Characterizing Computable Analysis with Differential Equations

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Outline









Ojakian, Campagnolo Characterizing Computable Analysis

Outline



- 2 Our model and our result
- 3 Discussion of the Proof
- 4 Conclusion

Ojakian, Campagnolo Characterizing Computable Analysis

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Computability over the Reals?

Suppose $f : \mathbb{R} \to \mathbb{R}$. When is f "computable"?

Let $f(x) = \begin{cases} 0, & \text{if } x \le 0; \\ 1, & \text{if } x > 0. \end{cases}$ Is it computable?

f is computable according to the BSS Model, and not according to Computable Analysis.

Is e^x computable?

e^x is not computable according to the BSS Model, and is computable according to Computable Analysis.

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Computable Analysis

f is computable according to "Naive" Computable Analysis iff:

There is a computable function $F^{x}(n)$ with an oracle for the real number x such that $F^{x}(n) \rightarrow f(x)$, as $n \rightarrow \infty$.

Definition

 $f \in \mathbf{C}_{\mathbb{R}}$ (Computable Analysis) iff:

There is a computable function $F^{x}(n)$ with an oracle for the real number x such that $|f(x) - F^{x}(n)| \le 1/n$.

Examples of functions in $C_{\mathbb{R}}$: e^x , π , sin x, log x, ...

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Other Models

- Shannon's circuit model.
- 2 Neural Networks.
- Hybrid systems.

There is not an agreed upon definition of computability over \mathbb{R} . There is no "basic theorem" for computability over \mathbb{R} !

Our work: Showing certain models equivalent.

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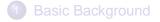
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3 Discussion of the Proof



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Function Algebras and Real Recursive Functions

Definition

A Function Algebra $FA[f_1, ..., f_k; op_1, ..., op_n]$ is the smallest set of functions containing $f_1, ..., f_k$, and closed under the operations $op_1, ..., op_n$.

Example: FA[0, 1, +, -, P; comp, \sum, \prod, μ], which is equal to the computable functions over the naturals.

Real Recursive Functions: Function algebras over the reals, introduced by C. Moore 1996.

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Some basic functions over the reals

• Constant functions: 0, 1, -1

• Projection functions "P" (example: U(x, y) = x)

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$$heta_k(x) = \left\{ egin{array}{cc} 0, & x < 0; \\ x^k, & x \ge 0. \end{array} \right.$$

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The Differential Equation Operation

Definition

ODE is the operation:

- Input: g(x), f(y, u, x).
- Output: The solution of the IVP

$$h(0,x) = g(x), \qquad \frac{\partial}{\partial y}h = f(y,h,x)$$

Definition

LI is the operation defined like ODE, except that *f* must be linear in *h*.

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A real function algebra and some examples

Definition

Let \mathcal{ODE}_k^* be the total functions of FA[0, 1, -1, θ_k , P; comp, ODE]

Some functions in ODE_k^* : e^x , (x + y), (xy), sin x, ...

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The Limit Operation

Definition

LIM is the operation:

- Input: $f(t, \bar{x})$
- **Output**: $F(\bar{x}) = \lim_{t \to \infty} f(t, \bar{x})$, if the limit exists, and $|F(\bar{x}) f(t, \bar{x})| \le 1/t$.

Definition

If $\mathcal F$ a set of functions, then $\mathcal F(\text{LIM})$ is $\mathcal F$ closed under the operation LIM.

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Our main theorem

Theorem (Main Theorem)

 $\mathbf{C}_{\mathbb{R}} = \mathcal{ODE}_{k}^{\star}(LIM)$ for $k \geq 2$.

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A function algebra based on searching

Definition

The operation UMU takes as **Input**: $f(t, \bar{x})$ such that

- For any \bar{x} , $f(t, \bar{x})$ increases in t, and
- 2 For any \bar{x} , there is a unique T such that $f(T, \bar{x}) = 0$ (and at that T, $\frac{\partial}{\partial t}f > 0$).

Output: Function $F(\bar{x})$ = the unique T such that $f(T, \bar{x}) = 0$.

Definition

Let \mathcal{UMU}_k be FA[0, 1, θ_k , P; comp, LI, UMU]

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Our mainer theorem

Theorem

For $k \geq 2$, $\mathbf{C}_{\mathbb{R}} = \mathcal{ODE}_{k}^{\star}(LIM) = \mathcal{UMU}_{k}(LIM)$.

Theorem (Bournez and Hainry 2006)

Roughly: $C^2 \cap [\mathbf{C}_{\mathbb{R}}] = [\mathcal{UMU}_k(LIM)], \text{ for } k \geq 3$

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More Motivation

Results of the form "CA = FA".

- Interesting to provide alternative models for Computable Analysis.
- Connects discrete-time and continuous-time.
- "Basic Theorem" over the reals? Church-Turing thesis over the reals?
- Could alternative models facilitate technical work? (e.g. like showing a function or operation is or is not computable)

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The "Main Step"

$\mathbf{C}_{\mathbb{R}} \subseteq \mathcal{UMU}_k(\mathsf{LIM})$

By Turing Machine simulation (Bournez and Hainry 2006).

Difficulties with this approach ...

- Forced into unnecessary restrictions.
- Appears less general.

Our approach: Approximation ...

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Approximation

Definition

•
$$f(\bar{x}) \leq t f^*(\bar{x}, t)$$
 means: $|f(\bar{x}) - f^*(\bar{x}, t)| < \frac{1}{t}$

For classes of functions A and B, A ≤ B means:
 For any f ∈ A there is f* ∈ B such that f ≤ f*.

Goal: $\mathbf{C}_{\mathbb{R}} \preceq \mathcal{UMU}_k$ Implies: $\mathbf{C}_{\mathbb{R}} \subseteq \mathcal{UMU}_k(\text{LIM})$

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Breaking up the proof

Lemma (Transitivity)

Suppose A, B, and C are classes of functions and suppose C is nice. Then $A \preceq B$ and $B \preceq C$ implies $A \preceq C$.

Definition

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Let \mathbf{C}_{\mathbb{Q}} be \{f_{|\mathbb{Q}} \mid f \in \mathbf{C}_{\mathbb{R}}\}
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$$\mathsf{C}_{\mathbb{Q}} \preceq \widetilde{\mathsf{dC}}_{\mathbb{Q}} \preceq \widetilde{\mathsf{dMU}}_{\mathbb{Q}} \preceq \mathcal{MU}_{\mathbb{Q}} \preceq \mathcal{UMU}_{k}$$

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A function algebra for the computable functions

Definition

Let $\mathcal{MU}_{\mathbb{N}}$ be FA[0, 1, +, -, P; comp, $\sum, \prod, MU].$

Definition

The operation MU. Input: $f(t, \bar{x})$ (over N) satisfying:

For each \bar{x} , there is a unique $T \ge 1$ such that $f(T, \bar{x}) = 0$, and otherwise

$$f(t,\bar{x}) = \begin{cases} -1, & \text{if } t < T; \\ 1, & \text{if } t > T. \end{cases}$$

Output: The function $g(\bar{x}) =$ the unique *T* such that $f(T, \bar{x}) = 0$.

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A function algebra over \mathbb{Q}

Definition

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Definition

Suppose OP takes a function $f : \mathbb{N}^k \to \mathbb{N}$ and returns a function $g : \mathbb{N}^m \to \mathbb{N}$. Then OP₀ is the following operation:

- OP_Q takes as input $f : \mathbb{Q}^k \to \mathbb{Q}$ such that $f_{|\mathbb{N}} : \mathbb{N}^k \to \mathbb{N}$.
- **OP**_Q then applies OP to $f_{\mathbb{N}}$ to get some $g : \mathbb{N}^m \to \mathbb{N}$.
- OP_Q outputs $Lin_Q(g)$.

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- OP_Q takes as input $f : \mathbb{Q}^k \to \mathbb{Q}$ such that $f_{|\mathbb{N}} : \mathbb{N}^k \to \mathbb{N}$.
- $\mathsf{OP}_{\mathbb{Q}}$ then applies OP to $f_{|\mathbb{N}}$ to get some $g: \mathbb{N}^m o \mathbb{N}$.
- 3 $OP_{\mathbb{Q}}$ outputs $Lin_{\mathbb{Q}}(g)$.

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A function algebra over \mathbb{Q}

Definition

Let $\mathcal{MU}_{\mathbb{Q}}$ be FA[0, 1, -1, +, P, *, div, θ_1 ; comp, $\sum_{\mathbb{Q}}, \prod_{\mathbb{Q}}, MU_{\mathbb{Q}}, Lin_{\mathbb{Q}}$].

Definition

Suppose OP takes a function $f : \mathbb{N}^k \to \mathbb{N}$ and returns a function $g : \mathbb{N}^m \to \mathbb{N}$. Then $OP_{\mathbb{Q}}$ is the following operation:

- OP_Q takes as input $f : \mathbb{Q}^k \to \mathbb{Q}$ such that $f_{|\mathbb{N}} : \mathbb{N}^k \to \mathbb{N}$.
- **2** OP_Q then applies OP to $f_{|\mathbb{N}}$ to get some $g : \mathbb{N}^m \to \mathbb{N}$.
 - OP_Q outputs $Lin_Q(g)$.

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The inductive proof

Proof proceeds inductively on the function algebra $\mathcal{MU}_{\mathbb{Q}}$:

- Show the basic functions of $\mathcal{MU}_{\mathbb{Q}}$ are approximated by \mathcal{UMU}_k .
- Show that the operations of $\mathcal{MU}_{\mathbb{Q}}$ preserve the approximation:

For $f \in \mathcal{MU}_{\mathbb{Q}}$ and $f^* \in \mathcal{UMU}_k$, suppose g = OP(f), and $f \leq f^*$. Then there is $g^* \in \mathcal{UMU}_k$ such that $g \leq g^*$.

The inductive proof

Proof proceeds inductively on the function algebra $\mathcal{MU}_{\mathbb{Q}}$:

- Show the basic functions of MU_Q are approximated by UMU_k.
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For $f \in \mathcal{MU}_{\mathbb{Q}}$ and $f^* \in \mathcal{UMU}_k$, suppose g = OP(f), and $f \leq f^*$. Then there is $g^* \in \mathcal{UMU}_k$ such that $g \leq g^*$.

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Basic Functions

- 0, 1, -1, +, P, *
- θ₁, div ...

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Composition

Generic and easy, but uses "modulus assumption" significantly

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Linearization

Generic, but involved

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Sums and Products

Specific, but uses earlier results

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Search Operation MU

Specific and significant Key Points:

- Use the particular shape.
- Need to process the approximation function.
- Can approximate linearization in well-behaved way.

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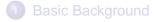
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Outline



- 2 Our model and our result
- 3 Discussion of the Proof



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Conclusion

We have a new characterization of Computable Analysis.

Further improvements?:

- Show it is useful.
- Simplify to "analytic version" by removing θ_k.
 Does C_ℝ = ODE*(LIM)?
- Develop a general theory.

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