Characterizing Computable Analysis with Differential Equations

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Outline

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Computability over the Reals?

Suppose $f : \mathbb{R} \to \mathbb{R}$. When is f "computable"?

Let $f(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ 1, & \text{if } x > 0. \end{cases}$ Is it computable?

f is computable according to the BSS Model, and not according to Computable Analysis.

Is e^x computable?

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Computable Analysis

f is computable according to "Naive" Computable Analysis iff:

There is a computable function $F^x(n)$ with an oracle for the real number x such that $F^x(n) \to f(x)$, as $n \to \infty$.

Definition

$f \in \mathbf{C}_{\mathbb{R}}$ (Computable Analysis) iff:

There is a computable function $F^x(n)$ with an oracle for the real number x such that $|f(x) - F^x(n)| \leq 1/n$.

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Other Models

1. Shannon's circuit model.
2. Neural Networks.
3. Hybrid systems.

There is not an agreed upon definition of computability over $\mathbb{R}$.
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Our work: Showing certain models equivalent.

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Function Algebras and Real Recursive Functions

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A Function Algebra $\mathrm{FA}[f_1, \ldots, f_k; \mathrm{op}_1, \ldots, \mathrm{op}_n]$ is the smallest set of functions containing $f_1, \ldots, f_k$, and closed under the operations $\mathrm{op}_1, \ldots, \mathrm{op}_n$.

Example: $\mathrm{FA}[0, 1, +, \dot{-}, \mathrm{P}; \mathrm{comp}, \sum, \prod, \mu]$, which is equal to the computable functions over the naturals.

Real Recursive Functions: Function algebras over the reals, introduced by C. Moore 1996.

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Some basic functions over the reals

- Constant functions: $0, 1, -1$
- Projection functions "P" (example: $U(x, y) = x$)
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ODE is the operation:

- **Input**: $g(x)$, $f(y,u,x)$.
- **Output**: The solution of the IVP

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Our main theorem

Theorem (Main Theorem)

$\mathbf{C}_{\mathbb{R}} = \mathcal{ODE}_k^{\star}(LIM)$ *for* $k \geq 2$.

A function algebra based on searching

Definition

The operation UMU takes as **Input**: $f(t, \bar{x})$ such that

1. For any $\bar{x}$, $f(t, \bar{x})$ increases in t, and
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For $k \geq 2$, $\mathbf{C}_{\mathbb{R}} = \mathcal{ODE}_k^{\star}(LIM)$ $= \mathcal{UMU}_k(LIM)$.

Theorem (Bournez and Hainry 2006)

Roughly: $\mathcal{C}^2 \cap [\mathbf{C}_{\mathbb{R}}] = [\mathcal{UMU}_k(LIM)]$, for $k \geq 3$

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More Motivation

Results of the form “$\mathcal{CA} = \text{FA}$”.

- Interesting to provide alternative models for Computable Analysis.
- Connects discrete-time and continuous-time.
- “Basic Theorem” over the reals?
  Church-Turing thesis over the reals?
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$\mathbf{C}_{\mathbb{R}} \subseteq \mathcal{UMU}_k(\mathrm{LIM})$

By Turing Machine simulation (Bournez and Hainry 2006).

Difficulties with this approach ...

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- $f(\bar{x}) \preceq^t f^*(\bar{x}, t)$ means: $|f(\bar{x}) - f^*(\bar{x}, t)| < \dfrac{1}{t}$
- For classes of functions $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \preceq \mathcal{B}$ means:

  For any f $\in \mathcal{A}$ there is $f^* \in \mathcal{B}$ such that $f \preceq f^*$.

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Breaking up the proof

Lemma (Transitivity)

Suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are classes of functions and suppose $\mathcal{C}$ is nice. Then $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{C}$ implies $\mathcal{A} \preceq \mathcal{C}$.

Definition

Let $\mathbf{C}_{\mathbb{Q}}$ be $\{f_{|\mathbb{Q}} \mid f \in \mathbf{C}_{\mathbb{R}}\}$

$\mathbf{C}_{\mathbb{Q}} \preceq \widetilde{\mathbf{dC}_{\mathbb{Q}}} \preceq \mathbf{d}\widetilde{\mathcal{MU}_{\mathbb{Q}}} \preceq \mathcal{MU}_{\mathbb{Q}} \preceq \mathcal{UMU}_k$

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The operation MU. **Input:** $f(t, \bar{x})$ (over $\mathbb{N}$) satisfying:

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Suppose OP takes a function $f : \mathbb{N}^k \to \mathbb{N}$ and returns a function $g : \mathbb{N}^m \to \mathbb{N}$. Then $\mathrm{OP}_{\mathbb{Q}}$ is the following operation:

1. $\mathrm{OP}_{\mathbb{Q}}$ takes as input $f : \mathbb{Q}^k \to \mathbb{Q}$ such that $f_{|\mathbb{N}} : \mathbb{N}^k \to \mathbb{N}$.
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The inductive proof

Proof proceeds inductively on the function algebra $\mathcal{MU}_{\mathbb{Q}}$:

- Show the basic functions of $\mathcal{MU}_{\mathbb{Q}}$ are approximated by $\mathcal{UMU}_k$.
- Show that the operations of $\mathcal{MU}_{\mathbb{Q}}$ preserve the approximation:

  For $f \in \mathcal{MU}_{\mathbb{Q}}$ and $f^* \in \mathcal{UMU}_k$, suppose $g = OP(f)$, and $f \preceq f^*$. Then there is $g^* \in \mathcal{UMU}_k$ such that $g \preceq g^*$.

The inductive proof

Proof proceeds inductively on the function algebra $\mathcal{MU}_{\mathbb{Q}}$:

- Show the basic functions of $\mathcal{MU}_{\mathbb{Q}}$ are approximated by $\mathcal{UMU}_k$.
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Basic Functions

- $0, 1, -1, +, \mathrm{P}, *$
- θ_1, div ...

Composition

Generic and easy, but uses “modulus assumption” significantly

....

Linearization

Generic, but involved ....

Sums and Products

Specific, but uses earlier results ....

Search Operation MU

Specific and significant .... Key Points:

- Use the particular shape.
- Need to process the approximation function.
- Can approximate linearization in well-behaved way.

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Outline

Conclusion

We have a new characterization of Computable Analysis.

Further improvements?:

- Show it is useful.
- Simplify to "analytic version" by removing θ_k.
  Does $\mathbf{C}_{\mathbb{R}} = \mathcal{ODE}^{\star}(\text{LIM})$?
- Develop a general theory.

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