Upper and Lower Ramsey Bounds in Bounded Arithmetic
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Abstract

Pudlák shows that bounded arithmetic (Buss' S_2) proves an upper bound on the Ramsey number $R_r(k)$ (the r refers to the number of colors, assigned to edges; the k refers to the size of the monochromatic set). We will strengthen this result by improving the bound. We also investigate lower bounds, obtaining a non-constructive lower bound for the special case of 2 colors (i.e. $r = 2$), by formalizing a use of the probabilistic method. A constructive lower bound is worked out for the case when the monochromatic set size is fixed to 3 (i.e. $k = 3$). The constructive lower bound is used to prove two "reversals." To explain this idea we note that the Ramsey upper bound proof for $k = 3$ (when the upper bound is explicitly mentioned) uses the weak pigeonhole principle (WPHP) in a significant way. The Ramsey upper bound proof for the case in which the upper bound is not explicitly mentioned, uses the totality of the exponentiation function (Exp) in a significant way. It turns out that the Ramsey upper bounds actually imply the respective principles (WPHP and Exp) used to prove them, indicating that these principles were in some sense necessary.

Key words: Bounded Arithmetic, Ramsey Theory, Probabilistic Method
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1 Introduction

In this paper we will formalize upper and lower Ramsey bounds in the context of bounded arithmetic. Bounded arithmetic is a natural place to consider finite combinatorics like Ramsey theory, since one of the key features of bounded arithmetic is the fact that the exponentiation function cannot be defined. If exponentiation is added as an axiom to bounded arithmetic, many proofs in finite combinatorics go through easily. Without exponentiation, just in bounded arithmetic, we are often forced to find a different proof, or a proof may not even be possible. Thus it is of interest to consider how much Ramsey theory can be carried out in bounded arithmetic.

We use the conventional arrow notation $n \rightarrow (k)_r$ to mean that if each of the edges of the complete graph on n vertices is assigned one of r colors then there is a size k subset X of the vertices, such that all of the edges with vertices in X are assigned the same color (X is called *monochromatic*). The smallest such n that makes the arrow relation hold is denoted $R_r(k)$.

The starting point for Ramsey theory in bounded arithmetic is Pudlák's result ([Pud 91]). He shows (in theorem 1) that in bounded arithmetic we can prove a formalization of $r^{rk} \rightarrow (k)_r$, or $R_r(k) \leq r^{rk}$. In section 3 we improve this to $R_r(k) \leq (1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r}$, where $\epsilon > 0$ is any fixed rational number (this is the convention throughout for ϵ unless otherwise mentioned). In Ramsey theory it is common to consider special cases for which the bounds can be improved. We consider the well-studied (in the combinatorics literature) special case where $k = 3$, and further improve the bound to $R_r(3) \leq 3(1+\epsilon)(r!)$. Both proofs follow [Pud 91], but improve the bound by counting more efficiently. Counting sets in bounded arithmetic is used throughout this paper; we raise this point more explicitly in the conclusion.

The lower bounds on $R_r(k)$ come in two varieties, constructive and non-constructive. By constructive, people mean that given any r and k, there is a number n and an explicit description of a particular r-coloring (i.e. a coloring of the edges using r colors) of the complete graph on n vertices, such that there is no size k monochromatic set, implying $R_r(k) > n$. By non-constructive, one means that there is an existence argument, but no explicit description of the coloring. A major issue in Ramsey theory is finding constructive lower bounds, even though the non-constructive bound is usually much better. The combinatorics literature generally relies on our intuition as far as what counts as an explicit description. What people generally have in mind is some description which allows you to feasibly decide the color of any given edge. A description which called on you to consider all possible graphs on n vertices, though explicit in some sense, would not be considered a constructive lower bound; thus the existence arguments using the probabilistic method are not considered constructive.

One of the main inspirations for this paper is to understand more precisely the distinction between constructive and non-constructive lower bounds in Ramsey theory. A natural context for such an investigation is a weak theory like bounded arithmetic. A constructive lower bound is then a coloring given by a formula. We can now say exactly how constructive a definition is according to the complexity of the formula exhibiting it. We can also go beyond just the issue of how complex the definition is, and consider how complex it is to prove that the definition has the correct properties, or in other words, what fragment of bounded arithmetic do we need. With these thoughts as our ultimate motivation, we take a small step in this direction in sections 4 and 5.

In section 4, we consider the special case of $k = 3$, proving a sort of converse to Pudlák's theorem. Pudlák's proof uses the weak pigeonhole principle (WPHP_n^m), which states, informally, that a function from $\{0, 1, \ldots, m-1\}$ to $\{0, 1, \ldots, n-1\}$ is not injective, as long as $m \geq 2n$. His result can be restated for the special case of $k = 3$, as saying that in bounded arithmetic WPHP_n^{2n} implies $(r^{3r} \to (3)_r)$. By putting an idea from Krajíček ([Kraj 2001]) into the context of bounded arithmetic, we shall essentially prove the converse (we call it a "reversal"), namely, that $(r^{3r} \to (3)_r)$ implies WPHP_n^{2n}. Krajíček develops his idea in the context of propositional proof complexity and is not concerned with having a constructive lower bound; for our case that will be essential. We then develop a second kind of reversal, this time for a modified Ramsey principle in which there is no explicit mention of the upper bound of r^{3r}; it essentially says $\forall r \exists n\ n \to (3)_r$. In this case adding an axiom stating that the exponentiation function is total (call this Exp) allows us to prove the Ramsey principle. Again we get a reversal, namely, the Ramsey principle implies Exp.

In section 5 we consider the special case of $r = 2$ and formalize a non-constructive lower bound whose proof uses the probabilistic method. In the earlier sections an extra predicate symbol is used to exhibit the coloring (we call them "*oracle*" versions), while section 5 uses a number to code the coloring (the "*number*" versions). Doing non-constructive lower bounds essentially forces this modification on us, since we need to be able to refer to arbitrary colorings.

Throughout the paper, we will work within the system of first-order bounded arithmetic, S_2 (or $I\Delta_0 + \Omega_1$) as described in [Kraj 95] (developed by Buss [Buss 86], though $I\Delta_0$ was originally presented in [Par 71]). We assume familiarity with this work, but briefly mention a few points. S_2 is a theory of arithmetic in which induction is only allowed on bounded formulae. By Σ_i^b we mean the set of bounded formulae with at most i alternations of bounded quantifiers (and any number of *sharply* bounded quantifiers, that is, quantifiers with a bound of essentially $(\log t)$ for some term t), beginning with an existential quantifier; Π_i^b is the same, except that we begin with a universal quantifier. For Φ being some such set of formulae, $\Phi-$IND is the set of all induction axioms for formulae in Φ; $\Phi-$LIND is defined similarly for length induction. We can now define T_2^i to be Σ_i^b-IND, plus some basic axioms; S_2^i is the same, except that LIND replaces IND. Thus there is a hierarchy of theories within S_2: $S_2^1 \subseteq T_2^1 \subseteq S_2^2 \subseteq T_2^2 \subseteq \ldots \subseteq S_2$. This can all be relativized to a new relation symbol R. By $\Sigma_i^b(R)$ we mean the set of Σ_i^b formula with R added to the language; $\Pi_i^b(R)$ is defined similarly. $S_2^i(R)$ and $T_2^i(R)$ result from replacing Σ_i^b-LIND and Σ_i^b-IND by $\Sigma_i^b(R)-$LIND and $\Sigma_i^b(R)-$IND, respectively. In any theory containing S_2^1 we may conservatively add functions which run in polynomial time in the length of their number inputs. Furthermore, for many such functions, we can prove that appropriate definitions have the expected properties. Thus we shall freely use such basic functions in this work.

2 Technical Preliminaries

We make precise some of the terminology that we will use throughout the paper. By $[x]$ we mean the set $\{0, 1, \ldots, x-1\}$. When we refer to a number x as the domain or range of a function we mean the set $[x]$. Sometimes for ease of readability we use $\exp(x, y)$ to mean x^y. We follow common terminology in bounded arithmetic, using $|\, x|$ to refer to the length of the binary representation of the number x (i.e. about $\log_2 x$). Informally, the weak pigeonhole principle ($\mathsf{WPHP}_n^{t(n)}$) says that a function mapping $t(n)$ to n is not injective. Formally, we have two variants to consider.

Definition 1 *Let R be a 2 place relation symbol and $t(n)$ a term.*

- (Functional Form) $\mathsf{WPHP}_n^{t(n)}(R)$ *is the formula:*
  $\forall x < t(n)\ \exists! d < n\ R(x, d)\ \rightarrow\ \exists x < y < t(n)\ \exists d < n\ R(x, d) \wedge R(y, d).$
- (Relational Form) *Let* $\mathsf{rWPHP}_n^{t(n)}(R)$ *be the same except that the* $\exists!$ *quantifier is replaced by* $\exists$.

The relational version is a stronger statement which does not require R to be a function. [PWW 88] first proved weak pigeon hole statements in bounded arithmetic. Their results were strengthened by [MPW 2000] in the following result.

Theorem 2 *For any $\Sigma_1^b(R)$ formula $\psi(R)$, $T_2^2(R)$ proves $\mathsf{rWPHP}_n^{2n}(\psi(R))$.*

It is immediate that $T_2^2(R)$ suffices for the various weaker statements, such as the functional form or the case of choosing $\psi(R)$ to be simply R. This result is sharp in terms of the standard hierarchy of theories within S_2. In [Kraj 92] it is shown that $S_2^2(R)$ does not even prove one of the weakest statements, $\mathsf{WPHP}_n^{2n}(R)$ (see also [Kraj 95], p. 216). We also note that the various forms of the WPHP_n^m for different values of m are essentially equivalent. The following theorem essentially comes from [PWW 88]. Provable connections of this sort are explicitly mentioned in [Kraj 2001] (theorem 6.1) and [Thap 2002] (lemma 2.1). We state the particular form we need.

Theorem 3 *Let $i \geq 1$ and $\phi(R)$ be a formula of complexity $\Sigma_i^b(R)$; let $t(n)$ be a term, and $\epsilon > 0$ be a fixed rational. Then there is a formula $\psi(R)$ of complexity $\Sigma_i^b(R)$ such that $S_2^i(R)$ proves $\mathsf{WPHP}_n^{t(n)}(\psi(R)) \rightarrow \mathsf{WPHP}_n^{(1+\epsilon)n}(\phi(R))$.*

In stating the theorems we will stick with the typical case of $m = 2n$, but use the above theorem freely. The theorem holds for the relational form too.

In many places, we will use numbers to code sequences, which will be used to talk about objects such as sets and colorings of graphs. Suppose we have a number x and we want to code a sequence of length x. The natural way to encode such a sequence would be with a number of size about 2^x, however we cannot carry out exponentiation in bounded arithmetic unless x is small enough.

Definition 4 *Let* small(x) *abbreviate* $\exists m\ x < |m|$.

If x is small (from now on, we use the word in the formal sense of the above definition), we can in fact find 2^x and so we can work with sequences of length x. In this paper, when a parameter x is small, we will freely refer to various objects coded by sequences of this length; we will also freely refer to operations on these objects, such as checking for membership in a number that codes a set.

Sometimes we explicitly mention the use of smallness, but when there are exponentially large terms (such as 2^x or $x!$) the assumption of smallness is implicit. So for example, the formula $\phi(2^x)$ abbreviates $\mathsf{small}(x) \to \phi(2^x)$.

Now we define the Ramsey principle. It uses a 3 argument relation symbol H to refer to the colors of edges; the first two arguments are vertices and the third argument is the color of the edge connecting them. The principle says that if all the edges of the complete graph on n vertices are colored with one of r colors (always assumed to be at least 2), then we can find a monochromatic set X of size k. We use the notation $\mathsf{size}(X)$ to refer to the cardinality of the set coded by the number X; in terms of its encoding as a sequence this simply amounts to the length of an appropriate sequence for the set X.

Definition 5 *Let H be a relation symbol with 3 arguments.*

- *Let* $\mathit{Coloring}(H, n, r)$ *be* $\forall x < y < n\ \exists! d < r\ H(x, y, d)$.
- *Let* $\mathit{Monochromatic}(H, X, n, r)$ *be* $\exists d < r\ \forall u, v \in X\ H(u, v, d)$.
- *Let* $\mathit{Ramsey}(H, n \to (k)_r)$ *be*
  $\mathit{Coloring}(H, n, r) \to \exists X \subseteq [n]\ \mathit{size}(X) = k \wedge \mathit{Monochromatic}(H, X, n, r)$.

3 Ramsey upper bounds

As discussed, Pudlák showed that bounded arithmetic can prove the upper bound of $R_r(k) \leq r^{rk}$. First we improve this bound to $R_r(k) \leq (1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r} \approx r^{rk}\frac{(1+\epsilon)}{r^{r-1/2}(k-1)^{(r-1)/2}\sqrt{2\pi}^{r-1}}$. The last approximate equality uses Stirling's formula, a form of which can be proven in S^1_2 by the work of [Jer 03] (in his appendix).

We note that for the special case of $r = 2$, we get $R_2(k) \leq (1+\epsilon)\binom{2k-2}{k-1}$, close to the best known upper bound (outside of bounded arithmetic) of $\frac{1}{\sqrt{k-1}}\binom{2k-2}{k-1}$ [Thom 88]. However, for the special case of $k = 3$, this bound yields only $R_r(3) \leq (1+\epsilon)\frac{(2r)!}{2^r}$, somewhat larger than the best known upper bound. We provide a special argument catered to this case that improves this further to $R_r(3) \leq 3(1+\epsilon)(r!)$; the best known bound (outside of bounded arithmetic) is $(e - \frac{1}{24})(r!)$ ([Folk 74] and [CG 83]).

We will see that the various Ramsey principles can be proved in $T_2^3(H)$. It can be shown that $S_2^2(H)$ does not prove these Ramsey principles, so it is unknown where exactly they fit in the hierarchy $S_2^1(H) \subseteq T_2^1(H) \subseteq S_2^2(H) \subseteq T_2^2(H) \subseteq \ldots \subseteq S_2(H)$. This will be further discussed at the end of section 4.

3.1 General Improvement

Now we consider the general case, showing the following.

Theorem 6 *There is a $\Sigma_2^b(H)$ formula $\psi(H)$ such that $S_2^2(H) + \mathsf{WPHP}_n^{2n}(\psi(H))$ proves*

$\mathsf{Ramsey}(H, (1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r} \rightarrow (k)_r)$.

First we note how this theorem leads to a corollary.

Corollary 7 *$T_2^3(H)$ proves $\mathsf{Ramsey}(H, (1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r} \rightarrow (k)_r)$.*

Proof

By theorem 6, it suffices to show that $T_2^3(H)$ proves $\mathsf{WPHP}_n^{2n}(\psi(H))$ for ψ being $\Sigma_2^b(H)$. First we write ψ equivalently as $(\exists x < t\ \phi(H))$, where $\phi(H)$ is $\Pi_1^b(H)$. We know that $T_2^2(R)$ proves $\mathsf{WPHP}_n^{2n}(\exists x < t\ R)$, so for $\phi(H)$ replacing R in the $\Sigma_2^b(R)$ induction axioms of the $T_2^2(R)$ proof, we can push negations in and pull out quantifiers to see that it suffices to have $\Sigma_3^b(H)$ induction.
$\square$

We now prove theorem 6, following Pudlák's argument (we also follow the presentation in Krajíček [Kraj 95], theorem 12.1.3).

Assuming the Ramsey principle is false allows us to define an injection from the set of vertices to a smaller set of sequences. Such an injection will violate the WPHP, giving us our desired contradiction. Our set of vertices is $[(1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r}]$, which we will map to the set of sequences with elements from $[r]$ and no more than $k-2$ of any number (let's call such sequences "good"). To this point the argument will be the same as Pudlák's. We shall then diverge by mapping the good sequences injectively to $\frac{(rk-r)!}{((k-1)!)^r}$, so the composition of these two maps is an injection from $(1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r}$ to $\frac{(rk-r)!}{((k-1)!)^r}$, violating $\mathsf{WPHP}_n^{(1+\epsilon)n}$. It will be helpful to have a more general notion of good sequences given by the following definition.

Definition 8 *For small $a_0, \ldots, a_{r-1}$ and r, let $\mathsf{Good}_{a_0,\ldots,a_{r-1}}$ be the set of sequences with* *elements from $[r]$ having at most a_i i's.*

This set can be exhibited by a Δ_1^b (in S_2^1) formula, which will allow the latter part of the proof to go through in S_2^1.

First, working in $S_2^2(H)$, we shall repeat Pudlák's argument, describing the first map, from the set of vertices, $(1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r}$, to the set of good sequences, $\mathsf{Good}_{k-2,\ldots,k-2}$, under the assumption that H is an r-coloring of the edges with no size k monochromatic set. To aid the process we define a $\Pi_1^b(H)$ relation E with 2 sequences as arguments. The first is a sequence of vertices; the second is a sequence of colors (i.e. numbers from $[r]$).

$E(\langle x_0,\ldots,x_h\rangle,\langle\delta_0,\ldots,\delta_{h-1}\rangle)$ holds if

$$\begin{array}{ll} x_0 = 0 & \wedge \\ x_0 < x_1 < \ldots < x_h & \wedge \\ \forall i < j \leq h \; H(x_i, x_j, \delta_i) & \wedge \\ \forall i < h \forall y < x_{i+1} \; (y > x_i \rightarrow \exists j \leq i \; \neg H(x_j, y, \delta_j)) & \end{array}$$

In words, E says that we start with $x_0 = 0$ and let x_1 be the smallest numbered vertex such that edge $\{x_0, x_1\}$ is colored δ_0; x_2 is the next smallest such that edge $\{x_1, x_2\}$ is colored δ_1 and edge $\{x_0, x_2\}$ is colored δ_0, and so on. Given $x < (1+\epsilon)\frac{(rk-r)!}{((k-1)!)^r}$ we define $F(x)$ to be the unique sequence $\overline{\delta}$ such that for some $\langle x_0,\ldots,x_h\rangle$ we have $E(\langle x_0,\ldots,x_h\rangle,\overline{\delta})$, where $x_h = x$. We can show, using $\Sigma_2^b(H)-$LIND, that F is a well-defined, injective function; we only need LIND because the pertinent parameter in the inductive proofs is the length of the sequences, and such lengths are small. To show its range is indeed $\mathsf{Good}_{k-2,\ldots,k-2}$ consider what would happen if $\overline{\delta}$ had any color repeated $k-1$ times, at say $\delta_{i_1} = \ldots = \delta_{i_{k-1}}$, for $i_1 < \ldots < i_{k-1} < h$. Then the edges of $\{x_{i_1},\ldots,x_{i_{k-1}},x_h\}$ would all be colored δ_{i_1}, thus yielding a size k monochromatic set, violating our assumption. For the formula $\psi(H,x,\overline{\delta})$ called for in the theorem, we take the $\Sigma_2^b(H)$ definition of $F(x) = \overline{\delta}$, namely $\exists\langle x_0,\ldots,x_h\rangle \; x_h = x \wedge E(\langle x_0,\ldots,x_h\rangle,\overline{\delta})$.

Now we define the map f, from $\mathsf{Good}_{k-2,\ldots,k-2}$ to $\frac{(rk-r)!}{((k-1)!)^r}$. We will refer to general small parameters a_i, as this facilitates the inductive arguments, and then in the end we will substitute $k-2$ for these parameters. For small parameters, let $G(a_0,\ldots,a_{r-1}) = \frac{(a_0+\ldots+a_{r-1}+r)!}{(a_0+1)!\ldots(a_{r-1}+1)!} - 1$, an approximation of the size of $\mathsf{Good}_{a_0,\ldots,a_{r-1}}$; an inductive proof shows that $G(a_0,\ldots,a_{r-1})$ is an integer. If any $a_i = -1$ we define $G(a_0,\ldots,a_{r-1}) = 0$. We can see that for $a_i \geq 0$, G satisfies the following recursive bound.

$$
\begin{aligned}
G(a_0,\ldots,a_{r-1}) \geq\, & 1 + G(a_0-1,a_1,\ldots,a_{r-1}) \\
& +G(a_0,a_1-1,a_2,\ldots,a_{r-1}) \\
& +\ldots \\
& +G(a_0,\ldots,a_{r-2},a_{r-1}-1)
\end{aligned}
$$

To avoid confusion, note that for what follows, at most one parameter among any particular list $a_0,\ldots,a_{r-1}$ may have the number 1 subtracted from it. We define the function $f_{a_0,\ldots,a_{r-1}}$ (a function from $\mathsf{Good}_{a_0,\ldots,a_{r-1}}$ to $[G(a_0,\ldots,a_{r-1})]$) with the following recursive formulas (Notation: $\langle\rangle$ is the empty sequence, and for strings u and v, $u|v$ is their concatenation).

Definition 9

- $f_{a_0,\ldots,a_{r-1}}(\langle\rangle) = \ 0$

- $$
\begin{aligned}
f_{a_0,\ldots,a_{r-1}}(x|i) = \ & 1 + f_{a_0,\ldots,a_i-1,\ldots,a_{r-1}}(x) \\
& + G(a_0-1,a_1,\ldots,a_{r-1}) \\
& + \ldots \\
& + G(a_0,\ldots,a_{i-1}-1,\ldots,a_{r-1})
\end{aligned}
$$

Note that the definition of $f_{a_0,\ldots,a_{r-1}}$ is Σ_1^b since we only require a recursion with $a_0+\ldots+a_{r-1}$ steps (a small number), which can be carried out using a short sequence that can be coded by a number.

Claim 10 *For $a_0,\ldots,a_{r-1} \leq k-2$, and any sequence x such that $x \in \mathsf{Good}_{a_0,\ldots,a_{r-1}}$, we have $f_{a_0,\ldots,a_{r-1}}(x) < G(a_0,\ldots,a_{r-1})$.*

Proof

The proof is by induction on the length of x and so can be carried out in S_2^1. We indicate how the inductive step works. Suppose x is an appropriate sequence whose last element is i; so $x = x'|i$.

$$
\begin{aligned}
f_{a_0,\ldots,a_{r-1}}(x) =\, & 1 + f_{a_0,\ldots,a_i-1,\ldots,a_{r-1}}(x') + G(a_0-1,a_1,\ldots,a_{r-1}) + \\
& \ldots + G(a_0,\ldots,a_{i-1}-1,\ldots,a_{r-1}) \\
<\, & 1 + G(a_0-1,a_1,\ldots,a_{r-1}) + \\
& \ldots + G(a_0,\ldots,a_i-1,\ldots,a_{r-1}) \\
\leq\, & G(a_0,\ldots,a_{r-1})
\end{aligned}
$$

The inductive hypothesis justifies the first inequality. The second inequality follows from the recurrence on G.

□

Claim 11 *For any $x, y \in \mathsf{Good}_{a_0,\ldots,a_{r-1}}$, such that $x \neq y$, $f_{a_0,\ldots,a_{r-1}}(x) \neq f_{a_0,\ldots,a_{r-1}}(y)$.*

Proof

We proceed by induction on the sequence lengths of x or y, again working in S^1_2. As in the last proof, we point out how the inductive step works. Suppose $x = x'|i$ and $y = y'|j$. If $i = j$ then once we use the definition of f, we can apply the inductive hypothesis. If $i \neq j$, assume $i < j$, and then we can calculate.

$$\begin{aligned} f_{a_0,\ldots,a_{r-1}}(x'|i) &= 1 + f_{a_0,\ldots,a_i-1,\ldots,a_{r-1}}(x') + G(a_0-1, a_1, \ldots, a_{r-1}) + \\ &\quad \ldots + G(a_0, \ldots, a_{i-1}-1, \ldots, a_{r-1}) \\ &< 1 + G(a_0-1, a_1, \ldots, a_{r-1}) + \\ &\quad \ldots + G(a_0, \ldots, a_i - 1, \ldots, a_{r-1}) \\ &\leq f_{a_0,\ldots,a_{r-1}}(y'|j) \end{aligned}$$

The first inequality follows from claim 10 and last from the definition of f.
□

Now we turn back to the particular case of $a_0 = \ldots = a_{r-1} = k-2$. The function $f_{k-2,\ldots,k-2}$ is injective by claim 11. From claim 10 we see that it has the desired the range of $\frac{(rk-r)!}{((k-1)!)^r}$. And so we have finished approximately counting the set of good sequences.

3.2 Special Case: $k = 3$

We now consider the special case of $k = 3$. It is already covered in the above case by counting the set $\mathsf{Good}_{1,\ldots,1}$ (r 1's), sequences with elements from $[r]$, having no repeated elements. We will now count this set more efficiently, bounding its size by $3(r!)$, which yields the following theorem.

Theorem 12 *There is a $\Sigma^b_2(H)$ formula $\psi(H)$ such that $S^2_2(H) + \mathsf{WPHP}^{2n}_n(\psi(H))$ proves*

$\mathsf{Ramsey}(H, 3(1+\epsilon)(r!) \to (3)_r)$.

The following corollary is proved in a similar way to corollary 7.

Corollary 13 *$T^3_2(H)$ proves $\mathsf{Ramsey}(H, 3(1+\epsilon)(r!) \to (3)_r)$.*

To prove theorem 12, we proceed as in the proof of theorem 6, working in $S^2_2(H)$ to map $3(1+\epsilon)(r!)$ to $\mathsf{Good}_{1,\ldots,1}$. Now we carry out the counting, describing an injective mapping ρ

from $\mathsf{Good}_{1,\ldots,1}$ to $3(r!)$. The rest of the argument can be carried out in S^1_2. We will use $(r)_m$ to denote the product $r(r-1)\ldots(r-m+1)$.

Let $\rho(\langle b_1,\ldots,b_h\rangle) = (r)_0+(r)_1+\ldots+(r)_{h-1}+g_r(\langle b_1,\ldots,b_h\rangle)$, where g_s is defined recursively as follows; the domain of g_s is $\mathsf{Good}_{1,\ldots,1}$ (s 1's).

- $g_s(\langle\rangle) = 0$
- $g_s(\langle b_1,\ldots,b_h\rangle) = (b_1)(s-1)_{(h-1)} + g_{s-1}(\langle b'_2,\ldots,b'_h\rangle)$, where $b'_i = \begin{cases} b_i & \text{if } b_i < b_1 \\ b_i - 1 & \text{if } b_i > b_1 \end{cases}$.

The definition of ρ is Σ^b_1 since the recursion is defined on short sequences. To show ρ is injective with proper range it suffices to prove the following three claims in S^1_2 (the parameters h and s are small). The next two claims both follow by induction on the length, h, of the sequence, similar to claims 10 and 11 of the last subsection.

Claim 14 *$g_s(\langle b_1,\ldots,b_h\rangle) < (s)_h$, for $\langle b_1,\ldots,b_h\rangle \in \mathsf{Good}_{1,\ldots,1}$ (s 1's) of length h.*

Claim 15 *For any fixed h, g_s is injective on the length h sequences in $\mathsf{Good}_{1,\ldots,1}$ (s 1's).*

Now to check that the range is correct we use the following bound (note that we use 3, rather than e since it allows for a simple inductive proof in S^1_2).

Claim 16 *For small r,* $(1/0!+1/1!+\ldots+1/r!) < 3$.

Proof

We show by induction on r show that $(1/0!+1/1!+\ldots+1/r!) < 3 - \frac{2}{(r+1)!}$. We note the inductive step.

$$\begin{aligned}(1/0!+1/1!+\ldots+1/r!+1/(r+1)!) &\leq 3 - \frac{2}{(r+1)!} + \frac{1}{(r+1)!} \\ &= 3 - \frac{1}{(r+1)!} \\ &\leq 3 - \frac{2}{(r+2)!}\end{aligned}$$

□

So the range is $[3(r!)]$ since

$$\begin{aligned}\rho(\langle b_1,\ldots,b_r\rangle) &< (r)_0+\ldots+(r)_r \\ &= r!(1/0!+1/1!+\ldots 1/r!) \\ &< 3(r!).\end{aligned}$$

4 The Reversals

We will consider two "reversals," spending the bulk of the section on the first one. In the first reversal we show how the Ramsey principle (for $k = 3$) implies the WPHP. We sketch the idea behind the proof now. Suppose for contradiction that WPHP_n^{2n} does not hold; in fact we can replace $2n$ by a larger quantity, $t(n)$ (a term defined in lemma 19). So we have an injective function from $t(n)$ to n. Assume that $n = 2^r$ for some r. We constructively exhibit an r-coloring of the graph on n vertices with no size 3 monochromatic set. Using our injective function we pull this coloring back to an isomorphic r-coloring for the graph on $t(n)$ vertices. Since the function is injective, this new coloring also has no size 3 monochromatic set. We in fact have that $t(n) > r^{3r}$, so since $(r^{3r} \to (3)_r)$ holds, we know there is a size 3 monochromatic set. We have arrived at a desired contradiction.

To prove our reversal we will formalize the constructive lower bound of $R_r(3) > 2^r$. The proof, as pointed out in [GRS 90] (p.145) goes through easily by induction on r. For the inductive step we start with two $(r-1)$ colored graphs, each with 2^{r-1} vertices and no monochromatic sets of size 3. They are joined by edges of a new color, giving us the appropriate graph on 2^r vertices. The construction we give, based on this argument, essentially takes as its vertices the binary strings of length r, and colors edges according to the first bit (from the right) at which two strings differ (let $\langle u \rangle_i$ refer to the i^{th} bit in the binary representation of u).

Definition 17 *Let* $\mathsf{Low}(x, y, k)$ *be the* Δ_1^b *formula:*

$$\forall i < \mathit{max}(|x|, |y|)(i < k \to \langle x \rangle_i = \langle y \rangle_i) \wedge \langle x \rangle_k \neq \langle y \rangle_k.$$

Now we can prove that Low really is a lower bound coloring.

Lemma 18 S_2^1 *proves* $\forall r \neg \mathit{Ramsey}(\mathit{Low}, 2^r \to (3)_r)$.

Proof

Fix any small r. S_2^1 proves for all $X = \{u, v, w\} \subseteq [2^r]$, $\neg\mathsf{Monochromatic}(\mathsf{Low}, X, 2^r, r)$. Consider any color $d < r$, and we show that X is not colored just by d. If $\langle u \rangle_d = \langle v \rangle_d$ then edge $\{u, v\}$ is not colored by d, so assume $\langle u \rangle_d \neq \langle v \rangle_d$. Then $\langle w \rangle_d$ has to equal one of $\langle u \rangle_d$ or $\langle v \rangle_d$, so not all the edges can be colored d.

We prove $\mathsf{Coloring}(\mathsf{Low}, 2^r, r)$ by LIND up to r, noting that for $x < y < 2^r$, the first bit where they differ will be at a unique position $d < r$.

□

Lemma 19 *Let* $t(n)$ *be the term* $(\log 2n)^{3 \log 2n}$. *Let* $i \geq 1$. *For any formula* $\phi(R)$ *of complexity* $\Sigma_i^b(R)$, *there is a formula* $\psi(R)$ *of complexity* $\Sigma_i^b(R)$ *such that*

$$S_2^1(R) + \forall r \mathit{Ramsey}(\psi(R), r^{3r} \to (3)_r) \text{ proves } \forall n \mathit{WPHP}_n^{t(n)}(\phi(R)).$$

Proof

Let $\psi(x_1, x_2, k)$ be the $\Sigma_i^b(R)$ formula $\exists y_1, y_2 < 2^r \phi(R, x_1, y_1) \wedge \phi(R, x_2, y_2) \wedge \mathsf{Low}(y_1, y_2, k)$. To show $\mathsf{WPHP}_n^{t(n)}(\phi(R))$ fix some n and find the r such that $2^{r-1} \leq n < 2^r$; note that r is small. Assume $\neg\mathsf{WPHP}_n^{t(n)}(\phi(R))$, so $\phi(R)$ is a total injective function from $t(n)$ to n. Showing $\neg\mathsf{Ramsey}(\psi, r^{3r} \to (3)_r)$ finishes the proof, so it suffices to show the following 2 claims.

Claim 20 *$S_2^1(R)$ proves $\mathsf{Coloring}(\psi, r^{3r}, r)$.*

Claim 21 *$S_2^1(R)$ proves $\forall x_1 < x_2 < x_3 < r^{3r} \neg\mathsf{Monochromatic}(\psi, \{x_1, x_2, x_3\}, r^{3r}, r)$.*

To prove the first claim, consider any edge given by $x_1, x_2 < r^{3r}$, $x_1 \neq x_2$. Note that $x_1 < r^{3r} = t(2^{r-1}) \leq t(n)$, therefore, x_1 is mapped by $\phi(R)$ to a number $y_1 < n < 2^r$. Similarly, x_2 is mapped to some $y_2 < 2^r$, $y_1 \neq y_2$. By lemma 18, Low assigns a unique color to the edge (y_1, y_2), so (x_1, x_2) is also assigned a unique color.

To prove the second claim, for $j = 1, 2, 3$, let $y_j < 2^r$ be such that $\phi(R, x_j, y_j)$. The set $\{y_1, y_2, y_3\}$ is not monochromatic by lemma 18, therefore neither is $\{x_1, x_2, x_3\}$, since the latter set is colored in the same manner as the former.

□

Now, lemma 19 together with theorem 3 immediately yields the following theorem, the reversal.

Theorem 22 (Reversal to WPHP) *Let $i \geq 1$. For any formula $\phi(R)$ of complexity $\Sigma_i^b(R)$, there is a formula $\psi(R)$ of complexity $\Sigma_i^b(R)$ such that*

$S_2^i(R) + \forall r \mathsf{Ramsey}(\psi(R), r^{3r} \to (3)_r)$ *proves* $\forall n \mathsf{WPHP}_n^{2n}(\phi(R))$.

Note a special case of particular interest if we take $\phi(R)$ to be simply R; then the above theorem tells us that there is a $\Sigma_1^b(R)$ formula with the appropriate proof going through in $S_2^1(R)$. From this fact we can obtain the following independence result.

Corollary 23 *There is a $\Sigma_1^b(R)$ formula $\psi(R)$ such that $S_2^2(R)$ does not prove*

$\mathsf{Ramsey}(\psi(R), r^{3r} \to (3)_r)$.

Proof

We take the ψ from theorem 22, for $i = 1$, so that if $S_2^2(R)$ did prove the Ramsey statement, it would also prove $WPHP_n^{2n}(R)$ (in fact R can be replaced by $\Sigma_1^b(R)$). But as mentioned earlier (see remarks following theorem 2), this is known to be unprovable in $S_2^2(R)$.

□

However, the argument from [CK 99] can be applied to give a proof of the following stronger result.

Theorem 24 *(essentially [CK 99]) $S_2^2(H)$ does not prove* $\mathsf{Ramsey}(H, r^{3r} \to (3)_r)$.

The reversal of theorem 22 can be mildly improved upon, using basically the same argument and the same lower bound. As long as k (the size of the monochromatic set we are looking for) is a constant larger than 2 (it was 3 above), or even a term r^c for some standard c, the lower bound used for $k = 3$ is good enough to get the reversal. However other reversals of this kind seem difficult to obtain. For example, in the case of $R_2(k)$, we have an upper bound of 2^{2k}, but the best known constructive lower bound is about $R_2(k) \geq e^{\log^2 k/4 \log\log k}$, from [FW 81]. The bounds are too far apart to obtain a reversal by these methods. For our reversal, notice we had the bounds $2^r < R_r(3) \leq r^{3r}$, which are close enough that for the term $t(n) = (\log 2n)^{3\log 2n}$ (from lemma 19), we have $r^{3r} \leq t(2^{r-1})$. For the existing bounds on $R_2(k)$, we do not have such a term in the language. Thus we see that obtaining the reversal in this manner requires two key ingredients: 1) The lower bound is constructive and 2) the upper bound and the lower bound are related appropriately by a term in the language.

We now consider a different kind of reversal, obtained from a different Ramsey principle. Up to this point we have considered Ramsey principles in which the upper bound is explicitly given. Now we consider the case in which the appropriate number is only asserted to exist, namely $\forall r \exists n \mathsf{Ramsey}(H, n \to (3)_r)$. This can easily be proven if we are given a principle stating that the exponentiation function is total, since the usual proof of Ramsey's theorem can be carried out directly, coding any objects in the proof using the exponentiation function. Let Exp abbreviate $\forall x \exists y\ x = |y|$, and we have the following result.

Claim 25 *$S_2^1(H) + \mathsf{Exp}$ proves $\forall r \exists n \mathsf{Ramsey}(H, n \to (3)_r)$.*

The reversal is something like a converse, where the Ramsey principle is replaced by a schema (we put Σ_0^b in place of the relation symbol H to indicate that the principle holds for all Σ_0^b formulae).

Theorem 26 (Reversal to Exp) *$S_2^1 + \forall r \exists n\ \mathsf{Ramsey}(\Delta_1^b, n \to (3)_r)$ proves Exp.*

Proof

> We actually only use the Ramsey principle for the formula Low (a Δ_1^b formula). Given r there is an n such that $\mathsf{Ramsey}(\mathsf{Low}, n \to (3)_r)$. To show Exp it suffices to show $r < |n|$. It is a general fact that for $b \geq a$, $\mathsf{Ramsey}(H, n \to (k)_b) \to \mathsf{Ramsey}(H, n \to (k)_a)$. So if $r \geq |n|$, we would arrive at $\mathsf{Ramsey}(\mathsf{Low}, n \to (3)_{|n|})$, but we in fact have that $\neg\mathsf{Ramsey}(\mathsf{Low}, n \to (3)_{|n|})$ (basically by the argument of lemma 18). Thus $r < |n|$ as desired.
> □

This is not an unexpected result, however note the significance of the constructive lower bound. It would seem difficult to prove related results with different Ramsey principles in which the constructive lower bounds are not good enough.

5 Formalizing The Probabilistic Method

The Probabilistic Method is used (in [Prob 92]) to show that there exists a 2-coloring of the complete graph on $n = 2^{k/2}$ vertices with no size k monochromatic set, or in other words $R_2(k) > 2^{k/2}$. Since this is an existence argument, to formalize this claim we will need to change our orientation so that we can assert the existence of graph colorings. Thus we will now consider colorings coded as numbers, rather than given by a predicate symbol. A graph coloring on n vertices will be coded by a number $G < \exp(2, \binom{n}{2})$ (recall $\exp(x, y)$ means x^y) , which can be interpreted as a binary string of length $\binom{n}{2}$, so each of the $\binom{n}{2}$ edges is colored "0" or "1" accordingly; note that the number of vertices n must be small, which will be implicit throughout our discussion. We will use the same formula "Ramsey" as before, except that now we will use the number G in place of the predicate symbol H. Finally we formalize the claim by proving the following.

Theorem 27 *T_2^2 proves* $\exists G < \exp(2, \binom{2^{k/2}}{2}) \neg \mathit{Ramsey}(G, 2^{k/2} \rightarrow (k)_2)$

To prove the theorem first consider the informal probabilistic argument restated as a counting argument. Call a coloring "bad" if it has a size k monochromatic set, and "good" otherwise. So our goal is to show that a good coloring exists. There are $\binom{n}{k}$ size k subsets of vertices and for each such fixed subset there are $\exp(2, \binom{n}{2} - \binom{k}{2} + 1)$ colorings that make it monochromatic. Thus the number of bad colorings is bounded by $\binom{n}{k} \exp(2, 1 - \binom{k}{2}) \exp(2, \binom{n}{2})$. Since $n = 2^{k/2}$, a calculation shows that $\binom{n}{k} \exp(2, 1 - \binom{k}{2}) < 1$, so the number of bad colorings is less than $\exp(2, \binom{n}{2})$, which is the total number of colorings. So there must be a good coloring.

The lack of exponentiation in bounded arithmetic precludes formalizing this argument directly since the above proof involves counting large sets of colorings. To formalize the theorem we reformulate the proof, using the structure of the counting argument to define a function on the set of all colorings, simulating the argument using the rWPHP.

Suppose for sake of contradiction that the theorem does not hold. So for some small k and n, where we let $n = 2^{k/2}$, every coloring $G < \exp(2, \binom{n}{2})$ is bad. We now define a multi-function, F, from the set of all colorings (i.e. numbers $< \exp(2, \binom{n}{2})$) to a set which counts all the bad colorings (numbers bounded by $(1/2) \exp(2, \binom{n}{2})$). F will be an injective multi-function and so violate rWPHP, which holds in T_2^2.

First we sketch the definition of F. Given a coloring $G < \exp(2, \binom{n}{2})$, F will take G and find a size k monochromatic set X. The function "SetNumber" (to be defined) will map the set X to s, $0 \leq s < \binom{n}{k}$; each set is mapped to a different number. Many colorings have X as its monochromatic set, so to uniquely identify G, we indicate which of the 2 colors X has, and for the remaining $\binom{n}{2} - \binom{k}{2}$ edges, we choose the appropriate coloring of the edges. This can be seen as a binary string of length $\binom{n}{2} - \binom{k}{2} + 1$, so we arrive at a number, say v, where $v < \exp(2, \binom{n}{2} - \binom{k}{2} + 1)$. We can obtain this v by a Σ_1^b definable (in S_2^1) function Rest, so $\mathsf{Rest}(G, X) = v$. G is then mapped to $s \cdot \exp(2, \binom{n}{2} - \binom{k}{2} + 1) + v$.

Now we define the function $\mathsf{SetNumber}$, a sort of lexicographic ordering which assigns set $\{0, 1, \ldots, k-1\}$ to 0, $\{n-k, \ldots, n-1\}$ to $\binom{n}{k} - 1$, and other sets to intermediary numbers in an injective manner.

Definition 28 *For $X \subseteq [n]$, let $\mathsf{SetNumber}(X) = f_0(X)$, where*

- $f_a(\{\}) = 0$ *for* $0 \leq a \leq n$

- $f_a(X) = \begin{cases} f_{a+1}(X - \{a\}) & \textit{if } a \in X, \\ \binom{n-a-1}{size(X)-1} + f_{a+1}(X) & \textit{otherwise.} \end{cases}$

This recursively defined function can be given by a Σ_1^b formula using sequences of small size n. We now prove some properties about it in S_2^1.

Claim 29 $f_a(X) < \binom{n-a}{size(X)}$, *for* $X \subseteq \{a, \ldots, n-1\}$.

Proof

We show this by induction on a from n down to 0. For the inductive step we assume the claim for $a+1$ and then need to show $f_a(X) < \binom{n-a}{size(X)}$, where $X \subseteq \{a, \ldots, n-1\}$.

If $a \in X$, then $f_a(X) = f_{a+1}(X - \{a\}) < \binom{n-a-1}{size(X)-1} \leq \binom{n-a}{size(X)}$.

Otherwise, $a \notin X$. For $X = \{\}$ we are done, otherwise we carry out the following calculation.

$f_a(X) = \binom{n-a-1}{size(X)-1} + f_{a+1}(X) < \binom{n-a-1}{size(X)-1} + \binom{n-a-1}{size(X)} = \binom{n-a}{size(X)}$.

□

This claim shows that the range of $\mathsf{SetNumber}$ (for $a = 0$) really is $[\binom{n}{k}]$. We can also see that it is injective on size k subsets of $[n]$. Consider two distinct sets $X, Y \subseteq [n]$, both of size k. Let b be the smallest element in one set, but not in the other; suppose $b \in X$, and $b \notin Y$. Since the recursive procedure will be the same up to b, we get that $f_0(X) = m + f_b(X')$ and $f_0(Y) = m + f_b(Y')$, for some number m; the sets X' and Y' come from X and Y, respectively, with the same elements from $\{0, \ldots, b-1\}$ removed by the recursive procedure (so in particular, the size of the primed sets are also the same). It now suffices to show that

$f_b(X') < f_b(Y')$. We have $f_b(X') = f_{b+1}(X' - \{b\}) < \binom{n-b-1}{size(X')-1} \leq f_{b+1}(Y') + \binom{n-b-1}{size(X')-1} = f_b(Y')$. We can now define the multi-function F, with the coloring G as input, and y as the output.

Definition 30 *Let $F(G, y)$ be the following Σ_1^b formula*

$$
\begin{aligned}
\exists X \subseteq [n]\ & \mathsf{size}(X) = k && \wedge \\
& \mathsf{Monochromatic}(G, X, n, 2) && \wedge \\
& \mathsf{SetNumber}(X) = s && \wedge \\
& \mathsf{Rest}(G, X) = v && \wedge \\
& y = s \cdot \exp(2, \tbinom{n}{2} - \tbinom{k}{2} + 1) + v
\end{aligned}
$$

By the assumption there exists such X, so F is a multi-function. F is injective, since SetNumber and Rest are injective and $v < \exp(2, \binom{n}{2} - \binom{k}{2} + 1)$. To show that the range of F is $(1/2)\exp(2, \binom{n}{2})$ we prove the following bound (essentially from [Prob 92]).

Claim 31 $y \leq (1/2)\exp(2, \binom{n}{2})$.

Proof

By claim 29 and the definition of s and v, $s \leq \binom{n}{k} - 1$ and $v < \exp(2, \binom{n}{2} - \binom{k}{2} + 1)$. Thus

$$
\begin{aligned}
y &< (\tbinom{n}{k} - 1)\exp(2, \tbinom{n}{2} - \tbinom{k}{2} + 1) + \exp(2, \tbinom{n}{2} - \tbinom{k}{2} + 1) \\
&= \tbinom{n}{k}\exp(2, \tbinom{n}{2} - \tbinom{k}{2} + 1) \\
&= \exp(2, \tbinom{n}{2})\tbinom{n}{k}\exp(2, 1 - \tbinom{k}{2}).
\end{aligned}
$$

We are done when we show that $\binom{n}{k}\exp(2, 1 - \binom{k}{2}) < 1/2$.

$$
\binom{n}{k}\exp(2, 1 - \binom{k}{2}) < \frac{n^k}{k!}\frac{\exp(2, 1 + k/2)}{\exp(2, k^2/2)} \leq \frac{\exp(2, 1 + k/2)}{k!}\frac{(\exp(2, k/2))^k}{\exp(2, k^2/2)},
$$

substituting $2^{k/2}$ for n.

The last expression is bounded by $1/2$ for most k ($k \geq 4$).

□

That finishes the entire proof since we have now arrived at a contradiction, namely, a multi-function F violating the weak pigeonhole principle. Since F is given by a Σ_1^b formula, theorem 2 shows that this proof goes through in T_2^2.

The technique used in this proof provides a recipe for formalizing such non-constructive counting arguments, as long as the counting argument is constructive enough to allow such a function to be defined. The complexity of the function definition affects what theory suffices.

6 Conclusion

We now review what we have done and present some thoughts on future work. We have seen another application of the weak pigeonhole principle in formalizing mathematics in bounded arithmetic (originally it was introduced in [PWW 88] in order to prove the infinitude of primes). In the reversal to WPHP we have shown a kind of formal equivalence between Ramsey and pigeonhole principles, thus making precise some of our informal connection between principles of these two kinds. We showed how to use the weak pigeonhole principle to simulate the probabilistic method, thus obtaining a non-constructive lower bound. However to obtain the reversals we found that constructive lower bounds were a crucial ingredient to our approach.

Proving constructive lower bounds in bounded arithmetic has inherent interest and in some cases, can be applied to obtain reversals. Many such lower bounds use set systems and the linear algebraic method (discussed in [BF 92]). Thus the formalization would require formalizing the appropriate aspects of linear algebra or somehow avoiding it. An example of this is the well known constructive lower bound of $R_2(k) \geq e^{\log^2 k/4 \log\log k}$ (from Frankl and Wilson, [FW 81]), mentioned at the end of section 4. When stated appropriately, such a lower bound looks hard, or perhaps impossible to prove in bounded arithmetic. If linear algebra plays a key role, the work of [SC 2003] on provably feasible matrix theorems could be of use.

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