb{Z}$, and then we linearize to approximate the functions everywhere. For θ_1, for $x \in \mathbb{Z}$, $\theta_1(x) = \mathsf{rd}(x)$, an exact approximation on $\mathbb{Z}$, and thus by corollary 4.10 we can approximate θ_1 on all of $\mathbb{Q}$. For div, we use the function $E(x,t) = \begin{cases} \dfrac{1 - e^{-tx}}{x}, & \text{if } x \neq 0; \\ t, & \text{if } x = 0. \end{cases}$, a function in $\pi\overrightarrow{\mathbf{RMU}}_k^{(0)}$ such that for $x \in [1, +\infty)$, we have $|1/x - E(x,t)| \leq 1/t$ (E was defined by Bournez and Hainry [5], directly before proposition 11). Let $f(x,t) = E(x,t) \cdot \mathsf{sg}(x) + \mathsf{sg}(-x+1)$, so $\mathsf{div}_{|\mathbb{Z}} \preceq f$, and thus by corollary 4.10 we can approximate div on all of $\mathbb{Q}$.
- **Composition.**
  Consider $f(g(x))$ in $\mathbf{QMU}^{(c)}$ (for composition with more functions, the argument is similar), where inductively we have $f^*(y, t_1), g^*(x, t_2)$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ such that $f \preceq f^*$ and $g \preceq g^*$, and $m_f(y,z), m_g(x,z)$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$, which are modulus functions for f and g, respectively. Note that $g(x) \leq g^*(x,1) + 1$, and since the latter function is in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ we can find a function b in $\pi\mathbf{RMU}_k^{(c)}$ such that $|g(x)| \leq b(x)$ and b is norm-increasing. Thus, $m_g(x, m_f(b(x), z))$ is in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ and it is a modulus for $f \circ g$. For the approximation we note that:

  $$f(g(x)) \preceq f^*(g^*(x, m_f(b(x), 2t)), 2t),$$

where the latter function is in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$.

- **Sums and Products.**

  We will use the following **claim** (follows from [9, lemma 4.7]):

  > If $F(x, \bar{a})$ is in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ and takes natural values on natural inputs, then there is a $G(y, \bar{a})$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ such that for $y, \bar{a} \in \mathbb{N}$, $G(y, \bar{a}) = \sum_{x=0}^{y} F(x, \bar{a})$; the claim also holds for *products* replacing *sums*.

  Consider $f(x, \bar{a})$ in $\mathbf{QMU}^{(c)}$ and let $g(y, \bar{a})$ be $\sum_{\mathbb{Q}}$ applied to f and the argument x; note that since $\sum_{\mathbb{Q}}$ can be applied, f must be natural valued on natural inputs. Inductively we have some $f^*(x, \bar{a}, t)$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ such that $f \preceq f^*$; thus $F(x, \bar{a}) = \mathsf{rd}(f^*(x, \bar{a}, 4))$ is in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$. Notice that $f(x, \bar{a}) = F(x, \bar{a})$ for $x, \bar{a} \in \mathbb{N}$. We apply the above claim to obtain some $G(y, \bar{a})$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(c)}$ such that $g(y, \bar{a}) = G(y, \bar{a})$, for $y, \bar{a} \in \mathbb{N}$. Using the function sg from above we can ensure that $G(y, \bar{a}) = 0$ for integer arguments $y, \bar{a}$ when at least one argument is negative; thus $g(y, \bar{a}) = G(y, \bar{a})$ for all $y, \bar{a} \in \mathbb{Z}$. By definition 3.22, $\sum_{\mathbb{Q}}$ returns a linearization, therefore $g = \mathsf{Lin}(g)$. Thus

  corollary 4.10 applied to G gives us an approximation function and modulus function for g. The argument for $\prod_{\mathbb{Q}}$ is basically the same.

- **Linearization (Lin).** This step follows immediately from corollary 4.10.

- **Search Operation ($\mathsf{MU}_{\mathbb{Q}}$).**

  For simplicity, we only consider a function of two variables (see the remark following lemma 4.11). Consider $f(y, z)$ in $\mathbf{QMU}^{(0)}$ satisfying the requirements of $\mathsf{MU}_{\mathbb{Q}}$, and we let $g(z) = \mathsf{MU}_{\mathbb{Q}}(f)$, so g is in $\mathbf{QMU}^{(1)}$. It will be useful to recall the example after definition 3.22.

  We let $y_z \in \mathbb{N}$ denote the unique *one* of f for $z \in \mathbb{N}$, i.e. the value y_z such that $f(y_z, z) = 1$ (recall that by definition 3.22, for $z \in \mathbb{N}$, we have $y_z \in \mathbb{N}$). Thus $f(y, z) = 0$ for natural $y \leq y_z - 1$, $f(y_z, z) = 1$, and $f(y, z) = 2$ for natural $y \geq y_z + 1$ (note that $y_z \geq 1$ by definition 3.12 so $f(0, z) = 0$). Inductively, we have some $f^*(y, z, t)$ in $\pi\overrightarrow{\mathbf{RMU}}_k^{(0)}$ such that $f \preceq f^*$. As we did in the case of *sums and products*, we consider the function $\mathsf{rd}(f^*(y, z, 4))$, which is exactly equal to $f(y, z)$ over the naturals. Using that function together with sg, we obtain a function $F(y, z)$ such that for natural y, z, $F(y, z) = f(y, z)$, for negative integers z, $F(y, z) = f(y, 0)$, and for negative integers y, $F(y, z) = 0$.

Since we have no control over how f (and thus F) behaves off of the integers, we apply lemma 4.11 to $F(y, z)$ to obtain $\mathbf{L}(y, z, t)$, an approximation of $\mathsf{Lin}(f)$. By conditions 2 and 3, the requirements of UMU are met, thus we can define $G(z, t) = \mathsf{UMU}(\mathbf{L}(y, z, t) - 1, y)$ which finds the root of $\mathbf{L}(y, z, t) - 1$, with respect to y. Since $\mathsf{Lin}(f) \preceq \mathbf{L}$ (by condition 1 of lemma 4.11), y_z, the *one* of f (with respect to y) is within $\frac{1}{t}$ of the zero of $\mathbf{L}(y, z, t) - 1$. This key point is guaranteed since the slope of $\mathsf{Lin}(f)$ with respect to y is 1 near y_z; *if the slope near y_z were very small, the zero of* $\mathbf{L}(y, z, t) - 1$ *could be far from y_z*. By condition 4, G is norm-increasing and so in $\overrightarrow{\pi\mathbf{RMU}}_k^{(1)}$. By definition 3.22, $\mathsf{MU}_{\mathbb{Q}}$ returns a linearization, therefore $g = \mathsf{Lin}(g)$; thus corollary 4.10 supplies us with a modulus in $\overrightarrow{\pi\mathbf{RMU}}_k^{(1)}$.

■

5. Conclusion

A major advance of the three characterizations of theorem 2.17 is that they use notions from analysis instead of *notions from classic computability*. In terms of the proof techniques, the advance is in the use of our method of approximation. At the end of the conclusion we present an open question that we hope can be solved using the method of approximation.

The characterizations of $\mathbf{C}_{\mathbb{R}}$ in the style of Ko [23], which have some similarity to our work, in fact rely heavily on classic computability. We can view our theorem 2.17 as exhibiting a dense subset of $\mathbf{C}_{\mathbb{R}}$, together with an appropriate way to "complete" the class. For example, given a function $f(x)$ in $\mathbf{C}_{\mathbb{R}}$, by our theorem we must have a function $f^*(x, t)$ in $\mathbf{ODE}_k^\star$ such that $f = \lim_{t\to\infty} f^*$. If we just consider $t \in \mathbb{N}$, we can view $f^*(x, t)$ as a sequence of functions converging uniformly to f. Ko also characterizes $\mathbf{C}_{\mathbb{R}}$ by exhibiting functions which converge uniformly to the functions of $\mathbf{C}_{\mathbb{R}}$ (see [23, theorem 2.15]). However in Ko's characterization, the uniform convergence is controlled by requiring that the convergence proceed computably; thus Ko's approach (and similarly envisioned modifications) use classical notions of computability in the characterization itself. In our approach we avoid the drawback of using computability in our definition of limits or elsewhere, thus $\mathbf{ODE}_k^\star(\mathsf{LIM})$ provides a more genuinely distinct characterization of $\mathbf{C}_{\mathbb{R}}$.

To highlight the significance of the proof technique of approximation, we compare our proof to the approach of Bournez and Hainry [6]. Most of the work, for us and them (though, they show something slightly different) goes into showing the main lemma 3.6, $\mathbf{C}_{\mathbb{R}} \subseteq \mathbf{RMU}_k(\mathsf{LIM})$. Starting with a function f in $\mathbf{C}_{\mathbb{R}}$, they proceed directly to an approximation $F(x, t)$ in $\mathbf{RMU}_k$, without passing through intermediary classes as we do (the notion of approximation is not stated explicitly in their work). Like us, they, apply LIM with respect to the argument t to obtain the result. Deviating from us, they use the fact that there is an elementary computable map which takes a modulus for f and a pair of integers (a_t, b_t) which code an approximation of x, and returns a pair of integers (p_t, q_t) which code an approximation of $f(x)$ (t is the approximation parameter). They show how that integer map can be embedded in $\mathbf{RMU}_k$, and they show how to *regularize* (similar to *linearizing*) the embedding to build $F(x,t)$ in $\mathbf{RMU}_k$, a *uniform* approximation for all x in the compact domain of f. The modulus of f is a crucial ingredient in the construction. Since the techniques applied in [6] do not exhibit a modulus for f (see [6, Remark 7.1]) they rely on the fact that C^2 functions on compact intervals have a (non constructive) trivial uniform modulus, which they use in the construction of F. Instead, we carry the modulus throughout our proof. Using the notion of one class of functions approximating another class of functions, we are able to show $\mathbf{C}_{\mathbb{R}} \preceq \pi\mathbf{RMU}_k^{(1)}$, with no additional restrictions, which implies the desired inclusion.

Since our approximation notion is transitive, we have broken up this approximation into a series of approximations, using some well-chosen intermediary classes of functions. The technique also allows us to work with function algebras rather than notions from classical computability (i.e. we avoid extensive work with the Turing machine definition of computable analysis). With function algebras, it is convenient to work with their inductive structure, a proof technique which works well with the method of approximation. The method appears amenable to generalization. As evidence of this, we see that in our proof, of the four approximations (recall figure 2), three are virtually the same as ones proved in our earlier paper [12]. In the context of a more general theory of approximation (future work), a number of approximations should follow from general facts, concentrating the work on the approximations that are important for the problem at hand. Thus we see this work as another step towards a more general theory that could have broad application to other problems of this sort.

We now discuss a possible improvement of our result. In our paper [10], we were able to remove the function θ_k from the basic functions, while maintaining the characterization of the elementary computable functions. This is nice because it makes the characterization even more simple, and furthermore the underlying function algebra (before the limit operation is applied) is then a collection of analytic functions. In our earlier work [10], we were able to dispense with the function θ_k, using our method of approximation, essentially showing that it suffices to work with an approximation of θ_k that can be built with other functions in the function algebra. This approach has difficulties in the context of this paper, and so we are led to the following interesting question (supposing $\mathbf{F}_k$ is one of the real function algebras we have considered in this paper, let $\mathbf{F}$ be the same function algebra, but *without* the basic function θ_k):

Question *Does* $\mathbf{C}_{\mathbb{R}} = \mathbf{RMU}(\mathit{LIM}) = \mathbf{INV}(\mathit{LIM}) = \mathbf{ODE}^\star(\mathit{LIM})$?

The last characterization by $\mathbf{ODE}^\star(\mathsf{LIM})$ would be especially interesting, since $\mathbf{ODE}^\star$ is essentially the widely studied class of real primitive recursive functions.

6. Acknowledgements

This work was partially supported by *Fundação para a Ciência e a Tecnologia* and EU FEDER POCTI/POCI, and grant SFRH / BPD / 16936 / 2004.

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