Representations of tangles by operators

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Abstract

The objective of this work is to represent tangles by calculable functions which are invariant under ambient isotopies.

In the first chapters we focus on the case of planar tangles up to planar isotopies. We construct a category using Boolean matrices (to represent connectivity relations) and monoids in such a way as to give us a representation of planar tangles. From this we can extract a numerical invariant for embeddings of a finite collection of disjoint circles and this invariant is, up to certain choices, complete.

In the final chapters we adapt the previous construction to the study of spatial tangles. First we give a linear representation for Temperley-Lieb algebras and then we use the Kauffman bracket skein relation to give a representation for tangles by means of linear operators which generalizes the Jones polynomial. The whole construction is quite distinct from the Turaev approach. We also give a representation for singular tangles using the Kauffman-Vogel polynomial for embedded 4-valent graphs.

Keywords:

-Tangles;

-Boolean matrices;

-Monoids;

-Representations;

-Temperley-Lieb algebras;

-Polynomial link invariants.

Resumo

O objectivo deste trabalho é representar emaranhados ("tangles") por funções calculáveis que são invariantes por isotopias ambientes.

Nos primeiros capítulos é focado o caso dos "emaranhados" planares a menos de isotopia planar. É construída uma categoria usando matrizes booleanas (para representar as relações de conectividade) e monóides de maneira a dar uma representação de emaranhados planares. Dessa categoria é possível extrair um invariante numérico para mergulhos de uma colecção finita de curvas fechadas disjuntas que é invariante, sobre certas escolhas, completo.

Nos últimos capítulos é adaptada a construção anterior para estudar emaranhados espaciais (os usuais na literatura especializada). Primeiro damos uma representação linear para as álgebras de Temperley-Lieb e usamos a "skein relation" do polinómio parênteses de Kauffman para obter uma representação para emaranhados por operadores lineares que generaliza o polinómio de Jones. Toda a construção é bastante distinta da usada por Turaev. Também é dada uma representação para emaranhados com singularidades usando o polinómio de Kauffman-Vogel para grafos 4-valentes mergulhados.

Palavras-chaves:

-Emaranhados;

-Matrizes booleanas;

-Monóides;

-Representações;

-Álgebras de Temperley-Lieb;

-Invariantes polinomiais para enlaces.

Resumo alargado em português

O objectivo deste trabalho é representar emaranhados ("tangles") por funções calculáveis que são invariantes por isotopias ambientes.

No primeiro capítulo a seguir à introdução é introduzido o conceito de "emaranhado" planar a menos de isotopia planar, e com este conceito define-se uma categoria (denominada $\mathbf{PT}$) cujos morfismos são emaranhados planares. É dada então uma apresentação para tal categoria definindo os geradores e as relações de modo a estudar uma possível representação de tal categoria. É também introduzida ao de leve a categoria (denominada $\mathbf{PI}_{\mathbb{M}}$) na qual será feita a representação. Esta categoria depende da escolha de um monóide $\mathbb{M}$.

No capítulo seguinte são introduzidos os conceitos algébricos envolvidos na categoria $\mathbf{PI}_{\mathbb{M}}$ e as propriedades destes necessárias para estudar a representação que pretendemos introduzir para a categoria $\mathbf{PT}$. Tais conceitos incluem matrizes booleanas, relações de equivalência sobre a perspectiva das matrizes booleanas, monóides e reticulados.

No quarto capítulo é definida a representação da categoria $\mathbf{PT}$ na categoria $\mathbf{PI}_{\mathbb{M}}$ e são explicadas as motivações topológicas por detrás desta. Para isso, definimos os morfismos que representam os geradores da categoria $\mathbf{PT}$ (esta representação depende, para além do monóide $\mathbb{M}$, da escolha de uma função com domínio e contradomínio $\mathbb{M}$). De seguida é verificado com todo o detalhe que tais morfismos estão bem definidos. Segue-se então a demonstração de que tais morfismos satisfazem as relações da categoria $\mathbf{PT}$, de modo que fica demonstrado tratar-se de facto de uma representação (um functor da categoria $\mathbf{PT}$ para a categoria $\mathbf{PI}_{\mathbb{M}}$). Cada emaranhado é representado por uma aplicação entre pares da forma $(R, \vec{v})$ onde R é uma matriz booleana descrevendo as relações de conectividade entre regiões e $\vec{v}$ é um vector de valores em $\mathbb{M}$ capturando informação sobre regiões enclausuradas.

No quinto capítulo é aplicada a representação do capítulo anterior ao caso particular de mergulhos de curvas fechadas disjuntas no plano de modo a obter invariantes numéricos para isotopias planares. Para tal, é apresentado um algoritmo que toma uma colecção de curvas disjuntas no plano representada por uma sequência de geradores da categoria $\mathbf{PT}$, e simplifica tal sequência através das relações da categoria. Deste modo torna-se mais fácil demonstrar que, escolhendo o monóide e a função apropriada, tais invariantes numéricos podem ser completos.

No sexto capítulo é abordado o estudo de álgebras de Temperley-Lieb sob a perspectiva da teoria introduzida nos capítulos anteriores. É dada então uma representação linear das álgebras de Temperley-Lieb a partir de uma adaptação da representação da categoria **PT**. Esta representação também se estende a emaranhados planares, embora com menor informação que a dos capítulos anteriores.

No sétimo capítulo é usada a representação linear das álgebras de Temperley-Lieb para obter uma representação linear para emaranhados espaciais orientados (ou simplesmente emaranhados orientados, na literatura usual na área de investigação). Esta representação generaliza o polinómio de Jones embora de uma maneira bastante distinta da abordada por Turaev. Isto é feito decompondo um emaranhado espacial numa combinação linear finita de emaranhados planares através da "skein relation" do polinómio parênteses de Kauffman, e depois representando estes por operadores lineares como é feito no capítulo anterior.

No oitavo capítulo é usado o mesmo raciocínio do sétimo capítulo para o polinómio de Kauffman-Vogel para grafos 4-valentes mergulhados. Para tal define-se uma nova categoria cujos morfismos são emaranhados planares com um número finito de singularidades transversais. É então estendida para esta categoria a representação linear dada no capítulo seis para a categoria **PT**. No entanto para que a "skein relation" do polinómio de Kauffman-Vogel possa ser usada é necessária restringir as variáveis deste para um caso particular.

Na conclusão são apresentadas conjecturas assim como possíveis desenvolvimentos.

No final da tese estão dois apêndices. O primeiro exemplifica vários métodos de cálculo com matrizes booleanas. No segundo é dado um exemplo do cálculo da representação definida no quarto capítulo para um sistema de curvas fechadas no plano, da qual se pode, de acordo com o que foi demonstrado no quinto capítulo, extrair um invariante numérico para tal sistema.

Acknowledgment - I wish to thank all of my friends and my family who supported me in writing this thesis. A special word of thanks to my supervisor Roger Picken, for his patience and availability, and for all his useful comments. I also want to thank P. Resende and L. Kauffman for their comments. Since I didn't follow all of their suggestions, they have no responsibility for any mistakes that may be found in the thesis, for which I am solely responsible.

Contents

1 Introduction

Despite all we know about braids, links and the relations between them, it is useful to work, sometimes, with tangles, which are generalizations of both of these. One of the advantages is that we can decompose a link into a composition of simpler tangles. In fact, tangles have the structure of a category[1] with a well-known presentation. Thus if we can represent the generators by operators which satisfy the relations of the presentation, then we will get a representation for all tangles (and in particular for all links). In this way we obtain a calculable invariant for links.

This approach can be used to address another problem: how to decide whether two embeddings of a collection of disjoint circles in the plane are ambient isotopic or not? For example, the Jordan lemma tells us that a single curve in the plane divides the plane into two regions (the outside and the inside of the curve). Thus the embedding of two curves with one inside the other is different, up to planar isotopy, to the embedding of two curves with each in the outside of the other. We are going to start with the study of this problem. Our approach is to take a Morse embedding and decompose it into small planar tangles which we can represent by means of certain functions. In this way an embedding of curves will be represented by a composition of functions whose values will give us a lot of information about the embedding.

The idea that we are going to pursue comes from the fact that a planar tangle divides a planar strip into finitely many regions (connected open sets in the planar strip). Some of these regions are surrounded by others like Russian dolls, so we should associate to the outer region a value that keeps track of the information concerning such enclosed regions. When we compose with another tangle some regions may glue, others may be enveloped by other regions and new regions may appear, so we need to know how we can capture all this information at the end of the process. That is what we are going to analyze in concrete terms in the first part of this thesis.

Afterwards, we will use some ideas of the previous construction to give a linear representation for Temperley-Lieb algebras via operators in a finite dimensional linear space. This means that two words in the generators of some Temperley-Lieb algebra which are the same element by the relations of the Temperley-Lieb algebra are represented by the same linear operator which may be determined by finite calculation. Fortunately, this linear rep-

[1] even a monoidal category.

resentation also works for the linearization of the category of planar tangles (which includes the Temperley-Lieb algebras as subcategories). Thus it is possible to use the skein relation of the Kauffman-bracket polynomial to obtain a linear representation for oriented tangles which generalizes the Jones polynomial.

Finally we can use the same reasoning to obtain a linear representation for singular tangles which generalizes the Kauffman-Vogel polynomial for 4-valent graphs embedded in three-dimensional space. The idea is to generalize the representation we got for the Temperley-Lieb algebras to the linearization of the category of singular planar tangles and to use the skein relation of the Kauffman-Vogel polynomial to extend the representation to singular (spatial) tangles. However this construction only works when a specific condition holds for the variables.

Our material is organized as follows. In chapter 2 we introduce the concept of planar tangle up to planar isotopy, and with this concept we define a category (called $\mathbf{PT}$) whose morphisms are planar tangles. Next we give a presentation for such category defining its generators and relations so as to be able to study possible representations for this category. We also introduce succinctly the category (called $\mathbf{PI}_{\mathbb{M}}$) where the representation will be done. This category depends on the choice of a monoid $\mathbb{M}$.

In the following chapter we talk about the algebraic concepts involved in the category $\mathbf{PI}_{\mathbb{M}}$ and its properties needed for the study of the representation that we want to introduce for the category $\mathbf{PT}$. Such concepts include Boolean matrices, equivalence relations from the perspective of Boolean matrix theory, monoids and lattices.

In chapter 4 we define a representation of the category $\mathbf{PT}$ in the category $\mathbf{PI}_{\mathbb{M}}$ and we explain the topological motivations for it. To do that, we define the morphisms which represent the generators of $\mathbf{PT}$ (this representation depends, not only on the monoid $\mathbb{M}$, but also on the choice of a function with domain and range in $\mathbb{M}$). Next we verify in detail that such morphisms are well-defined. Then follows the proof that such morphisms satisfy the relations of the category $\mathbf{PT}$, and therefore that the representation is well-defined. Each tangle is represented by a map between pairs of the form $(R, \vec{v})$ where R is a Boolean matrix describing connectivity relations between regions and $\vec{v}$ is an array of values in $\mathbb{M}$ capturing information about enclosed regions.

In chapter 5 we apply the representation of the previous chapter to the particular case of the embedding of disjoint closed curves in the plane so as to obtain numerical invariants for planar isotopies. To do that, we present an algorithm which simplifies an embedding of disjoint closed curves in the plane using the relations of the category **PT**. This makes it easier to show that, by choosing an appropriate monoid and function, such numerical invariants can be complete.

In chapter 6 we study the Temperley-Lieb algebras from the perspective of the theory introduced in the previous chapters. We give a linear representation for the Temperley-Lieb algebras from an adaptation of the representation previously presented for the category **PT**. This linear representation also extends to planar tangles, although with less information than the original one.

In chapter 7 we use the linear representation for the Temperley-Lieb algebras to obtain a linear representation for oriented tangles. This representation generalizes the Jones polynomial, though in a very different way to the approach adopted by Turaev [10]. The way we do this is by decomposing a tangle into a linear combination of planar tangles using the skein relation of the Kauffman bracket polynomial, and then representing these by linear operators like in the previous chapter. In the end we multiply by a factor depending on the writhe of the tangle so as to satisfy the relation associated to the first Reidemeister move.

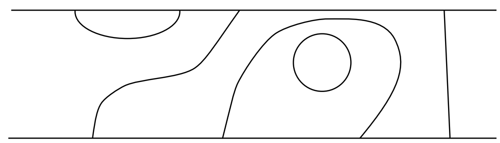
In chapter 8 we use the same reasoning as in chapter 7 for the Kauffman-Vogel polynomial for 4-valent embedded graphs. We define a new category whose morphisms are planar tangles with a finite number of singular crossings (4-valent vertices). Then we extend to this category the linear representation given in chapter 6 for the category **PT**. Finally we use the skein relation of the Kauffman-Vogel polynomial (with a certain restriction on its variables for this to work) to decompose a tangle (which may be singular, with 4-valent vertices) into a linear combination of singular planar tangles.

In the conclusions we present open problems as well as possible developments.

At the end of the thesis there are two appendices. In the first we give examples of some methods of calculation with Boolean matrices. In the second we give an example of the calculation of the representation defined in chapter 4 for a system of non-singular planar curves.

2 The category of non-singular planar tangles

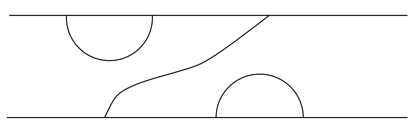
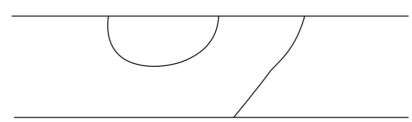
Let **PT** be the *category of non-singular planar tangles* whose objects are finite sets of points in the real line identified up to 1-dimensional isotopies[2], and whose morphisms between two objects O_1 and O_2 are piecewise regular 1-dimensional manifolds, with boundary $O_1 \times \{1\} \cup O_2 \times \{0\}$, embedded in $\mathbb{R} \times [0,1]$ identified up to planar isotopies:

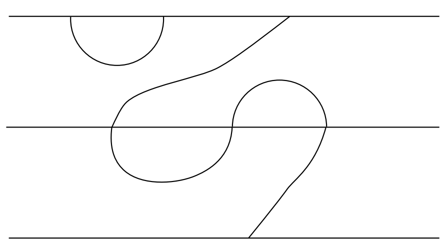
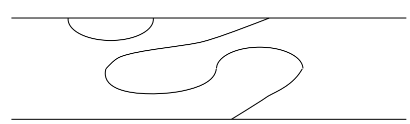


The composition of two morphisms t_1 and t_2 is defined by

$$t_2 \circ t_1 := g(f(t_1) \cup t_2)$$

where $f(x,y) = (x, y+1)$ and $g(x,y) = (x, y/2)$.

$t_1 =$  $t_2 =$ 

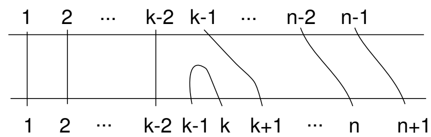
 $\longrightarrow t_2 \circ t_1 =$ 

In this paper, the downward direction composition is used, some authors use the opposite direction.

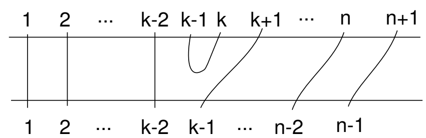
This category has the following presentation:

[2]these objects can be regarded as finite ordinal numbers $\emptyset, \{0\}, \{0,1\}, \ldots$

The generators are morphisms $\hat{t}_{n,k} \in \hom(\{1,...,n-1\},\{1,...,n+1\})$ connecting $(i,1)$ to $(i,0)$ if $i \leq k-2$, $(i,1)$ to $(i+2,0)$ if $i \geq k-1$ and $(k-1,0)$ to $(k,0)$:

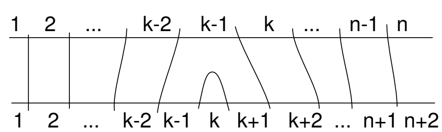
$\hat{t}_{n,k} =$ 

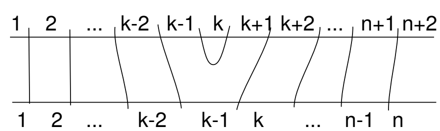
and $\check{t}_{n,k} \in \hom(\{1,...,n+1\},\{1,...,n-1\})$ connecting $(i,1)$ to $(i,0)$ if $i \leq k-2$, $(i+2,1)$ to $(i,0)$ if $i \geq k-1$ and $(k-1,1)$ to $(k,1)$:

$\check{t}_{n,k} =$ 

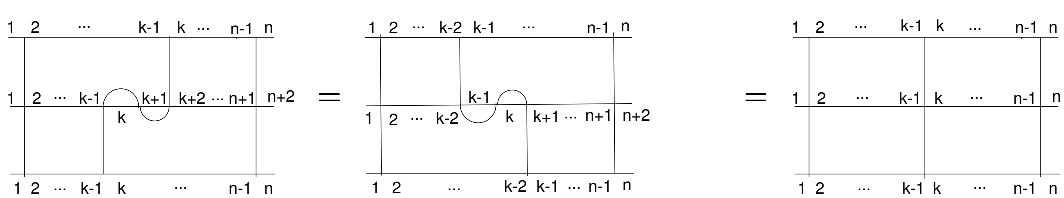
for any $k,n \in \mathbb{N}$ with $2 \leq k \leq n+1$.

For the rest of this paper it is better to number the intervals instead of the points:

$\hat{t}_{n,k} =$ 

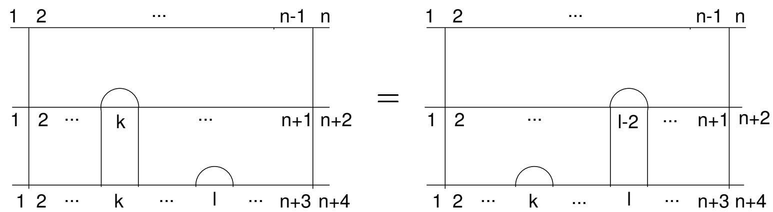
$\check{t}_{n,k} =$ 

These generators satisfy the following relations:



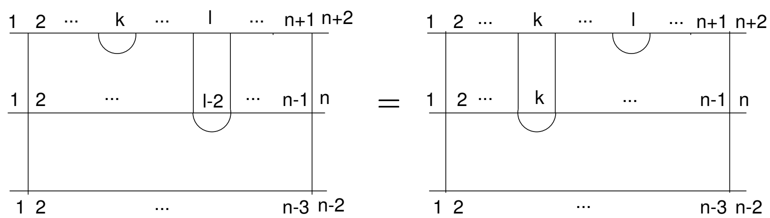
$$\check{t}_{n,k+1} \circ \hat{t}_{n,k} = \check{t}_{n,k-1} \circ \hat{t}_{n,k} = id_n$$

where $id_n := \{1, ..., n\} \times [0, 1]$ is the identity morphism on $\{1, ..., n\}$;



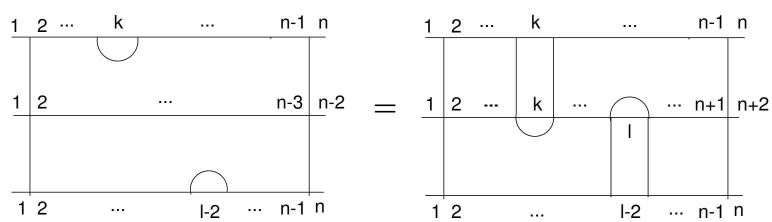
$$\hat{t}_{n+2,l} \circ \hat{t}_{n,k} = \hat{t}_{n+2,k} \circ \hat{t}_{n,l-2}$$

for $l \geq k + 2$;



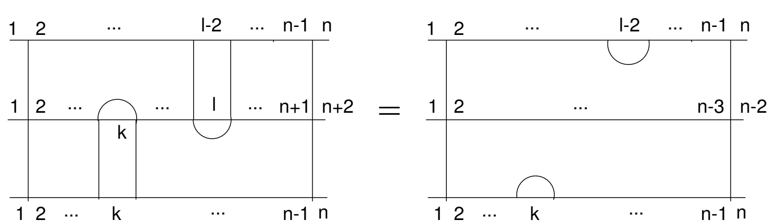
$$\check{t}_{n-2,l-2} \circ \check{t}_{n,k} = \check{t}_{n-2,k} \circ \check{t}_{n,l}$$

for $l \geq k + 2$;



$$\hat{t}_{n-2,l-2} \circ \check{t}_{n-2,k} = \check{t}_{n,k} \circ \hat{t}_{n,l}$$

for $l \geq k + 2$ and



$$\check{t}_{n,l} \circ \hat{t}_{n,k} = \hat{t}_{n-2,k} \circ \check{t}_{n-2,l-2}$$

for $l \geq k+2$.

Now we want to represent non-singular planar tangles by functions. That is, we want to find functions $\hat{T}_{n,k}$ and $\check{T}_{n,k}$ satisfying the following relations:

$$\check{T}_{n,k+1} \circ \hat{T}_{n,k} = \check{T}_{n,k-1} \circ \hat{T}_{n,k} = id$$

$$\hat{T}_{n+2,l} \circ \hat{T}_{n,k} = \hat{T}_{n+2,k} \circ \hat{T}_{n,l-2}$$

for $l \geq k+2$;

$$\check{T}_{n-2,l-2} \circ \check{T}_{n,k} = \check{T}_{n-2,k} \circ \check{T}_{n,l}$$

for $l \geq k+2$;

$$\hat{T}_{n-2,l-2} \circ \check{T}_{n-2,k} = \check{T}_{n,k} \circ \hat{T}_{n,l}$$

for $l \geq k+2$ and

$$\check{T}_{n,l} \circ \hat{T}_{n,k} = \hat{T}_{n-2,k} \circ \check{T}_{n-2,l-2}$$

for $l \geq k+2$.

Our proposal is to represent **PT** by the following category $\mathbf{PI}_{\mathbb{M}}$ whose objects are $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, ...$ where $\mathcal{O}_n$ is the set of pairs $(R, \overrightarrow{v})$ such that $R = [r_{i,j}]$ is a $n \times n$ Boolean matrix satisfying the following properties:

E1. $R \geq I$ (where I is the identity matrix);

E2. $R^t = R$ (the matrix is symmetric);

E3. $R^2 = R$ (the matrix is idempotent);

T1. If $r_{i,j} = 1$ then $|i-j|$ is even;

T2. For any $\alpha \leq \beta \leq \gamma \leq \delta$, $r_{\alpha,\gamma} r_{\beta,\delta} \leq r_{\alpha,\beta} r_{\beta,\gamma} r_{\gamma,\delta}$;

T3. For any $\alpha < \beta$ if $r_{\alpha,\beta} = 1$ then either $r_{\alpha+1,\beta-1} = 1$ or there exists γ between α and β such that $r_{\alpha,\gamma} = 1$.

and $\overrightarrow{v}$ is an array of n entries with values in a chosen lattice ordered monoid $\mathbb{M}$ such that it is fixed by the action induced by R which we will define later:

EC. $R * \overrightarrow{v} = \overrightarrow{v}$.

Note: The properties E1, E2 and E3 represent an equivalence relation, and the properties T1, T2 and T3 have topological motivations (see the explanation in section 4.1).

A morphism between $\mathcal{O}_m$ and $\mathcal{O}_n$ is just a set function between the sets $\mathcal{O}_m$ and $\mathcal{O}_n$.

3 Algebraic interlude

Definition 1 *The canonical Boolean algebra $\mathcal{B}$ is the set $\{0,1\}$ with two binary operations: the sum $+$ and the multiplication $\cdot$, and a unary operation the negation $\neg$ such that $(\{0,1\},+,\cdot)$ is the (unique) semi-ring with $1+1=1$, $\neg 0=1$ and $\neg 1=0$.*

Definition 2 *A Boolean matrix is a matrix with values in the canonical Boolean algebra.*

We define the operations sum, multiplication and transpose in the same way as on real matrices:

$$\text{Sum: } [a_{i,j}]+[b_{i,j}] := [c_{i,j}] \text{ where } c_{i,j}=a_{i,j}+b_{i,j}$$

$$\text{Multiplication: } [a_{i,j}][b_{i,j}] := [c_{i,j}] \text{ where } c_{i,j}=\sum_{k=1}^{n} a_{i,k}b_{k,j}$$

$$\text{Transpose: } [a_{i,j}]^t := [a_{j,i}]$$

There is a natural partial order relation on these matrices given in the following way:

$$[a_{i,j}] \leq [b_{i,j}] \text{ iff } a_{i,j} \leq b_{i,j} \quad \forall_{i,j}$$

These matrices have many of the properties of real matrices.

Proposition 1 *Let A, B and C be Boolean matrices with appropriate dimensions. We have:*

1. *(commutativity of the sum) $A+B=B+A$;*

2. *(associativity) $(A+B)+C=A+(B+C)$ and $(AB)C=A(BC)$;*

3. *(distributivity) $A(B+C)=AB+AC$ and $(A+B)C=AC+BC$;*

4. *(existence of the zero matrix) $A+O=A$, $AO=O$ and $OA=O$ where O is the matrix with all entries equal to zero;*

5. *(existence of the identity matrix) $AI=A$ and $IA=A$ where $I=[\delta_{i,j}]$ with $\delta_{i,j}=1 \Leftrightarrow i=j$;*

7. *(idempotency of the sum) $A+A=A$;*

8. *$A \le B \Leftrightarrow A + B = B$;*

9. *$(AB)^t = B^t A^t$ and $(A+B)^t = A^t + B^t$;*

10. *$A \le B \Rightarrow A + C \le B + C$, $CA \le CB$, $AC \le BC$ and $A^t \le B^t$;*

11. *$A \ge B$ and $A \ge C \Rightarrow A \ge B + C$.*

We can regard a square Boolean matrix $R = [r_{i,j}]$ of dimension n as a binary relation $\sim_R$ on the set $\{1, ..., n\}$:

$$i \sim_R j \Leftrightarrow r_{i,j} = 1$$

Then:

Proposition 2 *The binary relation $\sim_R$ represented by the matrix R is:*

i. *reflexive iff $I \le R$;*

ii. *symmetric iff $R^t = R$;*

iii. *transitive iff $R^2 \le R$.*

Thus we can transpose the notions of reflexivity, symmetry and transitivity from the binary relations to square Boolean matrices. Notice that reflexivity and transitivity imply idempotency of the product for Boolean matrices.

Proposition 3 *(definition) Let A be a square Boolean matrix and let*

$$\overline{A} := \sum_{n=1}^{\infty} A^n = A + A^2 + ...$$

Then:

i. *$\overline{A}$ is transitive;*

ii. *For any transitive matrix B, $A \le B \Rightarrow A \le \overline{A} \le B$;*

iii. *$A \le B \Rightarrow \overline{A} \le \overline{B}$;*

iv. *$\overline{\overline{A}} = \overline{A}$;*

v. *If $A \geq I$ then $\overline{A} = A^n$ for some natural number n.*[3]

$\overline{A}$ is called the transitive closure of A.

Next we will define a *lattice ordered additive monoid* to be a commutative monoid $(\mathbb{M}, \oplus, \emptyset)$, where $\oplus$ is the binary operation of the monoid and $\emptyset$ is the zero element, with a partial order relation $\leq$ such that $(\mathbb{M}, \leq)$ is a distributive lattice with minimum $\emptyset$ and where the sum $\oplus$ is distributive over the operations meet $\wedge$ and join $\vee$. Formally, it is a set $\mathbb{M}$ with three binary operations $\oplus$, $\vee$ and $\wedge$ and an element $\emptyset$ such that for any $a, b, c \in \mathbb{M}$:

M. $(\mathbb{M}, \oplus)$ is a commutative monoid:

- **M1.** (commutativity) $a \oplus b = b \oplus a$;
- **M2.** (associativity) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
- **M3.** (existence of the zero element) $\emptyset \oplus a = a$;

L. $(\mathbb{M}, \vee, \wedge)$ is a distributive lattice:

- **L1.** (idempotency) $a \vee a = a$ and $a \wedge a = a$;
- **L2.** (commutativity) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$;
- **L3.** (associativity) $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;
- **L4.** (absorption) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$;
- **L5.** (distributivity) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;

C. The lattice and monoid structures of $\mathbb{M}$ are compatible by the following axioms:

- **C1.** $\emptyset \vee a = a$ and $\emptyset \wedge a = \emptyset$;
- **C2.** $a \oplus (b \vee c) = (a \oplus b) \vee (a \oplus c)$ and $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$.

Remember that by definition $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$. Also we have $a \leq b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a$. Using the axioms of such monoids[4] we have the following properties:

[3] This is not true for matrices with infinite dimension.

[4] In this paper we will only consider monoids of this type and will refer to them simply as monoids.

P1. $a \leq a \oplus b$;

P2. $(a \vee b) \oplus (a \wedge b) = a \oplus b$.

The property P1 is very easy to prove and the proof of P2 follows from the following inequalities:

$$\begin{array}{rcl}(a \vee b) \oplus (a \wedge b) & = & [a \oplus (a \wedge b)] \vee [b \oplus (a \wedge b)] \\ & = & [(a \oplus a) \wedge (a \oplus b))] \vee [(b \oplus a) \wedge (b \oplus b)] \\ & \leq & (a \oplus b) \vee (b \oplus a) \\ & = & a \oplus b\end{array}$$

$$\begin{array}{rcl}(a \vee b) \oplus (a \wedge b) & = & [(a \vee b) \oplus a] \wedge [(a \vee b) \oplus b] \\ & = & [(a \oplus a) \vee (b \oplus a))] \wedge [(a \oplus b) \vee (b \oplus b)] \\ & \geq & (b \oplus a) \wedge (a \oplus b) \\ & = & a \oplus b\end{array}$$

Examples:

1. $\mathbb{M} := \mathbb{N}_0 = \{0, 1, 2, ...\}$, $\emptyset := 0$, $a \oplus b := a + b$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$;

2. $\mathbb{M} := \mathbb{N}_1 = \{1, 2, ...\}$, $\emptyset := 1$, $a \oplus b := ab$, $a \vee b := l.c.m.\{a, b\}$ and $a \wedge b := g.c.d.\{a, b\}$;

3. $\mathbb{M}$ a distributive lattice with minimum and $\oplus := \vee$.

Now we consider the following action of the canonical Boolean algebra $\mathcal{B}$ on a monoid $\mathbb{M}$:

$$\mathcal{B} \times \mathbb{M} \longrightarrow \mathbb{M}$$

$$(v, m) \longmapsto v * m$$

where

$$v * m := \begin{cases} m \text{ if } v = 1 \\ \emptyset \text{ if } v = 0 \end{cases}$$

Then we have:

i. $(v_1 v_2) * m = v_1 * (v_2 * m)\ \forall v_1, v_2 \in \mathcal{B}; m \in \mathbb{M}$;

ii. $(v_1 + v_2) * m = (v_1 * m) \vee (v_2 * m)\ \forall v_1, v_2 \in \mathcal{B}; m \in \mathbb{M};$

iii. $v * (m_1 \vee m_2) = (v * m_1) \vee (v * m_2)\ \forall v \in \mathcal{B}; m_1, m_2 \in \mathbb{M};$

iv. $v * (m_1 \oplus m_2) = (v * m_1) \oplus (v * m_2)\ \forall v \in \mathcal{B}; m_1, m_2 \in \mathbb{M}.$

Now we can define an action of Boolean matrices on arrays with values in the monoid $\mathbb{M}$.

Definition 3 *Let $[v_{i,j}]_{m\times n}$ be a Boolean matrix and $(a_j)_{j=1,\ldots,n}$ be an array in $\mathbb{M}^n$. We define*

$$[v_{i,j}] * (a_j) := (b_i)_{i=1,\ldots,m} \text{ where } b_i = \bigvee_{j=1}^{n} v_{i,j} * a_j$$

Proposition 4 *For any Boolean matrices A and B and any arrays $\overrightarrow{x}$ and $\overrightarrow{y}$ with values in $\mathbb{M}$, we have:*

1. $(AB) * \overrightarrow{x} = A * (B * \overrightarrow{x});$

2. $(A + B) * \overrightarrow{x} = (A * \overrightarrow{x}) \vee (B * \overrightarrow{x});$

3. $I * \overrightarrow{x} = \overrightarrow{x}$ *and* $O * \overrightarrow{x} = \overrightarrow{\emptyset};$

4. $A * (\overrightarrow{x} \vee \overrightarrow{y}) = (A * \overrightarrow{x}) \vee (A * \overrightarrow{y});$

5. $A * (\overrightarrow{x} \oplus \overrightarrow{y}) \leq (A * \overrightarrow{x}) \oplus (A * \overrightarrow{y}).$

where I is the identity matrix, O is the zero matrix, the operators $\vee$ and $\oplus$ are defined coordinate by coordinate in $\mathbb{M}^n$ and $\overrightarrow{\emptyset} = (\emptyset, ..., \emptyset)$.

4 Representation of the category PT on $\mathbf{PI}_{\mathbb{M}}$

To each object of **PT** with cardinality n we associate the object $\mathcal{O}_{n+1}$ of $\mathbf{PI}_{\mathbb{M}}$.

The motivation is the following. An object O of **PT** gives a decomposition of the real line into intervals, and each planar tangle that ends on O decomposes the strip $\mathbb{R} \times [0,1]$ into regions whose boundaries contain these intervals. Ordering the intervals in the natural way we will store in a Boolean matrix the information about which intervals are in the same region, that is, the intervals i and j are in the same region if and only if the (i,j) entry of the Boolean matrix is 1. Also to each interval we associate a value (in the given monoid $\mathbb{M}$) which is specific for the region to which the interval belongs. Thus in this way, intervals in the same region have the same value and therefore the array of values is fixed by the action of the matrix.

This should make clear the reason for the properties that the matrices in $\mathcal{O}_n$ have to satisfy. Indeed, the author conjectures that any matrix with the properties E1, E2, E3, T1, T2 and T3 has a geometric realization in this form.

To obtain a functor from the category **PT** to the category $\mathbf{PI}_{\mathbb{M}}$ we need to associate to each elementary tangle $\hat{t}_{n,k}$ and $\check{t}_{n,k}$ functions $\hat{T}_{n,k} : \mathcal{O}_n \longrightarrow \mathcal{O}_{n+2}$ and $\check{T}_{n,k} : \mathcal{O}_{n+2} \longrightarrow \mathcal{O}_n$ that satisfy the same relations as $\hat{t}_{n,k}$ and $\check{t}_{n,k}$.

We want these functions to preserve the motivation for the definition of $\mathcal{O}_n$. Specifically if $(R, \overrightarrow{v})$ is an element of $\mathcal{O}_n$, and R is the matrix of connectivity of the intervals for a specific tangle that ends on the object associated with $\mathcal{O}_n$ then the image $(R', \overrightarrow{v}')$ of $(R, \overrightarrow{v})$ by $\hat{T}_{n,k}$ (or $\check{T}_{n,k}$) has R' as the matrix of connectivity of the intervals which terminate the composition of the tangle $\hat{t}_{n,k}$ (or $\check{t}_{n-2,k}$) with the specific tangle. Furthermore, if $\overrightarrow{v}$ gives the values assigned to the intervals, then $\overrightarrow{v}'$ gives the values assigned to the intervals after composition with the tangle $\hat{t}_{n,k}$ (or $\check{t}_{n-2,k}$).

We will define $\hat{T}_{n,k}$ and $\check{T}_{n,k}$ as follows

$$\hat{T}_{n,k}(R, \overrightarrow{v}) = (R', \overrightarrow{v}')$$

where

$$R' = B_{n,k} R B_{n,k}^t + D_{n+2,k}$$

and

$$\overrightarrow{v}' = B_{n,k} * \overrightarrow{v}$$

$B_{n,k}$ is a Boolean matrix with $n+2$ rows and n columns defined by

$$B_{n,k} := [b_{i,j}] \text{ with } b_{i,j} = 1 \text{ iff } i = j < k \text{ or } i = j+2 > k.$$

$D_{n,k}$ is the diagonal square Boolean matrix of dimension n defined by

$$D_{n,k} := [d_{i,j}] \text{ with } d_{i,j} = 1 \text{ iff } i = j = k.$$

We can regard the matrix $B_{n,k}$ as the connectivity relation between the upper and lower intervals of the tangle $\hat{t}_{n,k}$, that is, $b_{i,j} = 1$ iff the upper interval j and the lower interval i are in the same region for the tangle $\hat{t}_{n,k}$ (or equivalently, iff the upper interval i and the lower interval j are in the same region for the tangle $\check{t}_{n,k}$).

In this sense the formula $R' = B_{n,k} R B^t_{n,k} + D_{n+2,k}$ means that two distinct intervals i and j are in the same region after the composition with $\hat{t}_{n,k}$ if $i \neq k$ and $j \neq k$ and the intervals $\hat{k}(i)$ and $\hat{k}(j)$ ($\hat{k}(i) = i$ if $i < k$, $\hat{k}(i) = i+2$ if $i > k$) are in the same region before the composition by $\hat{t}_{n,k}$. In other words two intervals which not k are in the same region if the respective intervals above them in the tangle $\hat{t}_{n,k}$ are in the same region before the composition with $\hat{t}_{n,k}$.

The formula $\overrightarrow{v}' = B_{n,k} * \overrightarrow{v}$ means that the extended regions (after the composition with the tangle $\hat{t}_{n,k}$) preserve the old values and the new region created over the interval k receives the value $\emptyset$.

Now we define $\check{T}_{n,k}$:

$$\check{T}_{n,k}(R, \overrightarrow{v}) = (R', \overrightarrow{v}')$$

where

$$R' = (B^t_{n,k} R B_{n,k})^2$$

and

$$\overrightarrow{v}' = R' * [(B^t_{n,k} * \overrightarrow{v}) \oplus (e_{n,k-1} * x_k)]$$

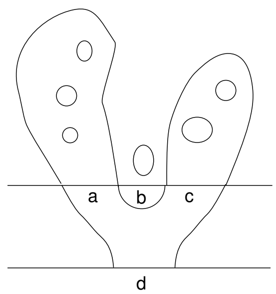
where $e_{n,k-1}$ is a 1-column Boolean matrix of dimension n defined by $e_{n,k-1} := [\epsilon_i]$ with $\epsilon_i = 1$ iff $i = k-1$ and x_k is a monoid value which depends on R and $\overrightarrow{v}$ (despite this, we normally use the symbol x_k instead of $x_k(R, \overrightarrow{v})$ to simplify the notation), given by the following formula:

$$\begin{aligned} x_k &= [r_{k-1,k+1} * \varphi(v_k)] \oplus [(\neg r_{k-1,k+1}) * (v_{k-1} \wedge v_{k+1})] \\ &= \begin{cases} \varphi(v_k) & \text{if} \quad r_{k-1,k+1} = 1 \\ v_{k-1} \wedge v_{k+1} & \text{if} \quad r_{k-1,k+1} = 0 \end{cases} \end{aligned}$$

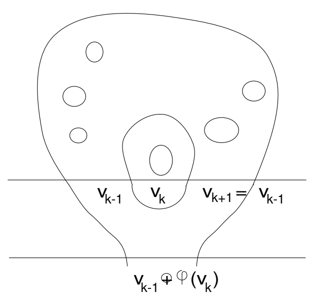
where $r_{k-1,k+1} = e^t_{n,k-1} R e_{n,k+1}$ (the $(k-1, k+1)$ entry of R) and $\varphi : \mathbb{M} \longrightarrow \mathbb{M}$ is a fixed function (without structure) independent[5] of $(R, \overrightarrow{v})$.

The idea behind the formula $R' = (B^t_{n,k} R B_{n,k})^2$ is the same as before. The matrix $B^t_{n,k} R B_{n,k}$ transfers the relation between two intervals of "belonging to the same region" from the top of the tangle $\check{t}_{n,k}$ to the bottom, and we need to take the square power because the matrix $B^t_{n,k} R B_{n,k}$ may not be transitive, since $\check{t}_{n,k}$ joins the regions associated to the intervals $k-1$ and $k+1$ (which may or may not be the same).

The formula $\overrightarrow{v}' = R' * [(B^t_{n,k} * \overrightarrow{v}) \oplus (e_{n,k-1} * x_k)]$ plays a crucial rule in the construction and needs a more careful explanation. What it says is that the interval $k-1$ receives the values of the old intervals $k-1$ and $k+1$ and if these intervals are in distinct regions then we sum them by the operation $\oplus$ (since $(v_{k-1} \vee v_{k+1}) \oplus (v_{k-1} \wedge v_{k+1}) = (v_{k-1} \oplus v_{k+1})$ by P2).



If they are in the same region then we take their common value and sum to it some modification (given by the function φ) of the value of the interval k corresponding to a region which is closed after the composition with $\check{t}_{n,k}$.



The other intervals receive their former value if they are not in the region associated to the interval $k-1$ or receive the new value of the interval $k-1$

[5]The representation depends on the choice of the function φ i.e. a different function φ gives a different representation.

if they are in the same region as that interval. This is why we take the action of the matrix R' on the array $(B^t_{n,k} * \overrightarrow{v}) \oplus (e_{n,k-1} * x_k)$ so as to transfer the value of the interval $k-1$ to others connected with it.

A better way of thinking about this may be that the array of values describes the histories of the regions associated to each interval, which keep track of the histories of any closed region inside them by means of the function φ. The Boolean matrix essentially plays the role of an assistant storing the information about which intervals are in the same region. For example, in the case of closed planar curves, which are morphisms from the empty set to itself, we get in the end a one-dimensional square matrix (which is unique by the condition E1) and a one-dimensional array (or simply a monoid value). So in this case the Boolean matrix doesn't matter at all and the only significant content is the monoid value.

See appendix B for an example of how to calculate the representation for an embedding of disjoint circles.

So as to simplify the notation we will always substitute $\hat{t}_{n,k}$, $\check{t}_{n,k}$, $\hat{T}_{n,k}$, $\check{T}_{n,k}$, $B_{n,k}$, $D_{n,k}$ and $e_{n,k}$ by $\hat{t}_k$, $\check{t}_k$, $\hat{T}_k$, $\check{T}_k$, B_k, D_k and e_k when n is implicit.

4.1 The well-definedness of the functions $\hat{T}_{n,k}$ and $\check{T}_{n,k}$

$$\begin{array}{rccc} \hat{T}_{n,k}: & \mathcal{O}_n & \longrightarrow & \mathcal{O}_{n+2} \\ & (R, \overrightarrow{v}) & \longmapsto & (R', \overrightarrow{v}') \end{array}$$

with $R' = B_k R B^t_k + D_k$ and $\overrightarrow{v}' = B_k * \overrightarrow{v}$.

We need to check that if $R = [r_{i,j}]$ is an n-dimensional matrix that satisfies the conditions:

E1. $R \geq I$;

E2. $R^t = R$;

E3. $R^2 = R$;

T1. If $r_{i,j} = 1$ then $|i - j|$ is even;

T2. For any $\alpha \leq \beta \leq \gamma \leq \delta$, $r_{\alpha,\gamma} r_{\beta,\delta} \leq r_{\alpha,\beta} r_{\beta,\gamma} r_{\gamma,\delta}$;

T3. For any $\alpha < \beta$ if $r_{\alpha,\beta} = 1$ then $r_{\alpha+1,\beta-1} = 1$ or there exists γ between α and β such that $r_{\alpha,\gamma} = 1$.

then the matrix $R' = B_k R B_k^t + D_k$ is an $n+2$-dimensional matrix that satisfies the same conditions.

Also we need to prove that if $\overrightarrow{v}$ is fixed by the action of R:

EC. $R * \overrightarrow{v} = \overrightarrow{v}$

then $\overrightarrow{v}' = B_k * \overrightarrow{v}$ is likewise fixed by the action of R'.

First we will check that R' satisfies the conditions E1, E2, E3, T1, T2 and T3.

E1:

$$R \geq I \Rightarrow R' = B_k R B_k^t + D_k \geq B_k I B_k^t + D_k \geq (I - D_k) + D_k = I$$

The operation *minus* on Boolean matrices is defined in the following way:

$[a_{i,j}] - [b_{i,j}] = [c_{i,j}]$ where $c_{i,j} = 1$ iff $a_{i,j} > b_{i,j}$(i.e. $a_{i,j} = 1$ and $b_{i,j} = 0$)

It easy to see that, for any matrices A and B, $(A - B) + B \geq A$ and $A \geq B \Rightarrow (A - B) + B = A$. To check the inequality $B_k B_k^t \geq I - D_k$ see appendix A.

E2:

$$R = R^t \Rightarrow R'^t = (B_k R B_k^t + D_k)^t = B_k R^t B_k^t + D_k^t = B_k R B_k^t + D_k = R'$$

E3:

$$\begin{aligned} R^2 = R \Rightarrow R'^2 &= (B_k R B_k^t + D_k)^2 \\ &= B_k R B_k^t B_k R B_k^t + B_k R B_k^t D_k + D_k B_k R B_k^t + D_k^2 \\ &= B_k R I R B_k^t + B_k R O + O R B_k + D_k \\ &= B_k R B_k^t + D_k \\ &= R' \end{aligned}$$

We leave it to the reader to check that $B_k^t B_k = I$, $B_k^t D_k = O$ and $D_k B_k = O$ (or see appendix A for the first two equations).

T1: The condition

$$r_{i,j} = 1 \Rightarrow i - j \in 2\mathbb{Z}$$

is equivalent to the condition

$$R \leq C_{n\times n}$$

where $C_{m\times n}$ is the *chess board matrix of dimension* $m \times n$ defined in the following way:

$$C_{m\times n} = [c_{i,j}]_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} \text{ with } c_{i,j} = 1 \text{ iff } i - j \in 2\mathbb{Z}$$

It is easy to see that $C_{l\times m}C_{m\times n} \leq C_{l\times n}$ (see appendix A)(in fact, this is an equality unless $m = 1$), and also we have $B_{n,k} \leq C_{(n+2)\times n}$ and $D_{n,k} \leq C_{(n+2)\times(n+2)}$. Thus $R \leq C_{n\times n} \Rightarrow R' = B_k R B_k^t + D_k \leq C_{(n+2)\times n}C_{n\times n}C_{n\times(n+2)} + C_{(n+2)\times(n+2)} \leq C_{(n+2)\times(n+2)}$.

T2: Let $[r_{i,j}] = R$ and $[r'_{i,j}] = R' = B_k R B_k^t + D_k$.

Suppose that:

Hypothesis: $\forall_{\alpha\leq\beta\leq\gamma\leq\delta} \qquad r_{\alpha,\gamma}r_{\beta,\delta} \leq r_{\alpha,\beta}r_{\beta,\gamma}r_{\gamma,\delta}$.

We want to prove that:

Thesis: $\forall_{\alpha\leq\beta\leq\gamma\leq\delta} \qquad r'_{\alpha,\gamma}r'_{\beta,\delta} \leq r'_{\alpha,\beta}r'_{\beta,\gamma}r'_{\gamma,\delta}$.

In the case $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \delta$ this assertion is true if $[r'_{i,j}] = R'$ satisfies the conditions of an equivalence relation (E1, E2 and E3):

(1) $r'_{i,i} = 1$ for any i;

(2) $r'_{i,j} = r'_{j,i}$ for any i and j;

(3) $r'_{i,j}r'_{j,k} \leq r'_{i,k}$ for any i, j and k.

which we have already seen to be true.

In fact, if $\alpha = \beta$ we have

$$r'_{\alpha,\gamma}r'_{\beta,\delta} = r'_{\alpha,\gamma}r'_{\alpha,\delta} = r'_{\gamma,\alpha}r'_{\alpha,\delta} \leq r'_{\gamma,\delta}$$

and

$$r'_{\alpha,\gamma}r'_{\beta,\delta} \leq r'_{\alpha,\gamma} = r'_{\beta,\gamma}$$

and

$$r'_{\alpha,\gamma}r'_{\beta,\delta} \leq 1 = r'_{\alpha,\alpha} = r'_{\alpha,\beta}$$

thus

$$r'_{\alpha,\gamma}r'_{\beta,\delta} \leq r'_{\alpha,\beta}r'_{\beta,\gamma}r'_{\gamma,\delta}$$

if $\beta = \gamma$ we have

$$r'_{\alpha,\gamma}r'_{\beta,\delta} = r'_{\alpha,\beta}r'_{\beta,\delta} = r'_{\alpha,\beta}r'_{\beta,\beta}r'_{\beta,\delta} = r'_{\alpha,\beta}r'_{\beta,\gamma}r'_{\gamma,\delta}$$

and if $\gamma = \delta$ we have

$$r'_{\alpha,\gamma}r'_{\beta,\delta} = r'_{\alpha,\gamma}r'_{\beta,\gamma} = r'_{\alpha,\gamma}r'_{\gamma,\beta} \leq r'_{\alpha,\beta}$$

and

$$r'_{\alpha,\gamma}r'_{\beta,\delta} \leq r'_{\beta,\delta} = r'_{\beta,\gamma}$$

and

$$r'_{\alpha,\gamma}r'_{\beta,\delta} \leq 1 = r'_{\gamma,\gamma} = r'_{\gamma,\delta}$$

thus

$$r'_{\alpha,\gamma}r'_{\beta,\delta} \leq r'_{\alpha,\beta}r'_{\beta,\gamma}r'_{\gamma,\delta}$$

So we are left with the case $\alpha < \beta < \gamma < \delta$. It easy to see that

$$r'_{i,j} = \begin{cases} r_{\hat{k}(i),\hat{k}(j)} & \text{if} \quad i,j \neq k \\ \delta_{k,j} & \text{if} \quad i = k \\ \delta_{i,k} & \text{if} \quad j = k \end{cases}$$

$$\text{where } \hat{k}(i) = \begin{cases} i & \text{if} \quad i < k \\ i-2 & \text{if} \quad i > k \end{cases} \quad \text{and } \delta_{i,k} = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases}$$

if $k \in \{\alpha, \beta, \gamma, \delta\}$ (with $\alpha < \beta < \gamma < \delta$) then

$$r'_{\alpha,\gamma}r'_{\beta,\delta} = 0 \leq r'_{\alpha,\beta}r'_{\beta,\gamma}r'_{\gamma,\delta}$$

if $k \notin \{\alpha, \beta, \gamma, \delta\}$ we have $\hat{k}(\alpha) \leq \hat{k}(\beta) \leq \hat{k}(\gamma) \leq \hat{k}(\delta)$ and then

$$r'_{\alpha,\gamma}r'_{\beta,\delta} = r'_{\hat{k}(\alpha),\hat{k}(\gamma)}r'_{\hat{k}(\beta),\hat{k}(\delta)} \leq r'_{\hat{k}(\alpha),\hat{k}(\beta)}r'_{\hat{k}(\beta),\hat{k}(\gamma)}r'_{\hat{k}(\gamma),\hat{k}(\delta)} = r'_{\alpha,\beta}r'_{\beta,\gamma}r'_{\gamma,\delta}$$

T3: Suppose by hypothesis that:

> Hypothesis: $\forall_{\alpha<\beta} \quad r_{\alpha,\beta} = 1 \Rightarrow r_{\alpha+1,\beta-1} = 1$ or $\exists_{\alpha<\gamma<\beta}$: $r_{\alpha,\gamma} = 1$.

We want to prove

> Thesis: $\forall_{\alpha<\beta} \quad r'_{\alpha,\beta} = 1 \Rightarrow r'_{\alpha+1,\beta-1} = 1$ or $\exists_{\alpha<\gamma<\beta}: \quad r'_{\alpha,\gamma} = 1$.

where $[r_{i,j}] = R$ and $[r'_{i,j}] = R' = B_k R B_k^t + D_k$. If $\beta = \alpha + 1$ then the thesis is true by the condition E2 or by the condition T1.

If $\beta = \alpha + 2$ then the thesis is true by the condition E1.

Now, we consider $\beta \geq \alpha + 3$. If $k = \alpha$ or $k = \beta$ we have $r'_{\alpha,\beta} = 0$, so the thesis is true. If $k = \alpha + 1$ then $r'_{\alpha,k+1} = r'_{k-1,k+1} = r_{\hat{k}(k-1),\hat{k}(k+1)} = r_{k-1,k-1} = 1$ and we have $\alpha = k - 1 < k + 1 < \beta$, so the thesis is true. If $k = \beta - 1$ then $r'_{\alpha,k-1} = r'_{\alpha,\beta} r'_{\beta,k-1} = r'_{\alpha,\beta} r'_{k+1,k-1} = r'_{\alpha,\beta}$ and we have $\alpha < k - 1 < k + 1 = \beta$, so the thesis is true. Now suppose that $k \notin \{\alpha, \alpha + 1, \beta - 1, \beta\}$. If $r'_{\alpha,\beta} = r_{\hat{k}(\alpha),\hat{k}(\beta)} = 1$ then, by hypothesis, $r_{\hat{k}(\alpha)+1,\hat{k}(\beta)-1} = 1$ or there exists $\hat{k}(\alpha) < \gamma' < \hat{k}(\beta)$ s.t. $r_{\hat{k}(\alpha),\gamma'} = 1$. Since $k \notin \{\alpha, \alpha + 1, \beta - 1, \beta\}$ we have that $\hat{k}(\alpha) + 1 = \hat{k}(\alpha + 1)$ and $\hat{k}(\beta) - 1 = \hat{k}(\beta - 1)$ ($k \notin \{\alpha, \alpha+1\} \Rightarrow \alpha+1 < k$ or $k < \alpha \Rightarrow \hat{k}(\alpha)+1 = \alpha + 1 = \hat{k}(\alpha + 1)$ or $\hat{k}(\alpha) + 1 = \alpha - 2 + 1 = \hat{k}(\alpha + 1)$, and the same argument for $\hat{k}(\beta) - 1 = \hat{k}(\beta - 1)$). Thus $r_{\hat{k}(\alpha)+1,\hat{k}(\beta)-1} = r'_{\alpha+1,\beta-1}$. Since $\hat{k} : \mathbb{N} \setminus \{k\} \longrightarrow \mathbb{N}$ is surjective and monotone, for any γ' between $\hat{k}(\alpha)$ and $\hat{k}(\beta)$ there exists γ between α and β such that $\gamma' = \hat{k}(\gamma)$. And then $r_{\hat{k}(\alpha),\gamma'} = r'_{\alpha,\gamma}$. Thus the thesis is true.

Now we only have to check the extra condition:

EC. $R' * \overrightarrow{v}' = \overrightarrow{v}'$

assuming that $\overrightarrow{v}$ is fixed by the action of R.

$$\begin{aligned}
R' * \overrightarrow{v}' &= (B_k R B_k^t + D_k) * (B_k * \overrightarrow{v}) \\
&= [(B_k R B_k^t + D_k) B_k] * \overrightarrow{v} \\
&= (B_k R B_k^t B_k + D_k B_k) * \overrightarrow{v} \\
&= (B_k R) * \overrightarrow{v} \\
&= B_k * (R * \overrightarrow{v}) \\
&= B_k * \overrightarrow{v} \\
&= \overrightarrow{v}'
\end{aligned}$$

Next we show that the function $\check{T}_k$ is well-defined. Recall that:

$$
\begin{array}{rccc}
\check{T}_{n,k}: & \mathcal{O}_{n+2} & \longrightarrow & \mathcal{O}_n \\
 & (R, \overrightarrow{v}) & \longmapsto & (R', \overrightarrow{v}')
\end{array}
$$

with

$$R' = (B_k^t R B_k)^2$$

and

$$\overrightarrow{v}' = R' * [(B_k^t * \overrightarrow{v}) \oplus (e_{k-1} * x_k)]$$

where

$$
\begin{array}{rcl}
x_k & = & [r_{k-1,k+1} * \varphi(v_k)] \oplus [(\neg r_{k-1,k+1}) * (v_{k-1} \wedge v_{k+1})] \\
 & = & \left\{ \begin{array}{ccc} \varphi(v_k) & \text{if} & r_{k-1,k+1} = 1 \\ v_{k-1} \wedge v_{k+1} & \text{if} & r_{k-1,k+1} = 0 \end{array} \right.
\end{array}
$$

Let us prove that R' satisfies the six conditions in $\mathcal{O}_n$ and that $\overrightarrow{v}'$ is fixed by the action of R'.

E1:

$$R \geq I \Rightarrow R' = (B_k^t R B_k)^2 \geq (B_k^t I B_k)^2 = I^2 = I$$

E2:

$$R = R^t \Rightarrow R'^t = [(B_k^t R B_k)^2]^t = [(B_k^t R B_k)^t]^2 = (B_k^t R^t B_k)^2 = (B_k^t R B_k)^2 = R'$$

E3: To check the transitivity of R' it is sufficient to prove that $(B_k^t R B_k)^3 \leq (B_k^t R B_k)^2$ (assuming that $R^2 = R$).

$$
\begin{array}{rcl}
(B_k^t R B_k)^3 & = & B_k^t R B_k B_k^t R B_k B_k^t R B_k \\
 & \leq & B_k^t R (I + B_k D_{k-1} B_k^t) R B_k B_k^t R B_k \\
 & = & B_k^t R I R B_k B_k^t R B_k + B_k^t R B_k D_{k-1} B_k^t R B_k B_k^t R B_k \\
 & \leq & B_k^t R B_k B_k^t R B_k + B_k^t R B_k D_{k-1} B_k^t R (I + B_k D_{k-1} B_k^t) R B_k \\
 & = & B_k^t R B_k B_k^t R B_k + B_k^t R B_k D_{k-1} B_k^t R^2 B_k \\
 & & + B_k^t R B_k D_{k-1} B_k^t R B_k D_{k-1} B_k^t R B_k \\
 & \leq & B_k^t R B_k B_k^t R B_k + B_k^t R B_k D_{k-1} B_k^t R B_k + B_k^t R B_k D_{k-1} B_k^t R B_k \\
 & = & B_k^t R B_k B_k^t R B_k \\
 & = & (B_k^t R B_k)^2
\end{array}
$$

We leave it to the reader to check that $B_k B_k^t \leq I + B_k D_{k-1} B_k^t$ and $D_{k-1} B_k^t R B_k D_{k-1} \leq D_{k-1}$.

T1:

$$\begin{aligned} R \leq C_{(n+2)\times(n+2)} \Rightarrow R' &= (B_k^t R B_k)^2 \leq (C_{n\times(n+2)} C_{(n+2)\times(n+2)} C_{(n+2)\times n})^2 \\ &\leq (C_{n\times n})^2 = C_{n\times n} \end{aligned}$$

.

T2: Using the "equality" $r_{i,j} = e_i^t R e_j$ we need to check that:

$$\text{Thesis: } \forall_{\alpha\leq\beta\leq\gamma\leq\delta} \qquad e_\alpha^t R' e_\gamma e_\beta^t R' e_\delta \leq e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta.$$

assuming the hypothesis:

$$\text{Hypothesis: } \forall_{\alpha\leq\beta\leq\gamma\leq\delta} \qquad e_\alpha^t R e_\gamma e_\beta^t R e_\delta \leq e_\alpha^t R e_\beta e_\beta^t R e_\gamma e_\gamma^t R e_\delta.$$

Since R' satisfies the equivalence relation conditions (E1, E2 and E3) the thesis is satisfied for $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \delta$. So we can assume that $\alpha < \beta < \gamma < \delta$.

We will substitute the hypothesis by a more appropriate hypothesis. But for that we need to introduce a new definition and same properties. Let u and v be two one-column non-zero matrices. We define:

$$u \prec v \text{ iff } \max\{i : u_i = 1\} < \min\{i : v_i = 1\}.$$

Proposition 5 *$\prec$ defines a strict order relation on the set of non-zero one-column matrices with the following properties:*

1. *$u \prec v, e_\alpha \leq u$ and $e_\beta \leq u \Rightarrow \alpha < \beta$;*

2. $(\forall_{\alpha<\beta<\gamma<\delta} \quad e_\alpha^t R e_\gamma e_\beta^t R e_\delta \leq e_\alpha^t R e_\beta e_\beta^t R e_\gamma e_\gamma^t R e_\delta) \Rightarrow$ $(\forall_{u\prec v\prec w\prec x} \quad u^t R w v^t R x \leq u^t R v v^t R w w^t R x)$;

3. $\alpha < \beta \Rightarrow B_k e_\alpha \prec B_k e_\beta$.

Now we take a new (weaker) hypothesis:

$$\text{N.H.: } \forall_{\alpha<\beta<\gamma<\delta}$$
$$e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \leq e_\alpha^t B_k^t R B_k e_\beta e_\beta^t B_k^t R B_k e_\gamma e_\gamma^t B_k^t R B_k e_\delta.$$

$$\begin{array}{rcl} e_\alpha^t R' e_\gamma e_\beta^t R' e_\delta & = & e_\alpha^t B_k^t R B_k B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k B_k^t R B_k e_\delta \\ & \leq & e_\alpha^t B_k^t R(I + A_k) R B_k e_\gamma e_\beta^t B_k^t R(I + A_k) R B_k e_\delta \\ & = & e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta + e_\alpha^t B_k^t R A_k R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\ & & + e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R A_k R B_k e_\delta \\ & & + e_\alpha^t B_k^t R A_k R B_k e_\gamma e_\beta^t B_k^t R A_k R B_k e_\delta \end{array}$$

where

$$A_k = B_k D_{k-1} B_k^t = B_k e_{k-1} e_{k-1}^t B_k^t$$

Now, we only need to prove that:

(i) $e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \leq e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta$;

(ii) $e_\alpha^t B_k^t R A_k R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \leq e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta$;

(iii) $e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R A_k R B_k e_\delta \leq e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta$;

(iv) $e_\alpha^t B_k^t R A_k R B_k e_\gamma e_\beta^t B_k^t R A_k R B_k e_\delta \leq e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta$.

For that it is useful to observe that, for arbitrary square matrices,

$$e_i^t X e_j e_k^t Y e_l = e_k^t Y e_l e_i^t X e_j,$$

$$e_i^t X e_j = e_j^t X^t e_i$$

and

$$(e_i^t X e_j)^2 = e_i^t X e_j,$$

since $e_i^t X e_j$ and $e_k^t Y e_l$ are one-dimensional square matrices.

(i)

$$\begin{array}{rcl} e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta & \leq & e_\alpha^t B_k^t R B_k e_\beta e_\beta^t B_k^t R B_k e_\gamma e_\gamma^t B_k^t R B_k e_\delta \\ & \leq & e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta \end{array}$$

Observe that

$$B_k^t R B_k \leq (B_k^t R B_k)^2 = R' \Rightarrow e_i^t B_k^t R B_k e_j \leq e_i^t R' e_j$$

(ii) $e_\alpha^t B_k^t R A_k R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta = e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta$.

If $k-1 < \beta$ then

$$\begin{aligned}
&e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\
&\le e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\beta e_\beta^t B_k^t R B_k e_\gamma e_\gamma^t B_k^t R B_k e_\delta \\
&\le e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta
\end{aligned}$$

If $k-1 = \beta$ then

$$\begin{aligned}
&e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\
&= e_\alpha^t B_k^t R B_k e_\beta e_\beta^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\
&\le e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\beta^t R' e_\delta \\
&= e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\beta^t R' e_\gamma e_\beta^t R' e_\delta \\
&= e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\beta e_\beta^t R' e_\delta \\
&\le e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta
\end{aligned}$$

If $\beta < k-1 < \delta$ then

$$\begin{aligned}
&e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\
&= e_\alpha^t B_k^t R B_k e_{k-1} e_\beta^t B_k^t R B_k e_\delta e_{k-1}^t B_k^t R B_k e_\gamma \\
&\le e_\alpha^t B_k^t R B_k e_\beta e_\beta^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\delta e_{k-1}^t B_k^t R B_k e_\gamma \\
&\le e_\alpha^t R' e_\beta e_\beta^t R' e_{k-1} e_{k-1}^t R' e_\delta e_{k-1}^t R' e_\gamma \\
&= e_\alpha^t R' e_\beta e_\beta^t R' e_{k-1} e_{k-1}^t R' e_\gamma e_\gamma^t R' e_{k-1} e_{k-1}^t R' e_\delta \\
&\le e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta
\end{aligned}$$

If $k-1 = \delta$ then

$$\begin{aligned}
&e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\
&= e_\alpha^t B_k^t R B_k e_\delta e_\delta^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\
&= e_\alpha^t B_k^t R B_k e_\delta (e_\delta^t B_k^t R B_k e_\beta)^2 e_\gamma^t B_k^t R B_k e_\delta \\
&\le e_\alpha^t R' e_\beta e_\beta^t R' e_\delta (e_\gamma^t R' e_\delta)^2 \\
&= e_\alpha^t R' e_\beta e_\beta^t R' e_\delta e_\delta^t R' e_\gamma e_\gamma^t R' e_\delta \\
&\le e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta
\end{aligned}$$

If $k-1 > \delta$ then

$$\begin{aligned}
&e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_\delta \\
&= e_\alpha^t B_k^t R B_k e_{k-1} e_\beta^t B_k^t R B_k e_\delta e_\gamma^t B_k^t R B_k e_{k-1} \\
&\le e_\alpha^t B_k^t R B_k e_{k-1} e_\beta^t B_k^t R B_k e_\gamma e_\gamma^t B_k^t R B_k e_\delta e_\delta^t B_k^t R B_k e_{k-1} \\
&\le e_\alpha^t R' e_{k-1} e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta e_\delta^t R' e_{k-1} \\
&= e_\alpha^t R' e_{k-1} e_{k-1}^t R' e_\delta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta \\
&\le e_\alpha^t R' e_\delta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta \\
&= e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta
\end{aligned}$$

(iii) $e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R A_k R B_k e_\delta = e_\delta^t B_k^t R A_k R B_k e_\beta e_\gamma^t B_k^t R B_k e_\alpha$.
Since all arguments in case (ii) are valid reversing the order of α, β, γ and δ, we have that:

$$e_\alpha^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R A_k R B_k e_\delta \leq e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta.$$

(iv)

$$\begin{aligned}
& e_\alpha^t B_k^t R A_k R B_k e_\gamma e_\beta^t B_k^t R A_k R B_k e_\delta \\
& = e_\alpha^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\gamma e_\beta^t B_k^t R B_k e_{k-1} e_{k-1}^t B_k^t R B_k e_\delta \\
& \leq e_\alpha^t R' e_{k-1} e_{k-1}^t R' e_\gamma e_\beta^t R' e_{k-1} e_{k-1}^t R' e_\delta \\
& = e_\alpha^t R' e_{k-1} e_{k-1}^t R' e_\beta e_\beta^t R' e_{k-1} e_{k-1}^t R' e_\gamma e_\gamma^t R' e_{k-1} e_{k-1}^t R' e_\delta \\
& \leq e_\alpha^t R' e_\beta e_\beta^t R' e_\gamma e_\gamma^t R' e_\delta
\end{aligned}$$

T3: Lemma 6 *If $R = [r_{i,j}]$ represents an equivalence relation, then the following statements are equivalent:*

(i) $\forall_{\alpha<\beta} \quad r_{\alpha,\beta} = 1 \Rightarrow r_{\alpha+1,\beta-1} = 1$ *or* $\exists_{\alpha<\gamma<\beta}: \quad r_{\alpha,\gamma} = 1$*;*

(ii) $\forall_{\alpha<\beta} \quad r_{\alpha,\beta} = 1 \Rightarrow \exists_{\alpha<\gamma\leq\beta}: \quad r_{\alpha,\gamma} = r_{\alpha+1,\gamma-1} = 1$*;*

(iii) $\forall_{\alpha<\beta} \quad r_{\alpha,\beta} = 1 \Rightarrow r_{\alpha+1,\beta-1} = 1$ *or* $\exists_{\alpha<\gamma<\beta}: \quad r_{\gamma,\beta} = 1$*;*

(iv) $\forall_{\alpha<\beta} \quad r_{\alpha,\beta} = 1 \Rightarrow \exists_{\alpha\leq\gamma<\beta}: \quad r_{\gamma,\beta} = r_{\gamma+1,\beta-1} = 1$.

Proof. It easy to see that $(ii) \Rightarrow (i)$ and $(iv) \Rightarrow (iii)$. To see that $(i) \Rightarrow (ii)$ we take $\gamma = \inf\{\delta \leq \beta : r_{\alpha,\delta} = 1\}$ and to see that $(iii) \Rightarrow (iv)$ we take $\gamma = \sup\{\delta \geq \alpha : r_{\delta,\beta} = 1\}$. $(i) \Leftrightarrow (iii)$ results from the transitivity of $[r_{i,j}]$. ■

Now we want to see that $[r'_{i,j}] = (B_k^t R B_k)^2$ satisfies one of the statements of the lemma (assuming that $R = [r_{i,j}]$ also satisfies the same statements and the remaining conditions on $\mathcal{O}_{n+2}$). Using the properties of $R = [r_{i,j}]$ and the relation $R' = [r'_{i,j}] = (B_k^t R B_k)^2$ we can set:

$$r'_{\alpha,\beta} = \begin{cases} r_{\alpha,\beta} + r_{\alpha,k+1} r_{k-1,\beta} & \text{if } \; k-1 > \beta \\ r_{\alpha,k-1} + r_{\alpha,k+1} & \text{if } \; k-1 = \beta \\ r_{\alpha,\beta+2} + r_{\alpha,k-1} r_{k+1,\beta+2} & \text{if } \; \alpha < k-1 < \beta \\ r_{k-1,\beta+2} + r_{k+1,\beta+2} & \text{if } \; k-1 = \alpha \\ r_{\alpha+2,\beta+2} + r_{\alpha+2,k+1} r_{k-1,\beta+2} & \text{if } \; k-1 < \alpha \end{cases}$$

Case A: $k-1 > \beta$. $r'_{\alpha,\beta} = r_{\alpha,\beta} + r_{\alpha,k+1} r_{k-1,\beta} = 1 \Rightarrow r_{\alpha,\beta} = 1$ or $r_{\alpha,k+1} = r_{k-1,\beta} = 1$.

A.1: $r_{\alpha,\beta} = 1 \Rightarrow \exists_{\alpha<\gamma_1\leq\beta} : r_{\alpha+1,\gamma_1-1} = r_{\alpha,\gamma_1} = 1 \Rightarrow \exists_{\alpha<\gamma_1\leq\beta} : r'_{\alpha+1,\gamma_1-1} = r'_{\alpha,\gamma_1} = 1$.

A.2: $r_{\alpha,k+1} = r_{k-1,\beta} = 1 \Rightarrow \exists_{\alpha<\gamma_2\leq k+1} : r_{\alpha+1,\gamma_2-1} = r_{\alpha,\gamma_2} = 1$.

A.2.1: If $\gamma_2 \leq \beta$ then we have $r'_{\alpha+1,\gamma_2-1} = r'_{\alpha,\gamma_2} = 1$ with $\alpha < \gamma_2 \leq \beta$.

A.2.2: If $\beta < \gamma_2 \leq k-1$ we can use the condition T2 to get $r_{\alpha,\beta} = 1$ (case A.1) since $r_{\alpha,\gamma_2} = 1$ and $r_{\beta,k-1} = 1$.

A.2.3: If $\gamma_2 = k$ we have $r_{\alpha,k} = 1$ which together with $r_{\alpha,k+1} = 1$ contradicts the condition T1.

A.2.4: If $\gamma_2 = k+1$ then $r_{\alpha+1,k} = 1 \Rightarrow \exists_{\alpha+1\leq\gamma_3<k} : r_{\gamma_3+1,k-1} = r_{\gamma_3,k} = 1$.

A.2.4.1: If $\beta \leq \gamma_3 \leq k-1$ we can use the condition T2 to get $r_{k-1,k} = 1$ (which contradicts the condition T1) since $r_{\beta,k-1} = 1$ and $r_{\gamma_3,k} = 1$.

A.2.4.2: If $\alpha+1 \leq \gamma_3 \leq \beta-1$ we have $r'_{\gamma_3+1,\alpha} = 1$ (since $r_{\gamma_3+1,k-1} = 1$ and $r_{k+1,\alpha} = 1$) and $r_{\gamma_3,\alpha+1} = 1$ (since $r_{\gamma_3,k} = 1$ and $r_{k,\alpha+1} = 1$). Taking $\gamma_4 = \gamma_3 + 1$ we have $r'_{\alpha+1,\gamma_4-1} = r'_{\alpha,\gamma_4} = 1$ with $\alpha < \gamma_4 \leq \beta$.

Case B: $k-1 = \beta$. $r'_{\alpha,\beta} = r_{\alpha,k-1} + r_{\alpha,k+1} = 1 \Rightarrow r_{\alpha,k-1} = 1$ or $r_{\alpha,k+1} = 1$.

B.1: $r_{\alpha,k-1} = 1 \Rightarrow \exists_{\alpha<\gamma_1\leq k-1} : r_{\alpha+1,\gamma_1-1} = r_{\alpha,\gamma_1} = 1 \Rightarrow \exists_{\alpha<\gamma_1\leq k-1} : r'_{\alpha+1,\gamma_1-1} = r'_{\alpha,\gamma_1} = 1$.

B.2: $r_{\alpha,k+1} = 1 \Rightarrow \exists_{\alpha<\gamma_2\leq k+1} : r_{\alpha+1,\gamma_2-1} = r_{\alpha,\gamma_2} = 1$.

B.2.1: If $\gamma_2 \leq k-1$ then we have $r'_{\alpha+1,\gamma_2-1} = r'_{\alpha,\gamma_2} = 1$ with $\alpha < \gamma_2 \leq k-1 = \beta$.

B.2.2: If $\gamma_2 = k$ we have $r_{\alpha,k} = 1$ which together with $r_{\alpha,k+1} = 1$ contradicts the condition T1.

B.2.3: If $\gamma_2 = k+1$ then $r_{\alpha+1,k} = 1 \Rightarrow \exists_{\alpha+1\leq\gamma_3<k} : r_{\gamma_3+1,k-1} = r_{\gamma_3,k} = 1$.

B.2.3.1: If $\gamma_3 = k-1$ then $r_{\gamma_3+1,k-1} = r_{k,k-1} = 1$ which contradicts the condition T1.

B.2.3.2: If $\alpha+1 \leq \gamma_3 \leq k-2$ we have $r'_{\gamma_3+1,\alpha} = 1$ (since $r_{\gamma_3+1,k-1} = 1$ and $r_{k+1,\alpha} = 1$) and $r_{\gamma_3,\alpha+1} = 1$ (since

$r_{\gamma_3,k} = 1$ and $r_{k,\alpha+1} = 1$). Taking $\gamma_4 = \gamma_3 + 1$ we have $r'_{\alpha+1,\gamma_4-1} = r'_{\alpha,\gamma_4} = 1$ with $\alpha < \gamma_4 \leq k-1$.

Case C: $\alpha < k-1 < \beta$. $r'_{\alpha,\beta} = r_{\alpha,\beta+2} + r_{\alpha,k+1}r_{k-1,\beta+2} = 1 \Rightarrow r_{\alpha,\beta+2} = 1$ or $r_{\alpha,k+1} = r_{k-1,\beta+2} = 1$.

C.1: $r_{\alpha,\beta+2} = 1 \Rightarrow \exists_{\alpha<\gamma_1\leq\beta+2} : r_{\alpha+1,\gamma_1-1} = r_{\alpha,\gamma_1} = 1$ and $\exists_{\alpha\leq\gamma_2<\beta+2} : r_{\gamma_2,\beta+2} = r_{\gamma_2+1,\beta+1} = 1$.

C.1.1: If $\gamma_1 \geq \gamma_2 + 1$ we can use the condition T2 to get $r_{\alpha,\beta+1} = 1$ since $r_{\alpha,\gamma_1} = 1$ and $r_{\gamma_2+1,\beta+1} = 1$. This together with $r_{\alpha,\beta+2} = 1$ contradicts the condition T1.

C.1.2: If $\gamma_1 \leq \gamma_2$ then $\gamma_1 < k$ or $\gamma_2 > k$ or $\gamma_1 = \gamma_2 = k$.

C.1.2.1: If $\gamma_1 < k$ then $r_{\alpha+1,\gamma_1-1} = r_{\alpha,\gamma_1} = 1 \Rightarrow r'_{\alpha+1,\gamma_1-1} = r'_{\alpha,\gamma_1} = 1$ with $\alpha < \gamma_1 < k \leq \beta$.

C.1.2.2: If $\gamma_2 > k$ then $r_{\gamma_2,\beta+2} = r_{\gamma_2+1,\beta+1} = 1 \Rightarrow r'_{\gamma_2,\beta+2} = r'_{\gamma_2+1,\beta+1} = 1$ with $\alpha \leq k-2 < \gamma_2 - 2 < \beta$.

C.1.2.3: If $\gamma_1 = \gamma_2 = k$ then $r_{\alpha+1,\gamma_1-1} = r_{\alpha+1,k-1} = 1$ and $r_{\gamma_2+1,\beta+1} = r_{k+1,\beta+1} = 1$ which implies $r'_{\alpha+1,\beta-1} = 1$.

C.2: $r_{\alpha,k-1} = r_{k+1,\beta+2} = 1 \Rightarrow r'_{\alpha,k-1} = 1$ with $\alpha < k-1 < \beta$.

Case D: $k-1 = \alpha$. This case is analogous to case B.

Case E: $k-1 < \alpha$. This case is analogous to case A.

Now we only have to check the condition:

EC. $R' * \overrightarrow{v}' = \overrightarrow{v}'$

which is obvious from the definition of $\overrightarrow{v}'$ because R' is idempotent.

4.2 Checking the relations on $\hat{T}_k$ and $\check{T}_k$

Now we are going to prove that $\hat{T}_k$ and $\check{T}_k$ satisfy the following relations:

1. $\check{T}_{k+1} \circ \hat{T}_k = \check{T}_{k-1} \circ \hat{T}_k = id$;

2. $\hat{T}_l \circ \hat{T}_k = \hat{T}_k \circ \hat{T}_{l-2}$ for $l \geq k+2$;

3. $\check{T}_k \circ \hat{T}_l = \hat{T}_{l-2} \circ \check{T}_k$ and $\check{T}_l \circ \hat{T}_k = \hat{T}_k \circ \check{T}_{l-2}$ for $l \geq k+2$;

4. $\check{T}_k \circ \check{T}_l = \check{T}_{l-2} \circ \check{T}_k$ for $l \geq k+2$.

We begin by checking the first relation.

1. $\check{T}_{k+1} \circ \hat{T}_k = \check{T}_{k-1} \circ \hat{T}_k = id.$

Let $(R_1, \overrightarrow{a}) \in \mathcal{O}_n$, $(R_2, \overrightarrow{b}) = \hat{T}_k(R_1, \overrightarrow{a})$ and $(R_3, \overrightarrow{c}) = \check{T}_{k+1}(R_2, \overrightarrow{b}) = \check{T}_{k+1} \circ \hat{T}_k(R_1, \overrightarrow{a})$.

We want to show that $(R_3, \overrightarrow{c}) = (R_1, \overrightarrow{a})$.

$$\left\{ \begin{array}{lcl} R_2 & = & B_k R_1 B_k^t + D_k = B_k R_1 B_k^t + I \\ R_3 & = & (B_{k+1}^t R_2 B_{k+1})^2 \end{array} \right. \Rightarrow R_3 = [B_{k+1}^t (B_k R_1 B_k^t + I) B_{k+1}]^2$$

$$R_3 = (B_{k+1}^t B_k R_1 B_k^t B_{k+1} + B_{k+1}^t B_{k+1})^2 = (R_1 + I)^2 = R_1^2 = R_1$$

We leave it to the reader to check the identities $B_{k+1}^t B_k = I$ and $B_{k+1}^t B_{k+1} = I$.

$$\left\{ \begin{array}{lcl} \overrightarrow{c} & = & R_3 * [(B_{k+1}^t * \overrightarrow{b}) \oplus (e_k * x_{k+1})] \\ \overrightarrow{b} & = & B_k * \overrightarrow{a} \end{array} \right. \Rightarrow$$
$$\Rightarrow \overrightarrow{c} = R_3 * [(B_{k+1}^t * (B_k * \overrightarrow{a})) \oplus (e_k * x_{k+1})]$$

where $x_{k+1} = [r_{k,k+2} * \varphi(b_{k+1})] \oplus [(\neg r_{k,k+2}) * (b_k \wedge b_{k+2})]$ with

$$r_{k,k+2} = e_k^t R_2 e_{k+2} = e_k^t (B_k R_1 B_k^t + D_k) e_{k+2} = e_k^t B_k R_1 B_k^t e_{k+2} + e_k^t D_k e_{k+2} = 0$$

thus $x_{k+1} = b_k \wedge b_{k+2} = \emptyset \wedge a_k = \emptyset$.

Then we have

$$\overrightarrow{c} = R_3 * [((B_{k+1}^t B_k) * \overrightarrow{a}) \oplus (e_k * \emptyset)] = R_1 * (\overrightarrow{a} \oplus \emptyset) = R_1 * \overrightarrow{a} = \overrightarrow{a}$$

To check the identity $\check{T}_{k-1} \circ \hat{T}_k = id$ we use the same procedure.

2. $\hat{T}_l \circ \hat{T}_k = \hat{T}_k \circ \hat{T}_{l-2}$ for $l \geq k+2$.

Let $(R_1, \overrightarrow{a}) \in \mathcal{O}_n$, $(R_2, \overrightarrow{b}) = \hat{T}_k(R_1, \overrightarrow{a})$ and $(R_3, \overrightarrow{c}) = \hat{T}_l(R_2, \overrightarrow{b}) = \hat{T}_l \circ \hat{T}_k(R_1, \overrightarrow{a})$, and let $(R_2', \overrightarrow{b}') = \hat{T}_{l-2}(R_1, \overrightarrow{a})$ and $(R_3', \overrightarrow{c}') = \hat{T}_k(R_2', \overrightarrow{b}') = \hat{T}_k \circ \hat{T}_{l-2}(R_1, \overrightarrow{a})$. We want to check that $(R_3', \overrightarrow{c}') = (R_3, \overrightarrow{c})$.

$$\begin{array}{lcl} R_3 & = & B_l R_2 B_l^t + D_l \\ & = & B_l (B_k R_1 B_k^t + D_k) B_l^t + D_l \\ & = & B_l B_k R_1 B_k^t B_l^t + B_l D_k B_l^t + D_l \\ & = & (B_l B_k) R_1 (B_l B_k)^t + D_k + D_l \end{array}$$

and

$$\begin{array}{rcl} R_3' & = & B_k R_2' B_k^t + D_k \\ & = & B_k(B_{l-2}R_1B_{l-2}^t + D_{l-2})B_k^t + D_k \\ & = & B_kB_{l-2}R_1B_{l-2}^tB_k^t + B_kD_{l-2}B_k^t + D_k \\ & = & (B_kB_{l-2})R_1(B_kB_{l-2})^t + D_l + D_k \end{array}$$

Thus $R_3' = R_3$. We leave it to the reader to check $B_lD_kB_l^t = D_k$, $B_kD_{l-2}B_k^t = D_l$ and $B_lB_k = B_kB_{l-2}$.

$$\left\{ \begin{array}{l} \overrightarrow{c} = B_l * \overrightarrow{b} = B_l * (B_k * \overrightarrow{a}) = (B_lB_k) * \overrightarrow{a} \\ \overrightarrow{c}' = B_k * \overrightarrow{b}' = B_k * (B_{l-2} * \overrightarrow{a}) = (B_kB_{l-2}) * \overrightarrow{a} \end{array} \right. \Rightarrow \overrightarrow{c}' = \overrightarrow{c}.$$

3. $\check{T}_k \circ \hat{T}_l = \hat{T}_{l-2} \circ \check{T}_k$ and $\check{T}_l \circ \hat{T}_k = \hat{T}_k \circ \check{T}_{l-2}$ for $l \geq k+2$.

Let $(R_1, \overrightarrow{a}) \in \mathcal{O}_n$, $(R_2, \overrightarrow{b}) = \hat{T}_l(R_1, \overrightarrow{a})$ and $(R_3, \overrightarrow{c}) = \check{T}_k(R_2, \overrightarrow{b}) = \check{T}_k \circ \hat{T}_l(R_1, \overrightarrow{a})$, and let $(R_2', \overrightarrow{b}') = \check{T}_k(R_1, \overrightarrow{a})$ and $(R_3', \overrightarrow{c}') = \hat{T}_{l-2}(R_2', \overrightarrow{b}') = \hat{T}_{l-2} \circ \check{T}_k(R_1, \overrightarrow{a})$.

We want to check that $(R_3', \overrightarrow{c}') = (R_3, \overrightarrow{c})$.

First the case $l > k+2$ (where we have the identity $B_k^tB_l = B_{l-2}B_k^t$).

$$\begin{array}{rcl} R_3 & = & (B_k^tR_2B_k)^2 \\ & = & [B_k^t(B_lR_1B_l^t + D_l)B_k]^2 \\ & = & (B_k^tB_lR_1B_l^tB_k + B_k^tD_lB_k)^2 \\ & = & (B_{l-2}B_k^tR_1B_kB_{l-2}^t + D_{l-2})^2 \\ & = & B_{l-2}B_k^tR_1B_kB_{l-2}^tB_{l-2}B_k^tR_1B_kB_{l-2}^t + B_{l-2}B_k^tR_1B_kB_{l-2}^tD_{l-2} \\ & & +D_{l-2}B_{l-2}B_k^tR_1B_kB_{l-2}^t + D_{l-2}^2 \\ & = & B_{l-2}B_k^tR_1B_kIB_k^tR_1B_kB_{l-2}^t + B_{l-2}B_k^tR_1B_kO + OB_k^tR_1B_kB_{l-2}^t + D_{l-2} \\ & = & B_{l-2}B_k^tR_1B_kB_k^tR_1B_kB_{l-2}^t + D_{l-2} \\ & = & B_{l-2}(B_k^tR_1B_k)^2B_{l-2}^t + D_{l-2} \\ & = & B_{l-2}R_2'B_{l-2}^t + D_{l-2} \\ & = & R_3' \end{array}$$

We leave it to the reader to check $B_k^tD_lB_k = D_{l-2}$.

$\overrightarrow{c} = R_3 * [(B_k^t * \overrightarrow{b}) \oplus (e_{k-1} * x_k)]$ where $x_k = [r_{k-1,k+1}^{(2)} * \varphi(b_k)] \oplus [(\neg r_{k-1,k+1}^{(2)}) * (b_{k-1} \wedge b_{k+1})]$ with $r_{k-1,k+1}^{(2)} = e_{k-1}^tR_2e_{k+1}$ and $\overrightarrow{b} = B_l * \overrightarrow{a}$.

In this way:

$$\begin{aligned}
\overrightarrow{c} &= R_3 * [(B_k^t B_l * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \\
&= R_3' * [(B_{l-2} B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \\
&= (B_{l-2} R_2' B_{l-2}^t + D_{l-2}) * [(B_{l-2} B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \\
&= \{(B_{l-2} R_2' B_{l-2}^t) * [(B_{l-2} B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)]\} \\
&\quad \vee \{D_{l-2} * [(B_{l-2} B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)]\}
\end{aligned}$$

Lemma 7 *Let $M = [\mu_{i,j}]$ be a Boolean matrix and let $\overrightarrow{v}$ and $\overrightarrow{w}$ be two arrays. If, for each index i, we have one of the following situations:*

(1) *$\mu_{i,j} * w_j = \emptyset$ for all j (or $\mu_{i,j} * v_j = \emptyset$ for all j);*

(2) *$\mu_{i,j} = 1$ for, at most, a single index j;*

then $M * (\overrightarrow{v} \oplus \overrightarrow{w}) = (M * \overrightarrow{v}) \oplus (M * \overrightarrow{w})$.

Proof. Let $\overrightarrow{x} := M * (\overrightarrow{v} \oplus \overrightarrow{w})$ and $\overrightarrow{y} := (M * \overrightarrow{v}) \oplus (M * \overrightarrow{w})$. We have

$$x_i = \bigvee_{j=1}^{n} \mu_{i,j} * (v_j \oplus w_j) = \bigvee_{j=1}^{n} [(\mu_{i,j} * v_j) \oplus (\mu_{i,j} * w_j)]$$

and

$$y_i = (\bigvee_{j=1}^{n} \mu_{i,j} * v_j) \oplus (\bigvee_{j=1}^{n} \mu_{i,j} * w_j)$$

(1) If $\mu_{i,j} * w_j = \emptyset$ for all j then

$$x_i = \bigvee_j [(\mu_{i,j} * v_j) \oplus \emptyset] = \bigvee_j \mu_{i,j} * v_j$$

and

$$y_i = (\bigvee_j \mu_{i,j} * v_j) \oplus (\bigvee_j \emptyset) = \bigvee_j \mu_{i,j} * v_j$$

(2) If there exists k such that $\mu_{i,j} = 1 \Rightarrow j = k$ then

$$x_i = \bigvee_j [(\mu_{i,j} * v_j) \oplus (\mu_{i,j} * w_j)] = (\mu_{i,k} * v_k) \oplus (\mu_{i,k} * w_k)$$

and

$$y_i = (\bigvee_j \mu_{i,j} * v_j) \oplus (\bigvee_{j=1}^{n} \mu_{i,j} * w_j) = (\mu_{i,k} * v_k) \oplus (\mu_{i,k} * w_k)$$

■

With this lemma we have that:

$$B_{l-2}^t * [(B_{l-2}B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] = (B_{l-2}^t B_{l-2} B_k^t * \overrightarrow{a}) \oplus (B_{l-2}^t e_{k-1} * x_k)$$

by the condition (2) of the lemma for $i \neq l-3$, and by the condition (1) for $i = l-3$; and

$$D_{l-2} * [(B_{l-2}B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] = (D_{l-2} B_{l-2} B_k^t * \overrightarrow{a}) \oplus (D_{l-2} e_{k-1} * x_k)$$

by the condition (2) of the lemma.

Thus

$$\begin{aligned}
\overrightarrow{c} &= \{(B_{l-2}R_2' B_{l-2}^t) * [(B_{l-2}B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)]\} \\
&\quad \vee \{D_{l-2} * [(B_{l-2}B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)]\} \\
&= \{(B_{l-2}R_2') * [(B_{l-2}^t B_{l-2} B_k^t * \overrightarrow{a}) \oplus (B_{l-2}^t e_{k-1} * x_k)]\} \\
&\quad \vee [(D_{l-2}B_{l-2}B_k^t * \overrightarrow{a}) \oplus (D_{l-2}e_{k-1} * x_k)] \\
&= \{(B_{l-2}R_2') * [(B_k^t * \overrightarrow{a}) \oplus (B_{l-2}^t e_{k-1} * x_k)]\} \\
&\quad \vee [(OB_k^t * \overrightarrow{a}) \oplus (O * x_k)] \\
&= (B_{l-2}R_2') * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)]
\end{aligned}$$

On the other hand

$$\begin{aligned}
\overrightarrow{c}' &= B_{l-2} * \overrightarrow{b}' \\
&= B_{l-2} * \{R_2' * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k')]\} \\
&= (B_{l-2}R_2') * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k')]
\end{aligned}$$

where $x_k' = [r_{k-1,k+1}^{(1)} * \varphi(a_k)] \oplus [(\neg r_{k-1,k+1}^{(1)}) * (a_{k-1} \wedge a_{k+1})]$ with $r_{k-1,k+1}^{(1)} = e_{k-1}^t R_1 e_{k+1}$.

To check $\overrightarrow{c}' = \overrightarrow{c}$ we only need to prove that $x_k' = x_k$.

$$\begin{aligned}
r_{k-1,k+1}^{(2)} &= e_{k-1}^t R_2 e_{k+1} \\
&= e_{k-1}^t (B_l R_1 B_l^t + D_l) e_{k+1} \\
&= e_{k-1}^t B_l R_1 B_l^t e_{k+1} + e_{k-1}^t D_l e_{k+1} \\
&= e_{k-1}^t R_1 e_{k+1} \\
&= r_{k-1,k+1}^{(1)}
\end{aligned}$$

$\overrightarrow{b} = B_l * \overrightarrow{a}$ and $l > k+2$ implies that $b_{k-1} = a_{k-1}$, $b_k = a_k$ and $b_{k+1} = a_{k+1}$.

Then

$$\begin{aligned} x_k &= [r^{(2)}_{k-1,k+1} * \varphi(b_k)] \oplus [(\neg r^{(2)}_{k-1,k+1}) * (b_{k-1} \wedge b_{k+1})] \\ &= [r^{(1)}_{k-1,k+1} * \varphi(a_k)] \oplus [(\neg r^{(1)}_{k-1,k+1}) * (a_{k-1} \wedge a_{k+1})] \\ &= x'_k \end{aligned}$$

Now, let us study the case $l = k+2$.

In this case, in contrast with the case $l > k+2$, we don't have $B^t_k B_l = B_{l-2} B^t_k$. In fact, $B^t_k B_{k+2} = (I - D_k) + Q_{k-1,k+1}$ and $B_k B^t_k = (I - D_k) + Q_{k-1,k+1} + Q_{k+1,k-1}$ where $Q_{\alpha,\beta} = e_\alpha e^t_\beta$ (i.e. all entries of $Q_{\alpha,\beta}$ are zero except the entry (α, β)). Thus $B^t_k B_{k+2} < B_k B^t_k$.

$$\begin{aligned} R_3 &= (B^t_k R_2 B_k)^2 \\ &= [B^t_k (B_{k+2} R_1 B^t_{k+2} + D_{k+2}) B_k]^2 \\ &= (B^t_k B_{k+2} R_1 B^t_{k+2} B_k + B^t_k D_{k+2} B_k)^2 \\ &= (B^t_k B_{k+2} R_1 B^t_{k+2} B_k + D_k)^2 \\ &= B^t_k B_{k+2} R_1 B^t_{k+2} B_k B^t_k B_{k+2} R_1 B^t_{k+2} B_k + B^t_k B_{k+2} R_1 B^t_{k+2} B_k D_k \\ &\quad + D_k B^t_k B_{k+2} R_1 B^t_{k+2} B_k + D^2_k \\ &= B^t_k B_{k+2} R_1 B^t_{k+2} B_k B^t_k B_{k+2} R_1 B^t_{k+2} B_k + D_k \end{aligned}$$

because

$$B^t_k B_{k+2} R_1 B^t_{k+2} B_k D_k \leq B^t_k B_{k+2} R_1 B_k B^t_k D_k = B^t_k B_{k+2} R_1 B_k O = O$$

and

$$D_k B^t_k B_{k+2} R_1 B^t_{k+2} B_k = (B^t_k B_{k+2} R_1 B^t_{k+2} B_k D_k)^t = O$$

$$\begin{aligned} R'_3 &= B_k R'_2 B^t_k + D_k \\ &= B_k (B^t_k R_1 B_k)^2 B^t_k + D_k \\ &= B_k B^t_k R_1 B_k B^t_k R_1 B_k B^t_k + D_k \end{aligned}$$

Since $B^t_{k+2} B_k \leq B_k B^t_k$ and $B^t_k B_{k+2} \leq B_k B^t_k$, we have

$$\begin{aligned} R_3 &= B^t_k B_{k+2} R_1 B^t_{k+2} B_k B^t_k B_{k+2} R_1 B^t_{k+2} B_k + D_k \\ &\leq B_k B^t_k R_1 B_k B^t_k B_k B^t_k R_1 B_k B^t_k + D_k \\ &= R'_3 \end{aligned}$$

On the other hand, since $B_{k+2}^t B_k B_k^t B_{k+2} = B_k B_k^t$, we have

$$\begin{array}{rcl} R_3' & = & B_k B_k^t R_1 B_k B_k^t R_1 B_k B_k^t + D_k \\ & = & B_{k+2}^t B_k B_k^t B_{k+2} R_1 B_{k+2}^t B_k B_k^t B_{k+2} R_1 B_{k+2}^t B_k B_k^t B_{k+2} + D_k \\ & = & B_{k+2}^t B_k (B_k^t B_{k+2} R_1 B_{k+2}^t B_k B_k^t B_{k+2} R_1 B_{k+2}^t B_k + D_k) B_k^t B_{k+2} + D_k \\ & = & B_{k+2}^t B_k R_3 B_k^t B_{k+2} + D_k \\ & \leq & B_k B_k^t R_3 B_k B_k^t + D_k \\ & \leq & R_3^3 + D_k \\ & = & R_3 \end{array}$$

Thus $R_3 = R_3'$.

Here we make use of the following inequalities:

$$B_{k+2}^t B_k D_k B_k^t B_{k+2} \leq B_k B_k^t D_k B_k^t B_{k+2} = O$$

and

$$\begin{array}{rcl} R_3 & = & (B_k^t R_2 B_k)^2 \\ & = & [B_k^t (B_{k+2} R_1 B_{k+2}^t + D_{k+2}) B_k]^2 \\ & \geq & (B_k^t B_{k+2} R_1 B_{k+2}^t B_k)^2 \\ & \geq & (B_k^t B_{k+2} B_{k+2}^t B_k)^2 \\ & = & (B_k B_k^t)^2 \\ & = & B_k B_k^t \end{array}$$

$$\begin{array}{rcl} \overrightarrow{c} & = & R_3 * [(B_k^t * \overrightarrow{b}) \oplus (e_{k-1} * x_k)] \\ & = & R_3' * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \\ & = & (B_k R_2' B_k^t + D_k) * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \\ & = & \{(B_k R_2' B_k^t) * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)]\} \\ & & \vee \{D_k * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)]\} \\ & = & (B_k R_2' B_k^t) * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \end{array}$$

because, using lemma 7, we have

$$D_k * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] = (D_k B_k^t B_{k+2} * \overrightarrow{a}) \oplus (D_k e_{k-1} * x_k) = \emptyset$$

since $D_k B_k^t B_{k+2} \leq D_k B_k B_k^t = O$ and $D_k e_{k-1} = O$.

$$\begin{array}{rcl} \overrightarrow{c}' & = & B_k * \overrightarrow{b}' \\ & = & B_k * \{R_2' * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k')]\} \\ & = & (B_k R_2') * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k')] \end{array}$$

Lemma 8 *Let $\overrightarrow{v}$ be an array with values in a monoid. If $v_{k+1} \leq v_{k-1}$ then $B_k^t * \overrightarrow{v} = X_k * \overrightarrow{v}$ where $X_k = B_k^t(I - D_{k+1})$.*

Proof. X_k differs from B_k^t only in the entry $(k-1, k+1)$ which is 0 in X_k and 1 in B_k^t. Thus we only need to check the equality $B_k^t * \overrightarrow{v} = X_k * \overrightarrow{v}$ for the $k-1$ coordinate which is $v_{k-1} \vee v_{k+1}$ in $B_k^t * \overrightarrow{v}$ and v_{k-1} in $X_k * \overrightarrow{v}$. Since, by hypothesis, $v_{k+1} \leq v_{k-1}$ we have the equality. ■

The $k+1$ coordinate of $(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)$ is

$$\begin{aligned} e_{k+1}^t * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] &= (e_{k+1}^t B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k+1}^t e_{k-1} * x_k) \\ &= (e_{k+1}^t * \overrightarrow{a}) \oplus (0 * x_k) \\ &= a_{k+1} \end{aligned}$$

and the $k-1$ coordinate of $(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)$ is

$$\begin{aligned} e_{k-1}^t * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] &= (e_{k-1}^t B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1}^t e_{k-1} * x_k) \\ &= [(e_{k-1}^t + e_{k+1}^t) * \overrightarrow{a}] \oplus (1 * x_k) \\ &= (a_{k-1} \vee a_{k+1}) \oplus x_k \end{aligned}$$

Thus, we may apply the previous lemma

$$\begin{aligned} B_k^t * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] &= X_k * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \\ &= (X_k B_k^t B_{k+2} * \overrightarrow{a}) \oplus (X_k e_{k-1} * x_k) \\ &= (B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k) \end{aligned}$$

We leave it to the reader to check that $X_k B_k^t B_{k+2} = B_k^t$. Therefore

$$\begin{aligned} \overrightarrow{c} &= (B_k R_2' B_k^t) * [(B_k^t B_{k+2} * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \\ &= (B_k R_2') * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x_k)] \end{aligned}$$

and thus $\overrightarrow{c} = \overrightarrow{c}'$ if $x_k = x_k'$ and we can check this using the same proof as was used in the case $l > k+2$.

Now, let us study the other relation $\check{T}_l \circ \hat{T}_k = \hat{T}_k \circ \check{T}_{l-2}$ for $l \geq k+2$.

We know that $\check{T}_{l'} \circ \hat{T}_{k'} = \hat{T}_{k'-2} \circ \check{T}_{l'}$ for $k' \geq l'+2$ (substituting k by l' and l by k' in the relation we have already proved).

We will use a mirror symmetry between these two relations. For that purpose, let us introduce the function *mirror symmetry* defined as follow:

$$\begin{array}{rccc} M_n : & \mathcal{O}_n & \longrightarrow & \mathcal{O}_n \\ & (R, \overrightarrow{v}) & \longmapsto & (S_n R S_n, S_n * \overrightarrow{v}) \end{array}$$

where S_n is the n-dimensional square matrix defined by

$$S_n := [s_{i,j}] \text{ with } s_{i,j} = 1 \text{ iff } i + j = n + 1.$$

We have that $M_n \circ M_n$ is the identity function on $\mathcal{O}_n$ since $S_n^2 = I$. Also, hearing in mind the relations $S_{n+2} B_{n,k} S_n = B_{n,n+3-k}$ and $S_{n+2} D_{n+2,k} S_{n+2} = D_{n+2,n+3-k}$, it is not too hard[6] to check the commutativity of the following squares:

$$\begin{tikzcd} \mathcal{O}_n \arrow[r, "\hat{T}_k"] \arrow[d, leftrightarrow, "M_n"'] & \mathcal{O}_{n+2} \arrow[d, leftrightarrow, "M_{n+2}"] \\ \mathcal{O}_n \arrow[r, "\hat{T}_{n+3-k}"'] & \mathcal{O}_{n+2} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{n+2} \arrow[r, "\check{T}_l"] \arrow[d, leftrightarrow, "M_{n+2}"'] & \mathcal{O}_n \arrow[d, leftrightarrow, "M_n"] \\ \mathcal{O}_{n+2} \arrow[r, "\check{T}_{n+3-l}"'] & \mathcal{O}_n \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_n \arrow[r, "\check{T}_{l-2}"] \arrow[d, leftrightarrow, "M_n"'] & \mathcal{O}_{n-2} \arrow[d, leftrightarrow, "M_{n-2}"] \\ \mathcal{O}_n \arrow[r, "\check{T}_{n+3-l}"'] & \mathcal{O}_{n-2} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{n-2} \arrow[r, "\hat{T}_k"] \arrow[d, leftrightarrow, "M_{n-2}"'] & \mathcal{O}_n \arrow[d, leftrightarrow, "M_n"] \\ \mathcal{O}_{n-2} \arrow[r, "\hat{T}_{n+1-k}"'] & \mathcal{O}_n \end{tikzcd}$$

Thus the outside square in the following diagram:

[6] It is only necessary to check the commutativity of the two first squares since the other two are obtained from these by a change of variables. For the second square it is useful to check first the identity $x_{n+3-l}(S_{n+2} R S_{n+2}, S_{n+2} * \overrightarrow{v}) = x_l(R, \overrightarrow{v})$.

$$
\begin{tikzcd}
\mathcal{O}_n \arrow[rrr, "\hat{T}_k"] \arrow[ddd, "\check{T}_{l-2}"'] \arrow[dr, leftrightarrow, "M_n"] & & & \mathcal{O}_{n+2} \arrow[ddd, "\check{T}_l"] \arrow[dl, leftrightarrow, "M_{n+2}"'] \\
& \mathcal{O}_n \arrow[r, "\hat{T}_{n+3-k}"] \arrow[d, "\check{T}_{n+3-l}"'] & \mathcal{O}_{n+2} \arrow[d, "\check{T}_{n+3-l}"] & \\
& \mathcal{O}_{n-2} \arrow[r, "\hat{T}_{n+1-k}"'] \arrow[dl, leftrightarrow, "M_{n-2}"'] & \mathcal{O}_n \arrow[dr, leftrightarrow, "M_n"] & \\
\mathcal{O}_{n-2} \arrow[rrr, "\hat{T}_k"] & & & \mathcal{O}_{n+2}
\end{tikzcd}
$$

commutes (i.e. $\check{T}_l \circ \hat{T}_k = \hat{T}_k \circ \check{T}_{l-2}$) if and only if the inside square commutes (i.e. $\check{T}_{n+3-l} \circ \hat{T}_{n+3-k} = \hat{T}_{n+1-k} \circ \check{T}_{n+3-l} \Leftrightarrow \check{T}_{l'} \circ \hat{T}_{k'} = \hat{T}_{k'-2} \circ \check{T}_{l'}$ by making the change of variables: $l' = n+3-l$ and $k' = n+3-k$) which we know to be true since $l \geq k+2 \Rightarrow k' \geq l'+2$.

4. $\check{T}_k \circ \check{T}_l = \check{T}_{l-2} \circ \check{T}_k$ for $l \geq k+2$.

Let $(R_1, \overrightarrow{a}) \in \mathcal{O}_n$, $(R_2, \overrightarrow{b}) = \check{T}_l(R_1, \overrightarrow{a})$ and $(R_3, \overrightarrow{c}) = \check{T}_k(R_2, \overrightarrow{b}) = \check{T}_k \circ \check{T}_l(R_1, \overrightarrow{a})$, and let $(R_2', \overrightarrow{b}') = \check{T}_k(R_1, \overrightarrow{a})$ and $(R_3', \overrightarrow{c}') = \check{T}_{l-2}(R_2', \overrightarrow{b}') = \check{T}_{l-2} \circ \check{T}_k(R_1, \overrightarrow{a})$.

We want to see that $(R_3', \overrightarrow{c}') = (R_3, \overrightarrow{c})$.

$$R_3 = (B_k^t R_2 B_k)^2 = \overline{B_k^t R_2 B_k} = \overline{B_k^t \overline{B_l^t R_1 B_l} B_k} \overset{(*)}{=} \overline{B_k^t B_l^t R_1 B_l B_k}$$

$$R_3' = \overline{B_{l-2}^t R_2' B_{l-2}} = \overline{B_{l-2}^t \overline{B_k^t R_1 B_k} B_{l-2}} \overset{(*)}{=} \overline{B_{l-2}^t B_k^t R_1 B_k B_{l-2}}$$

and since $B_l B_k = B_k B_{l-2}$ we have $R_3 = R_3'$.

Note that $\overline{A}$ means the transitive closure of A.

(*) Now let us prove that $\overline{B_k^t \overline{B_l^t R_2 B_l} B_k} = \overline{B_k^t B_l^t R_2 B_l B_k}$ and $\overline{B_{l-2}^t \overline{B_k^t R_2 B_k} B_{l-2}} = \overline{B_{l-2}^t B_k^t R_2 B_k B_{l-2}}$.

Lemma 9 *Let A be a square matrix, then*

$$\overline{B_k^t A B_k} = \overline{B_k^t \overline{(I-D_k) A (I-D_k)} B_k}$$

Proof. For any natural number n, we have

$$\begin{array}{lcl} \overline{B_k^t A B_k} & \geq & (B_k^t A B_k)^n \\ & = & B_k^t A B_k B_k^t A B_k ... B_k^t A B_k B_k^t A B_k \\ & \geq & B_k^t A(I-D_k)A(I-D_k)...(I-D_k)A(I-D_k)AB_k \\ & = & B_k^t(I-D_k)A(I-D_k)A(I-D_k)...(I-D_k)A(I-D_k)A(I-D_k)B_k \\ & = & B_k^t[(I-D_k)A(I-D_k)]^n B_k \end{array}$$

therefore $\overline{B_k^t A B_k} \geq B_k^t \overline{(I-D_k)A(I-D_k)} B_k$ and thus

$$\overline{B_k^t A B_k} \geq \overline{B_k^t \overline{(I-D_k)A(I-D_k)} B_k}.$$

On the other hand,

$$B_k^t A B_k = B_k^t (I-D_k)A(I-D_k)B_k \leq B_k^t \overline{(I-D_k)A(I-D_k)} B_k$$

and then $\overline{B_k^t A B_k} \leq \overline{B_k^t \overline{(I-D_k)A(I-D_k)} B_k}$. ■

Corollary 10 *If* $(I-D_k)\overline{A}(I-D_k) \leq \overline{(I-D_k)A(I-D_k)}$ *then* $\overline{B_k^t \overline{A} B_k} = \overline{B_k^t A B_k}$.

Proof.

$$\begin{array}{lcl} \overline{B_k^t \overline{A} B_k} & = & \overline{B_k^t (I-D_k)\overline{A}(I-D_k)B_k} \\ & \leq & \overline{B_k^t \overline{(I-D_k)A(I-D_k)} B_k} \\ & = & \overline{B_k^t A B_k} \end{array}$$

On the other hand,

$$\overline{A} \geq A \Rightarrow B_k^t \overline{A} B_k \geq B_k^t A B_k \Rightarrow \overline{B_k^t \overline{A} B_k} \geq \overline{B_k^t A B_k}$$

■

Claim: $(I-D_k)\overline{B_l^t R_1 B_l}(I-D_k) \leq \overline{(I-D_k)B_l^t R_1 B_l (I-D_k)}$.

Proof.

$$\begin{array}{lcl} \overline{(I-D_k)B_l^t R_1 B_l (I-D_k)} & \geq & [(I-D_k)B_l^t R_1 B_l (I-D_k)]^2 \\ & = & (I-D_k)B_l^t R_1 B_l (I-D_k) B_l^t R_1 B_l (I-D_k) \end{array}$$

$$
\begin{aligned}
(I-D_k)\overline{B_l^t R_1 B_l}(I-D_k) &= (I-D_k)B_l^t R_1 B_l B_l^t R_1 B_l (I-D_k) \\
&= (I-D_k)B_l^t R_1 B_l [D_k + (I-D_k)] B_l^t R_1 B_l (I-D_k) \\
&= (I-D_k)B_l^t R_1 B_l D_k B_l^t R_1 B_l (I-D_k) \\
&\quad +(I-D_k)B_l^t R_1 B_l (I-D_k) B_l^t R_1 B_l (I-D_k)
\end{aligned}
$$

$$
\begin{aligned}
(I-D_k)B_l^t R_1 B_l D_k B_l^t R_1 B_l (I-D_k) &= (I-D_k)B_l^t R_1 D_k R_1 B_l (I-D_k) \\
&\leq (I-D_k)B_l^t R_1 B_l (I-D_k)
\end{aligned}
$$

$$
\begin{aligned}
&(I-D_k)B_l^t R_1 B_l (I-D_k) B_l^t R_1 B_l (I-D_k) \geq \\
&\quad \geq (I-D_k)B_l^t R_1 B_l (I-D_k) B_l^t B_l (I-D_k) \\
&\quad = (I-D_k)B_l^t R_1 B_l (I-D_k)
\end{aligned}
$$

thus

$$(I-D_k)B_l^t R_1 B_l D_k B_l^t R_1 B_l (I-D_k) \leq (I-D_k)B_l^t R_1 B_l (I-D_k) B_l^t R_1 B_l (I-D_k).$$

And therefore,

$$
\begin{aligned}
(I-D_k)\overline{B_l^t R_1 B_l}(I-D_k) &= \overline{(I-D_k)B_l^t R_1 B_l (I-D_k)} B_l^t R_1 B_l (I-D_k) \\
&\leq (I-D_k)B_l^t R_1 B_l (I-D_k)
\end{aligned}
$$

■

Using the same argument we can also prove the following claim.

Claim: $(I-D_{l-2})\overline{B_k^t R_1 B_k}(I-D_{l-2}) \leq \overline{(I-D_{l-2})B_k^t R_1 B_k (I-D_{l-2})}$.

Therefore we have $\overline{B_k^t \overline{B_l^t R_2 B_l} B_k} = \overline{B_k^t B_l^t R_2 B_l B_k}$ and $\overline{B_{l-2}^t \overline{B_k^t R_2 B_k} B_{l-2}} = \overline{B_{l-2}^t B_k^t R_2 B_k B_{l-2}}$.
Now let us see that $\overrightarrow{c}' = \overrightarrow{c}$. We will check $c_i' = c_i$ for each index i.

Lemma 11 *Let $R = [r_{i,j}]$ be a matrix with the properties E1, E2 and E3 (i.e. an equivalence relation) and $\overrightarrow{v}$ an array fixed by the action of R.*

1. *If* $r_{i,j} = 1$ *then* $e_i^t * \overrightarrow{v} = e_j^t * \overrightarrow{v}$ *(i.e.* $v_i = v_j$*).*

Now let $\dot{R} = [\dot{r}_{i,j}] = (B_\alpha^t R B_\alpha)^2$.

2. *If* $\dot{r}_{i,\alpha-1} = 0$ *then*

$$e_i^t * \{\dot{R} * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)]\} = e_i^t B_\alpha^t * \overrightarrow{v} = \begin{cases} v_i & \text{if } i < \alpha - 1 \\ v_{i+2} & \text{if } i > \alpha - 1 \end{cases}$$

3. $e_{\alpha-1}^t * \{\dot{R} * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)]\} = (v_{\alpha-1} \vee v_{\alpha+1}) \oplus x$.

Proof.

1.

$$\begin{array}{rcll} r_{i,j} = 1 \Rightarrow e_i^t * \overrightarrow{v} & = & e_i^t R * \overrightarrow{v} & (\text{since } \overrightarrow{v} = R * \overrightarrow{v}) \\ & \geq & e_i^t R e_j e_j^t * \overrightarrow{v} & (\text{since } e_j e_j^t = D_j \leq I) \\ & = & e_j^t * \overrightarrow{v} & (\text{since } e_i R e_j^t = r_{i,j} = 1) \end{array}$$

Since R is symmetric, we have also $e_j^t R * \overrightarrow{v} \geq e_i^t R * \overrightarrow{v}$.

2.

$$\begin{array}{rcl} \dot{r}_{i,\alpha-1} = 0 \Rightarrow e_i^t \dot{R} & = & e_i^t \dot{R}[(I - D_{\alpha-1}) + D_{\alpha-1}] \\ & = & e_i^t \dot{R}(I - D_{\alpha-1}) + e_i^t \dot{R} D_{\alpha-1} \\ & = & e_i^t \dot{R}(I - D_{\alpha-1}) + e_i^t \dot{R} e_{\alpha-1} e_{\alpha-1}^t \\ & = & e_i^t \dot{R}(I - D_{\alpha-1}) \end{array}$$

Thus

$$\begin{aligned} & e_i^t \dot{R} * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)] = \\ & \quad = e_i^t \dot{R}(I - D_{\alpha-1}) * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)] \\ & \quad = e_i^t \dot{R} * \{[(I - D_{\alpha-1}) B_\alpha^t * \overrightarrow{v}] \oplus [(I - D_{\alpha-1}) e_{\alpha-1} * x]\} \\ & \quad = e_i^t \dot{R}(I - D_{\alpha-1}) B_\alpha^t * \overrightarrow{v} \\ & \quad = e_i^t B_\alpha^t R B_\alpha B_\alpha^t R B_\alpha (I - D_{\alpha-1}) B_\alpha^t * \overrightarrow{v} \\ & \quad \leq e_i^t B_\alpha^t R B_\alpha B_\alpha^t R * \overrightarrow{v} \\ & \quad = e_i^t B_\alpha^t R B_\alpha B_\alpha^t * \overrightarrow{v} \end{aligned}$$

Since

$$e_i^t B_\alpha^t R B_\alpha e_{\alpha-1} \leq e_i^t \dot{R} e_{\alpha-1} = 0 \Rightarrow e_i^t B_\alpha^t R B_\alpha = e_i^t B_\alpha^t R B_\alpha (I - D_{\alpha-1})$$

we have that

$$\begin{array}{rcl} e_i^t \dot{R} * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)] & \leq & e_i^t B_\alpha^t R B_\alpha B_\alpha^t * \overrightarrow{v} \\ & = & e_i^t B_\alpha^t R B_\alpha (I - D_{\alpha-1}) B_\alpha^t * \overrightarrow{v} \\ & \leq & e_i^t B_\alpha^t R * \overrightarrow{v} \\ & = & e_i^t B_\alpha^t * \overrightarrow{v} \end{array}$$

On the other hand, we have

$$e_i^t \dot{R} * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)] \geq e_i^t \dot{R} * (B_\alpha^t * \overrightarrow{v}) \geq e_i^t B_\alpha^t * \overrightarrow{v}$$

3.

$$\begin{array}{rcl} e_{\alpha-1}^t \dot{R} * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)] & \geq & e_{\alpha-1}^t * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)] \\ & = & (e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1}^t e_{\alpha-1} * x) \\ & = & (v_{\alpha-1} \vee v_{\alpha+1}) \oplus x \end{array}$$

On the other hand

$$\begin{array}{rcl} e_{\alpha-1}^t \dot{R} * [(B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1} * x)] & \leq & (e_{\alpha-1}^t \dot{R} B_\alpha^t * \overrightarrow{v}) \oplus (e_{\alpha-1}^t \dot{R} e_{\alpha-1} * x) \\ & = & (e_{\alpha-1}^t \dot{R} B_\alpha^t * \overrightarrow{v}) \oplus x \\ & \overset{(*)}{=} & (v_{\alpha-1} \vee v_{\alpha+1}) \oplus x \end{array}$$

(*) $e_{\alpha-1}^t \dot{R} B_\alpha^t * \overrightarrow{v} \geq e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v} = v_{\alpha-1} \vee v_{\alpha+1}$, and on the other hand

$$\begin{array}{rcl} e_{\alpha-1}^t \dot{R} B_\alpha^t * \overrightarrow{v} & = & e_{\alpha-1}^t B_\alpha^t R B_\alpha B_\alpha^t R B_\alpha B_\alpha^t * \overrightarrow{v} \\ & \leq & e_{\alpha-1}^t B_\alpha^t R B_\alpha B_\alpha^t R (I + B_\alpha D_{\alpha-1} B_\alpha^t) * \overrightarrow{v} \\ & = & (e_{\alpha-1}^t B_\alpha^t R B_\alpha B_\alpha^t R * \overrightarrow{v}) \\ & & \vee (e_{\alpha-1}^t B_\alpha^t R B_\alpha B_\alpha^t R B_\alpha e_{\alpha-1} e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v}) \\ & = & (e_{\alpha-1}^t B_\alpha^t R B_\alpha B_\alpha^t * \overrightarrow{v}) \vee (e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v}) \\ & \leq & [e_{\alpha-1}^t B_\alpha^t R (I + B_\alpha D_{\alpha-1} B_\alpha^t) * \overrightarrow{v}] \vee (e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v}) \\ & = & (e_{\alpha-1}^t B_\alpha^t R * \overrightarrow{v}) \vee (e_{\alpha-1}^t B_\alpha^t R B_\alpha e_{\alpha-1} e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v}) \\ & & \vee (e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v}) \\ & = & e_{\alpha-1}^t B_\alpha^t * \overrightarrow{v} \end{array}$$

■

Lemma 12 *Let $R = [r_{i,j}]$ be a matrix with the properties E1, E2 and E3 (i.e. an equivalence relation) and let $\dot{R} = [\dot{r}_{i,j}] = (B_\alpha^t R B_\alpha)^2$. Then:*

(a) *For any* $j \neq \alpha - 1$, $\dot{r}_{\alpha-1,j} = r_{\alpha-1,\breve{\alpha}(j)} + r_{\alpha+1,\breve{\alpha}(j)}$ *where* $\breve{\alpha}(j) = \begin{cases} j & \textit{if} \;\; j < \alpha - 1 \\ j+2 & \textit{if} \;\; j > \alpha - 1 \end{cases}$;

(b) *If* $i \neq \alpha - 1$ *and* $j \neq \alpha - 1$ *then* $\dot{r}_{i,j} \geq r_{\breve{\alpha}(i),\breve{\alpha}(j)}$;

(c) *If* $i \neq \alpha - 1$, $j \neq \alpha - 1$ *and* $\dot{r}_{i,\alpha-1} = 0$ *(or* $\dot{r}_{j,\alpha-1} = 0$*) then* $\dot{r}_{i,j} = r_{\breve{\alpha}(i),\breve{\alpha}(j)}$.

Proof.

(a)

$$\begin{aligned} \dot{r}_{\alpha-1,j} &= e^t_{\alpha-1}\dot{R}e_j \\ &= e^t_{\alpha-1}(B^t_\alpha R B_\alpha)^2 e_j \\ &\geq e^t_{\alpha-1}B^t_\alpha R B_\alpha e_j \\ &= (e^t_{\alpha-1} + e^t_{\alpha+1}) R B_\alpha e_j \\ &= r_{\alpha-1,\breve{\alpha}(j)} + r_{\alpha+1,\breve{\alpha}(j)} \end{aligned}$$

On the other hand

$$\begin{aligned} \dot{r}_{\alpha-1,j} &= e^t_{\alpha-1}B^t_\alpha R B_\alpha B^t_\alpha R B_\alpha e_j \\ &\leq e^t_{\alpha-1}B^t_\alpha R(I + B_\alpha D_{\alpha-1}B^t_\alpha) R B_\alpha e_j \\ &= e^t_{\alpha-1}B^t_\alpha R B_\alpha e_j + e^t_{\alpha-1}B^t_\alpha R B_\alpha e_{\alpha-1} e^t_{\alpha-1} B^t_\alpha R B_\alpha e_j \\ &= e^t_{\alpha-1}B^t_\alpha R B_\alpha e_j \\ &= r_{\alpha-1,\breve{\alpha}(j)} + r_{\alpha+1,\breve{\alpha}(j)} \end{aligned}$$

(b)

$$\begin{aligned} \dot{r}_{i,j} &= e^t_i (B^t_\alpha R B_\alpha)^2 e_j \\ &\geq e^t_i B^t_\alpha R B_\alpha e_j \\ &= e^t_{\breve{\alpha}(i)} R e_{\breve{\alpha}(j)} \\ &= r_{\breve{\alpha}(i),\breve{\alpha}(j)} \end{aligned}$$

(c)

$$\begin{aligned} \dot{r}_{i,j} &= e^t_i (B^t_\alpha R B_\alpha)^2 e_j \\ &= e^t_i B^t_\alpha R B_\alpha B^t_\alpha R B_\alpha e_j \\ &\leq e^t_i B^t_\alpha R(I + B_\alpha D_{\alpha-1} B^t_\alpha) R B_\alpha e_j \\ &= e^t_i B^t_\alpha R B_\alpha e_j + e^t_i B^t_\alpha R B_\alpha e_{\alpha-1} e^t_{\alpha-1} B^t_\alpha R B_\alpha e_j \\ &\overset{(*)}{=} e^t_i B^t_\alpha R B_\alpha e_j \\ &= r_{\breve{\alpha}(i),\breve{\alpha}(j)} \end{aligned}$$

(*) $e_i^t B_\alpha^t R B_\alpha e_{\alpha-1} \leq e_i^t (B_\alpha^t R B_\alpha)^2 e_{\alpha-1} = \dot{r}_{i,\alpha-1} = 0$ (or $e_{\alpha-1}^t B_\alpha^t R B_\alpha e_j \leq \dot{r}_{\alpha-1,j} = 0$).

And thus, by (b), we have $\dot{r}_{i,j} = r_{\breve{\alpha}(i),\breve{\alpha}(j)}$.

■

Now, we are going to prove that $c_{k-1} = c'_{k-1}$ for the case $\ddot{r}_{k-1,l-3} = 0$.

Convention: $R_1 = [r_{i,j}]$, $R_2 = [\dot{r}_{i,j}] = (B_l^t R_1 B_l)^2$, $R_3 = [\ddot{r}_{i,j}] = (B_k^t R_2 B_k)^2$, $R'_2 = [\dot{r}'_{i,j}] = (B_k^t R_1 B_k)^2$ and $R'_3 = [\ddot{r}'_{i,j}] = (B_{l-2}^t R'_2 B_{l-2})^2$.

By lemma 11, we have:

$$\begin{array}{rcl} c_{k-1} & = & e_{k-1}^t R_3 * [(B_k^t * \overrightarrow{b}) \oplus (e_{k-1} * x_k)] \\ & = & (b_{k-1} \vee b_{k+1}) \oplus x_k \end{array}$$

where $x_k = [\dot{r}_{k-1,k+1} * \varphi(b_k)] \oplus [(\neg \dot{r}_{k-1,k+1}) * (b_{k-1} \wedge b_{k+1})]$ with $\dot{r}_{k-1,k+1} = e_{k-1}^t R_2 e_{k+1}$.

By lemma 12, $\ddot{r}_{k-1,l-3} = 0 \Rightarrow \dot{r}_{k-1,l-1} = 0$ and $\dot{r}_{k+1,l-1} = 0$. And then, by lemma 11,

$$b_{k-1} = e_{k-1}^t R_2 * [(B_l^t * \overrightarrow{a}) \oplus (e_{l-1} * x_l)] = a_{k-1}$$

and

$$b_{k+1} = e_{k+1}^t R_2 * [(B_l^t * \overrightarrow{a}) \oplus (e_{l-1} * x_l)] = a_{k+1}$$

Thus

$$c_{k-1} = (a_{k-1} \vee a_{k+1}) \oplus x_k$$

Since $R'_3 = R_3$, $\ddot{r}'_{k-1,l-3} = \ddot{r}_{k-1,l-3} = 0$. Then, by lemma 11,

$$\begin{array}{rcl} c'_{k-1} & = & e_{k-1}^t R'_3 * [(B_{l-2}^t * \overrightarrow{b}) \oplus (e_{l-3} * x'_{l-2})] \\ & = & b'_{k-1} \\ & = & e_{k-1}^t R'_2 * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x'_k)] \\ & = & (a_{k-1} \vee a_{k+1}) \oplus x'_k \end{array}$$

where

$$x'_k = [r_{k-1,k+1} * \varphi(a_k)] \oplus [(\neg r_{k-1,k+1}) * (a_{k-1} \wedge a_{k+1})]$$

with $r_{k-1,k+1} = e_{k-1}^t R_1 e_{k+1}$.

Thus $c_{k-1} = c'_{k-1}$ if $x_k = x'_k$.

$$\begin{aligned} x_k &= [\dot{r}_{k-1,k+1} * \varphi(b_k)] \oplus [(\neg \dot{r}_{k-1,k+1}) * (b_{k-1} \wedge b_{k+1})] \\ &= [\dot{r}_{k-1,k+1} * \varphi(b_k)] \oplus [(\neg \dot{r}_{k-1,k+1}) * (a_{k-1} \wedge a_{k+1})] \\ &\overset{(*)}{=} [\dot{r}_{k-1,k+1} * \varphi(a_k)] \oplus [(\neg \dot{r}_{k-1,k+1}) * (a_{k-1} \wedge a_{k+1})] \\ &\overset{(\dagger)}{=} [r_{k-1,k+1} * \varphi(a_k)] \oplus [(\neg r_{k-1,k+1}) * (a_{k-1} \wedge a_{k+1})] \\ &= x'_k \end{aligned}$$

(*) If $\dot{r}_{k-1,k+1} = 0$ then $\dot{r}_{k-1,k+1} * \varphi(b_k) = \emptyset = \dot{r}_{k-1,k+1} * \varphi(a_k)$.
If $\dot{r}_{k-1,k+1} = 1$ then $\dot{r}_{k,l-1} = 0$ by the properties T1 and T2 of R_2.
Thus $b_k = e_k^t R_2 * [(B_l^t * \overrightarrow{a}) \oplus (e_{l-1} * x_l)] = a_k$.

(†) $\ddot{r}_{k-1,l-3} = 0 \Rightarrow \dot{r}_{k-1,l-1} = 0$ and $\dot{r}_{k+1,l-1} = 0$ by lemma 11.
$\dot{r}_{k-1,l-1} = 0 \Rightarrow \dot{r}_{k-1,k+1} = r_{k-1,k+1}$ by lemma 11.

We now prove $c_{k-1} = c'_{k-1}$ for the case $\ddot{r}_{k-1,l-3} = 1$.

By lemma 12,

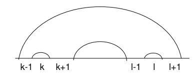
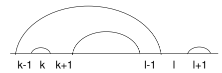
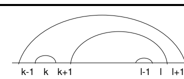
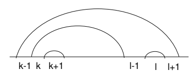
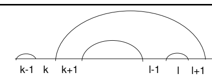
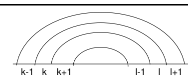
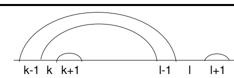
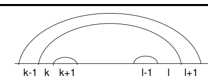
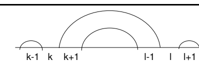
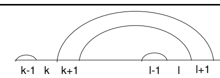
$$\begin{aligned} \ddot{r}_{k-1,l-3} = 1 &\Rightarrow \dot{r}_{k-1,l-1} = 1 \text{ or } \dot{r}_{k+1,l-1} = 1 \\ &\Rightarrow r_{k-1,l-1} = 1 \text{ or } r_{k-1,l+1} = 1 \text{ or } r_{k+1,l-1} = 1 \text{ or } r_{k+1,l+1} = 1 \end{aligned}$$

Now we have, in theory, 15 cases to study:

$$(r_{k-1,l-1}, r_{k-1,l+1}, r_{k+1,l-1}, r_{k+1,l+1}) \in \{0,1\}^4 \setminus \{(0,0,0,0)\}$$

but we can exclude the cases $(r_{k-1,l-1}, r_{k-1,l+1}, r_{k+1,l-1}, r_{k+1,l+1}) = (1,1,1,0)$, $(1,1,0,1)$, $(1,0,1,1)$ and $(0,1,1,1)$, because $R_1 = [r_{i,j}]$ satisfies the properties E1, E2 and E3, and the case $(r_{k-1,l-1}, r_{k-1,l+1}, r_{k+1,l-1}, r_{k+1,l+1}) = (1,0,0,1)$, because the inequalities $k-1 < k+1 \leq l-1 < l+1$ and property T2 of R_1 imply that if $r_{k-1,l-1} = r_{k+1,l+1} = 1$ then $r_{k+1,l-1} = 1$ and $r_{k-1,l+1} = 1$.

Thus, we have the following ten cases to study:

$r_{k-1,l-1}$	$r_{k-1,l+1}$	$r_{k+1,l-1}$	$r_{k+1,l+1}$	one geometric realization
1	1	1	1	
1	0	1	0	
0	1	0	1	
1	1	0	0	
0	0	1	1	
0	1	1	0	
1	0	0	0	
0	1	0	0	
0	0	1	0	
0	0	0	1	

Using the properties of $R_1 = [r_{i,j}]$ and $R_2 = [\dot{r}_{i,j}]$, and the relations between them ($R_2 = (B_l^t R_1 B_l)^2$), it is easy to prove the following statements:

1. If $r_{k-1,l-1} = 1$ or $r_{k-1,l+1} = 1$ or $r_{k+1,l-1} = 1$ or $r_{k+1,l+1} = 1$ then:

$$r_{k-1,k+1} = 1 \Leftrightarrow (r_{k-1,l-1} = r_{k+1,l-1} \text{ and } r_{k-1,l+1} = r_{k+1,l+1});$$

2. If $r_{k-1,l-1} = 1$ or $r_{k-1,l+1} = 1$ or $r_{k+1,l-1} = 1$ or $r_{k+1,l+1} = 1$ then:

$$r_{l-1,l+1} = 1 \Leftrightarrow (r_{k-1,l-1} = r_{k-1,l+1} \text{ and } r_{k+1,l-1} = r_{k+1,l+1});$$

3. $\dot{r}_{k-1,k+1} = r_{k-1,k+1} + r_{k-1,l+1} r_{k+1,l-1}$;

4. $\dot{r}_{k-1,l-1} = r_{k-1,l-1} + r_{k-1,l+1}$;

5. $\dot{r}_{k+1,l-1} = r_{k+1,l-1} + r_{k+1,l+1}$.

Also, using lemma 11, we can easily check:

$$c_{k-1} = (b_{k-1} \vee b_{k+1}) \oplus x_k = \begin{cases} b_{k-1} \oplus b_{k+1} & \text{if} \quad \dot{r}_{k-1,k+1} = 0 \\ b_{k-1} \oplus \varphi(b_k) & \text{if} \quad \dot{r}_{k-1,k+1} = 1 \end{cases}$$

$$b_{k-1} = \begin{cases} a_{k-1} & \text{if} \quad \dot{r}_{k-1,l-1} = 0 \\ b_{l-1} & \text{if} \quad \dot{r}_{k-1,l-1} = 1 \end{cases}$$

$$b_{k+1} = \begin{cases} a_{k+1} & \text{if} \quad \dot{r}_{k+1,l-1} = 0 \\ b_{l-1} & \text{if} \quad \dot{r}_{k+1,l-1} = 1 \end{cases}$$

$$b_{l-1} = (a_{l-1} \vee a_{l+1}) \oplus x_l = \begin{cases} a_{l-1} \oplus a_{l+1} & \text{if} \quad r_{l-1,l+1} = 0 \\ a_{l-1} \oplus \varphi(a_l) & \text{if} \quad r_{l-1,l+1} = 1 \end{cases}$$

Since $\ddot{r}_{k-1,l-3} = 1 \Rightarrow k-1-(l-3) \in 2\mathbb{Z}$ we have $\dot{r}_{k,l-1} = 0$ and then $b_k = a_k$.

With this, we can construct the following table:

$r_{k-1,l-1}$	$r_{k-1,l+1}$	$r_{k+1,l-1}$	$r_{k+1,l+1}$	$r_{k-1,k+1}$	$\dot{r}_{k-1,k+1}$	c_{k-1}
1	1	1	1	1	1	$b_{k-1} \oplus \varphi(b_k)$
1	0	1	0	1	1	$b_{k-1} \oplus \varphi(b_k)$
0	1	0	1	1	1	$b_{k-1} \oplus \varphi(b_k)$
1	1	0	0	0	0	$b_{k-1} \oplus b_{k+1}$
0	0	1	1	0	0	$b_{k-1} \oplus b_{k+1}$
0	1	1	0	0	1	$b_{k-1} \oplus \varphi(b_k)$
1	0	0	0	0	0	$b_{k-1} \oplus b_{k+1}$
0	1	0	0	0	0	$b_{k-1} \oplus b_{k+1}$
0	0	1	0	0	0	$b_{k-1} \oplus b_{k+1}$
0	0	0	1	0	0	$b_{k-1} \oplus b_{k+1}$

$r_{l-1,l+1}$	b_{l-1}	$\dot{r}_{k-1,l-1}$	b_{k-1}	$\dot{r}_{k+1,l-1}$	b_{k+1}	c_{k-1}
1	$a_{l-1} \oplus \varphi(a_l)$	1	$a_{l-1} \oplus \varphi(a_l)$	1	$a_{l-1} \oplus \varphi(a_l)$	$[a_{l-1} \oplus \varphi(a_l)] \oplus \varphi(a_k)$
0	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	$[a_{l-1} \oplus a_{l+1}] \oplus \varphi(a_k)$
0	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	$[a_{l-1} \oplus a_{l+1}] \oplus \varphi(a_k)$
1	$a_{l-1} \oplus \varphi(a_l)$	1	$a_{l-1} \oplus \varphi(a_l)$	0	a_{k+1}	$[a_{l-1} \oplus \varphi(a_l)] \oplus a_{k+1}$
1	$a_{l-1} \oplus \varphi(a_l)$	0	a_{k-1}	1	$a_{l-1} \oplus \varphi(a_l)$	$a_{k-1} \oplus [a_{l-1} \oplus \varphi(a_l)]$
0	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	$[a_{l-1} \oplus a_{l+1}] \oplus \varphi(a_k)$
0	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	0	a_{k+1}	$[a_{l-1} \oplus a_{l+1}] \oplus a_{k+1}$
0	$a_{l-1} \oplus a_{l+1}$	1	$a_{l-1} \oplus a_{l+1}$	0	a_{k+1}	$[a_{l-1} \oplus a_{l+1}] \oplus a_{k+1}$
0	$a_{l-1} \oplus a_{l+1}$	0	a_{k-1}	1	$a_{l-1} \oplus a_{l+1}$	$a_{k-1} \oplus [a_{l-1} \oplus a_{l+1}]$
0	$a_{l-1} \oplus a_{l+1}$	0	a_{k-1}	1	$a_{l-1} \oplus a_{l+1}$	$a_{k-1} \oplus [a_{l-1} \oplus a_{l+1}]$

We can construct an analogous table for the value c'_{k-1}. All we need to know is that:

1. If $r_{k-1,l-1} = 1$ or $r_{k-1,l+1} = 1$ or $r_{k+1,l-1} = 1$ or $r_{k+1,l+1} = 1$ then:

$$r_{k-1,k+1} = 1 \Leftrightarrow (r_{k-1,l-1} = r_{k+1,l-1} \text{ and } r_{k-1,l+1} = r_{k+1,l+1});$$

2. If $r_{k-1,l-1} = 1$ or $r_{k-1,l+1} = 1$ or $r_{k+1,l-1} = 1$ or $r_{k+1,l+1} = 1$ then:

$$r_{l-1,l+1} = 1 \Leftrightarrow (r_{k-1,l-1} = r_{k-1,l+1} \text{ and } r_{k+1,l-1} = r_{k+1,l+1});$$

3. $\dot{r}'_{l-3,l-1} = r_{l-1,l+1} + r_{k-1,l+1} r_{k+1,l-1}$;

4. $\dot{r}'_{k-1,l-3} = r_{k-1,l-1} + r_{k+1,l-1}$;

5. $\dot{r}'_{k-1,l-1} = r_{k-1,l+1} + r_{k+1,l+1}$.

$$\ddot{r}'_{k-1,l-3} = \ddot{r}_{k-1,l-3} = 1 \Rightarrow$$

$$c'_{k-1} = c'_{l-3} = (b'_{l-3} \vee b'_{l-1}) \oplus x'_{l-2} = \begin{cases} b'_{l-3} \oplus b'_{l-1} & \text{if } \dot{r}'_{l-3,l-1} = 0 \\ b'_{l-3} \oplus \varphi(b'_{l-2}) & \text{if } \dot{r}'_{l-3,l-1} = 1 \end{cases}$$

$$b'_{l-3} = \begin{cases} a_{l-1} & \text{if } \dot{r}'_{k-1,l-3} = 0 \\ b'_{k-1} & \text{if } \dot{r}'_{k-1,l-3} = 1 \end{cases}$$

$$b'_{l-1} = \begin{cases} a_{l+1} & \text{if } \dot{r}'_{k-1,l-1} = 0 \\ b'_{k-1} & \text{if } \dot{r}'_{k-1,l-1} = 1 \end{cases}$$

$$b'_{k-1} = (a_{k-1} \vee a_{k+1}) \oplus x'_k = \begin{cases} a_{k-1} \oplus a_{k+1} & \text{if } r_{k-1,k+1} = 0 \\ a_{k-1} \oplus \varphi(a_k) & \text{if } r_{k-1,k+1} = 1 \end{cases}$$

and $b'_{l-2} = a_l$ since $\ddot{r}'_{k-1,l-3} = 1 \Rightarrow \dot{r}'_{k-1,l-2} = 0$.

$r_{k-1,l-1}$	$r_{k-1,l+1}$	$r_{k+1,l-1}$	$r_{k+1,l+1}$	$r_{l-1,l+1}$	$\dot{r}'_{l-3,l-1}$	c'_{k-1}
1	1	1	1	1	1	$b'_{l-3} \oplus \varphi(b'_{l-2})$
1	0	1	0	0	0	$b'_{l-3} \oplus b'_{l-1}$
0	1	0	1	0	0	$b'_{l-3} \oplus b'_{l-1}$
1	1	0	0	1	1	$b'_{l-3} \oplus \varphi(b'_{l-2})$
0	0	1	1	1	1	$b'_{l-3} \oplus \varphi(b'_{l-2})$
0	1	1	0	0	1	$b'_{l-3} \oplus \varphi(b'_{l-2})$
1	0	0	0	0	0	$b'_{l-3} \oplus b'_{l-1}$
0	1	0	0	0	0	$b'_{l-3} \oplus b'_{l-1}$
0	0	1	0	0	0	$b'_{l-3} \oplus b'_{l-1}$
0	0	0	1	0	0	$b'_{l-3} \oplus b'_{l-1}$

$r_{k-1,k+1}$	b'_{k-1}	$\dot{r}'_{l-3,k-1}$	b'_{l-3}	$\dot{r}'_{k-1,l-1}$	b'_{l-1}	c'_{k-1}
1	$a_{k-1} \oplus \varphi(a_k)$	1	$a_{k-1} \oplus \varphi(a_k)$	1	$a_{k-1} \oplus \varphi(a_k)$	$[a_{k-1} \oplus \varphi(a_k)] \oplus \varphi(a_l)$
1	$a_{k-1} \oplus \varphi(a_k)$	1	$a_{k-1} \oplus \varphi(a_k)$	0	a_{l+1}	$[a_{k-1} \oplus \varphi(a_k)] \oplus a_{l+1}$
1	$a_{k-1} \oplus \varphi(a_k)$	0	a_{l-1}	1	$a_{k-1} \oplus \varphi(a_k)$	$a_{l-1} \oplus [a_{k-1} \oplus \varphi(a_k)]$
0	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	$[a_{k-1} \oplus a_{k+1}] \oplus \varphi(a_l)$
0	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	$[a_{k-1} \oplus a_{k+1}] \oplus \varphi(a_l)$
0	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	$[a_{k-1} \oplus a_{k+1}] \oplus \varphi(a_l)$
0	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	0	a_{l+1}	$[a_{k-1} \oplus a_{k+1}] \oplus a_{l+1}$
0	$a_{k-1} \oplus a_{k+1}$	0	a_{l-1}	1	$a_{k-1} \oplus a_{k+1}$	$a_{l-1} \oplus [a_{k-1} \oplus a_{k+1}]$
0	$a_{k-1} \oplus a_{k+1}$	1	$a_{k-1} \oplus a_{k+1}$	0	a_{l+1}	$[a_{k-1} \oplus a_{k+1}] \oplus a_{l+1}$
0	$a_{k-1} \oplus a_{k+1}$	0	a_{l-1}	1	$a_{k-1} \oplus a_{k+1}$	$a_{l-1} \oplus [a_{k-1} \oplus a_{k+1}]$

In order to facilitate the comparison of c_{k-1} and c'_{k-1} we will substitute each a_i (with $i \in \{k+1, l-1, l+1\}$) appearing in the expressions of c_{k-1} and c'_{k-1} by a_j where $j \in \{k-1, k+1, l-1, l+1\}$ is the smallest index such that $r_{i,j} = 1$.

The results can be seen in the following table:

$r_{k-1,l-1}$	$r_{k-1,l+1}$	$r_{k+1,l-1}$	$r_{k+1,l+1}$	$r_{k-1,k+1}$	$r_{l-1,l+1}$
1	1	1	1	1	1
1	0	1	0	1	0
0	1	0	1	1	0
1	1	0	0	0	1
0	0	1	1	0	1
0	1	1	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0

a_{k+1}	a_{l-1}	a_{l+1}	c_{k-1}	c'_{k-1}
a_{k-1}	a_{k-1}	a_{k-1}	$a_{k-1} \oplus \varphi(a_l) \oplus \varphi(a_k)$	$a_{k-1} \oplus \varphi(a_k) \oplus \varphi(a_l)$
a_{k-1}	a_{k-1}	a_{l+1}	$a_{k-1} \oplus a_{l+1} \oplus \varphi(a_k)$	$a_{k-1} \oplus \varphi(a_k) \oplus a_{l+1}$
a_{k-1}	a_{l-1}	a_{k-1}	$a_{l-1} \oplus a_{k-1} \oplus \varphi(a_k)$	$a_{l-1} \oplus a_{k-1} \oplus \varphi(a_k)$
a_{k+1}	a_{k-1}	a_{k-1}	$a_{k-1} \oplus \varphi(a_l) \oplus a_{k+1}$	$a_{k-1} \oplus a_{k+1} \oplus \varphi(a_l)$
a_{k+1}	a_{k+1}	a_{k+1}	$a_{k-1} \oplus a_{k+1} \oplus \varphi(a_l)$	$a_{k-1} \oplus a_{k+1} \oplus \varphi(a_l)$
a_{k+1}	a_{k+1}	a_{k-1}	$a_{k+1} \oplus a_{k-1} \oplus \varphi(a_k)$	$a_{k-1} \oplus a_{k+1} \oplus \varphi(a_l)$
a_{k+1}	a_{k-1}	a_{l+1}	$a_{k-1} \oplus a_{l+1} \oplus a_{k+1}$	$a_{k-1} \oplus a_{k+1} \oplus a_{l+1}$
a_{k+1}	a_{l-1}	a_{k-1}	$a_{l-1} \oplus a_{k-1} \oplus a_{k+1}$	$a_{l-1} \oplus a_{k-1} \oplus a_{k+1}$
a_{k+1}	a_{k+1}	a_{l+1}	$a_{k-1} \oplus a_{k+1} \oplus a_{l+1}$	$a_{k-1} \oplus a_{k+1} \oplus a_{l+1}$
a_{k+1}	a_{l-1}	a_{k+1}	$a_{k-1} \oplus a_{l-1} \oplus a_{k+1}$	$a_{l-1} \oplus a_{k-1} \oplus a_{k+1}$

We can easily see that $c_{k-1} = c'_{k-1}$ for all rows except the sixth row where $c_{k-1} = c'_{k-1}$ if $a_k = a_l$. But, since $r_{k-1,l+1} = r_{k+1,l-1} = 1$ and $r_{k-1,k+1} = 0$ in this case, we have, by the topological properties T2 and T3, that $r_{k,l} = 1$ and then $a_k = a_l$.

Now, to see that $c_{l-3} = c'_{l-3}$ we proceed in the same way as we did to show $c_{k-1} = c'_{k-1}$ for the case $\ddot{r}_{k-1,l-3} = 0$. For the case $\ddot{r}_{k-1,l-3} = 1$, we have $c_{l-3} = c_{k-1} = c'_{k-1} = c'_{l-3}$ (by lemma 11).

For another generic index i, we have:

If $\ddot{r}_{i,k-1} = 1$ then $c_i = c_{k-1} = c'_{k-1} = c'_i$.

If $\ddot{r}_{i,l-3} = 1$ then $c_i = c_{l-3} = c'_{l-3} = c'_i$.

If $\ddot{r}_{i,k-1} = \ddot{r}_{i,l-3} = 0$ then we have:

$$c_i = e_i^t R_3 * [(B_k^t * \overrightarrow{b}) \oplus (e_{k-1} * x_k)] = e_i^t B_k^t * \overrightarrow{b} = b_{\check{k}(i)}$$

$$\text{where} \quad \check{k}(i) = \begin{cases} i & \text{if} \quad i < k-1 \\ i+2 & \text{if} \quad i > k-1 \end{cases}$$

$\ddot{r}_{i,l-3} = 0 \Rightarrow \dot{r}_{\check{k}(i),l-1} = 0$, by lemma 11. Thus

$c_i = b_{\check{k}(i)} = e_{\check{k}(i)}^t R_2 * [(B_l^t * \overrightarrow{a}) \oplus (e_{l-1} * x_l)]\} = e_{\check{k}(i)}^t B_l^t * \overrightarrow{a} = e_i^t B_k^t B_l^t * \overrightarrow{a}$

$$c'_i = e_i^t R'_3 * [(B_{l-2}^t * \overrightarrow{b}') \oplus (e_{l-3} * x'_{l-2})]\} = e_i^t B_{l-2}^t * \overrightarrow{b}' = b'_{\check{l-2}(i)}$$

$\ddot{r}'_{i,k-1} = \ddot{r}_{i,k-1} = 0 \Rightarrow \dot{r}'_{\check{l-2}(i),k-1} = 0$, by lemma 11. Thus

$c'_i = b'_{\check{l-2}(i)} = e_{\check{l-2}(i)}^t R'_2 * [(B_k^t * \overrightarrow{a}) \oplus (e_{k-1} * x'_k)]\} = e_{\check{l-2}(i)}^t B_k^t * \overrightarrow{a} = e_i^t B_{l-2}^t B_k^t * \overrightarrow{a}$

Since $B_{l-2}^t B_k^t = B_k^t B_l^t$ we have $c_i = c'_i$.

5 Systems of non-singular planar curves

Now, we are going to study this representation in the particular case of systems of non-singular planar curves. By a system of non-singular planar curves we mean an immersion in the plane of a finite number of disjoint circles which may be regarded as a morphism from the empty partition of the line to itself (i.e. an element of $\hom(\emptyset, \emptyset)$).

Any morphism $t \in \hom(\emptyset, \emptyset)$ is a word made of generators $\hat{t}_{n,k}$ and $\check{t}_{n,k}$. Let us make the substitution: $\hat{t}_{n,k} \mapsto (2, 2k-n-3)$ and $\check{t}_{n,k} \mapsto (-2, 2k-n-3)$ for each word. The number $2k-n-3$ counts the number of strings on the left of the local maximum (minimum) minus the number of strings on the right.

The following lemma shows that no information is lost after this substitution. But first, let us introduce a notation for the following sets:

$$S_+ := \{(2, n) : n \in 2\mathbb{Z}\}$$

and

$$S_- := \{(-2, n) : n \in 2\mathbb{Z}\}$$

Lemma 13 *After the substitution: $\hat{t}_{n,k} \mapsto (2, 2k-n-3)$ and $\check{t}_{n,k} \mapsto (-2, 2k-n-3)$ of a morphism $t \in \hom(\emptyset, \emptyset)$ we get a word $(c_1, d_1)(c_2, d_2)...(c_n, d_n)$ in $S_+ \cup S_-$ satisfying the following condition:*

C. *For any index i:*

if $c_i = 2$ then $|d_i| \leq -(\sum_{j<i} c_j) - 2 = \sum_{j>i} c_j$,
if $c_i = -2$ then $|d_i| \leq -\sum_{j<i} c_j = (\sum_{j>i} c_j) - 2$.

Also, if a word in $S_+ \cup S_-$ satisfies this condition then there exists a unique morphism $t \in \hom(\emptyset, \emptyset)$ that becomes this word after the substitution: $\hat{t}_{n,k} \mapsto (2, 2k-n-3)$ and $\check{t}_{n,k} \mapsto (-2, 2k-n-3)$.

Proof. Observing that each (c_i, d_i) substitutes one generator $\hat{t}_{n_i,k_i}$ or $\check{t}_{n_i,k_i}$ it easy to see that $-\sum_{j<i} c_j$ is the number of points at the bottom of the generator ($\hat{t}_{n_i,k_i}$ or $\check{t}_{n_i,k_i}$) and that $\sum_{j>i} c_j$ is number of points at the top of the generator. The condition follows naturally from this fact.

For the second part of the lemma, we make the inverse substitution $(c_i, d_i) = (2, k) \mapsto \hat{t}_{n',k'}$ with $n' = (\sum_{j>i} c_j) + 1$ and $k' = \frac{k+n'+3}{2}$, and $(c_i, d_i) = (-2, k) \mapsto \check{t}_{n',k'}$ with $n' = (\sum_{j>i} c_j) - 1$ and $k' = \frac{k+n'+3}{2}$. ■

So we have a new language for morphisms in $\hom(\emptyset, \emptyset)$ and in this language the local relations become:

1 $...(c_i, d_i)(-2, k)(2, k{+}2)(c_{i+3}, d_{i+3})... = ...(c_i, d_i)(c_{i+3}, d_{i+3})... = ...(c_i, d_i)(-2, k)$ $(2, k-2)(c_{i+3}, d_{i+3})...$ (i.e. $\check{t}_{k'+1} \circ \hat{t}_{k'} = id = \check{t}_{k'-1} \circ \hat{t}_{k'}$);

2 $...(2, k)(2, l)... = ...(2, l+2)(2, k+2)...$ for $k \leq l-2$ (i.e. $\hat{t}_{k'} \circ \hat{t}_{l'} = \hat{t}_{l'+2} \circ \hat{t}_{k'}$ for $l' \geq k'+2$);

3.1 $...(-2, k)(2, l)... = ...(2, l-2)(-2, k+2)...$ for $k \leq l-4$ (i.e. $\check{t}_{k'} \circ \hat{t}_{l'} = \hat{t}_{l'-2} \circ \check{t}_{k'}$ for $l' \geq k'+2$)

3.2 $...(2, k)(-2, l)... = ...(-2, l{+}2)(2, k{-}2)...$ for $k \leq l$ (i.e. $\hat{t}_{k'} \circ \check{t}_{l'} = \check{t}_{l'+2} \circ \hat{t}_{k'}$ for $l' \geq k'+2$)

4 $...(-2, k)(-2, l)... = ...(-2, l-2)(-2, k-2)$ for $k \leq l-2$ (i.e. $\check{t}_{l'-2} \circ \check{t}_{k'} = \check{T}_{k'} \circ \check{T}_{l'}$ for $l' \geq k'+2$).

Note that, by the previous lemma, these relations preserve the condition C.

Proposition 14 *Any word in $S_+ \cup S_-$ satisfying the condition C is equivalent, by the previous relations, to a word of symbols $(2, 0)$ and $(-2, 0)$.*

Proof. We use the following algorithm to transform any word in $S_+ \cup S_-$ satisfying the condition C into a word of symbols $(2, 0)$ and $(-2, 0)$.

ALGORITHM:

input: Take a word in $S_+ \cup S_-$ satisfying the condition C;

step 1. Apply the relations 3.1 ($...(2, l-2)(-2, k+2)... = ...(-2, k)(2, l)...$ for $k \leq l-4$) and 3.2 ($...(2, k)(-2, l)... = ...(-2, l+2)(2, k-2)...$ for $k \leq l$) so as to put all the symbols of the form $(-2, k)$ on the left side of the word and all the symbols of the form $(2, k)$ on the right;

step 2. Use the relations 2 ($...(2, k)(2, l)... = ...(2, l+2)(2, k+2)...$ for $k \leq l-2$) and 4 ($...(-2, k)(-2, l)... = ...(-2, l{-}2)(-2, k{-}2)$ for $k \leq l{-}2$) to order in semi-decreasing order (according to d_i) each block consisting of (c_i, d_i) of the form $(2, k)$ or $(-2, k)$;

step 3. If there exists a sequence of the type $(-2,k)(2,k+2)$ in the word we apply the relation 1 $(...(c_i,d_i)(-2,k)(2,k+2)(c_{i+3},d_{i+3})... = ...(c_i,d_i)(c_{i+3},d_{i+3})...)$ and go back to step 1;

step 4. If there exists a sequence of the type $(-2,k)(2,l)$ with $k \leq l-4$ then we apply the relation 3.1 $(...(-2,k)(2,l)... = ...(2,l-2)(-2,k+2)...)$ and go back to step 3;

step 5. If there doesn't exist a sequence of the type $(-2,k)(2,l)$ with $k \leq l-2$ in the word we **output** this.

Claim 1. The algorithm always terminates in a finite number of steps.

We will prove this by induction on the number of symbols since the algorithm doesn't increase this.

It easy to see that any word satisfying the condition C has the same number of symbols of the form $(2,k)$ as of the form $(-2,k)$, and always begins with $(-2,0)$ and ends with $(2,0)$. So $(-2,0)(2,0)$ is the unique word of two symbols and this passes through the algorithm without any changes.

Now, assuming that the algorithm terminates for any word with $2n$ symbols, we take a word with $2n+2$ symbols. We consider for any word $w = (c_1,d_1)(c_2,d_2)...(c_n,d_n)$ the "potential"

$$E(w) = (\sum_{c_i=2} i, \sum_{c_i=-2} d_i - \sum_{c_i=2} d_i)$$

which takes values in $\mathbb{Z}^2$ with lexicographic order (i.e. $(a,b) \leq (c,d)$ iff $a \leq c$ or $a = c$ and $b \leq d$). We can see that, after being increased in step 1, the potential of the word is always decreased until (if it occurs) the algorithm returns to step 1 (after step 3), but in this case the word has $2n$ symbols and then by the induction hypothesis the algorithm terminates. The condition C implies that there are a finite number of potentials for a word with $2n+2$ symbols (or less), and thus the algorithm terminates in a finite number of steps.

Claim 2. The output word has only $(2,0)$ and $(-2,0)$ as symbols.

For a word $w = (c_1,d_1)...(c_{2n},d_{2n})$ let $\alpha_1 < ... < \alpha_n$ be the indices such that $c_{\alpha_i} = 2$ and $\beta_1 < ... < \beta_n$ be the indices such that $c_{\beta_i} = -2$. We consider a new "potential" E_2 defined by the formula:

$$E_2(w) = \max\left(\{d_{\alpha_{i+1}} - d_{\alpha_i} - 2(\alpha_{i+1} - \alpha_i - 1) : 1 \leq i \leq n-1\} \cup\right.$$

$$\{d_{\beta_{i+1}} - d_{\beta_i} - 2(\beta_{i+1} - \beta_i - 1) : 1 \leq i \leq n-1\})$$

After step 2 the potential of the word becomes non-positive and step 4 doesn't change this. This means that the output word contains no sequence of the type $(2,k)(2,l)$ or $(-2,k)(-2,l)$ with $k < l$. It is a condition of the algorithm that the output word contains no sequence of the type $(-2,k)(2,l)$ with $k < l$.

Thus, if it does not contain a sequence of the type $(2,k)(-2,l)$ with $k < l$, then the word is ordered by semi-decreasing order and, since it has to begin with $(-2,0)$ and to end with $(2,0)$, the word has only $(2,0)$ and $(-2,0)$ as symbols.

To show that the output word contains no sequence of the type $(2,k)(-2,l)$ with $k < l$, it is enough to observe that after step 1 there is no sequence of the type $(2,k)(-2,l)$ and that steps 2 and 4 do not produce any new sequences of the type $(2,k)(-2,l)$ with $k < l$. ■

> NOTE: This algorithm was not conceived to be the most efficient but to guarantee an easy proof that it terminates. The author conjectures that there exist more efficient algorithms.

Next, we observe that the family of non-singular planar curves (i.e. $\hom(\emptyset, \emptyset)$) together with the composition has a structure of a commutative monoid. We will see that any irreducible element of this monoid (i.e. a system of curves that is not a composition of other systems of curves) is a system of curves encircled by another curve. Note that to encircle a system of curves by another curve is, in the $(\pm 2, k)$ words language, the same as adding a $(-2,0)$ at the beginning and a $(2,0)$ at the end of the corresponding word. Thus if a word $(c_1,0)(c_2,0)...(c_n,0)$ of symbols $(-2,0)$ and $(2,0)$[7] satisfying the condition C (and thus $(c_1,0) = (-2,0)$ and $(c_n,0) = (2,0)$) doesn't come from a system of curves encircled by another curve (this means that the word $(c_2,0)...(c_{n-1},0)$ doesn't satisfy the condition C) then there exists $2 < k < n$ such that $\sum_{j<k} c_j = 0$, which implies that $(c_{k-1},0) = (2,0)$, $(c_k,0) = (-2,0)$ and the words $(c_1,0)...(c_{k-1},0)$ and $(c_k,0)...(c_n,0)$ satisfy the condition C. Therefore the morphism in $\hom(\emptyset,\emptyset)$ (corresponding to the word $(c_1,0)...(c_n,0)$) is the composition of two morphisms in $\hom(\emptyset,\emptyset)$ (corresponding to the words $(c_1,0)...(c_{k-1},0)$ and $(c_k,0)...(c_n,0)$).

[7] By the previous proposition, any system of non-singular planar curves can be represented in this way.

Note that we have just proved that an irreducible morphism is another morphism encircled by an exterior curve, but we haven't proved yet that a morphism encircled by an exterior curve is an irreducible morphism. This is because the last proposition says nothing about when two words of symbols $(-2,0)$ and $(2,0)$ represent equivalent words (by the relations 1, 2, 3 and 4).

However we do know that, using the composition of morphisms and the operation of encircling, we can generate all morphisms in $\hom(\emptyset,\emptyset)$ from the identity.

With this in mind, we are going to see what happens in the representation when we compose two morphisms (see corollary 17) or encircle one by a circle (see corollary 21).

Consider, for a fixed value $\mathbf{m} \in \mathbb{M}$ and for each natural number n, the following morphism:

$$\begin{array}{rccc} \Psi_{n,\mathbf{m}}: & \mathcal{O}_n & \longrightarrow & \mathcal{O}_n \\ & (R, \overrightarrow{v}) & \longmapsto & (R, R * (e_1 * \mathbf{m} \oplus \overrightarrow{v})) \end{array}$$

The purpose of this morphism is to add the value $\mathbf{m}$ to the region associated to the first interval.

Lemma 15 *For any* $2 \leq k \leq n+1$, $\hat{T}_{n,k} \circ \Psi_{n,\mathbf{m}} = \Psi_{n+2,\mathbf{m}} \circ \hat{T}_{n,k}$ *and* $\check{T}_{n,k} \circ \Psi_{n+2,\mathbf{m}} = \Psi_{n,\mathbf{m}} \circ \check{T}_{n,k}$

Proof.

First equation: $\hat{T}_{n,k} \circ \Psi_{n,\mathbf{m}} = \Psi_{n+2,\mathbf{m}} \circ \hat{T}_{n,k}$.

For an arbitrary $(R, \overrightarrow{v}) \in \mathcal{O}_n$, let $(R_1, \overrightarrow{v}_1) = \Psi_{n,\mathbf{m}}(R, \overrightarrow{v})$, $(R_2, \overrightarrow{v}_2) = \hat{T}_{n,k}(R_1, \overrightarrow{v}_1)$, $(R'_1, \overrightarrow{v}'_1) = \hat{T}_{n,k}(R, \overrightarrow{v})$ and $(R'_2, \overrightarrow{v}'_2) = \Psi_{n+2,\mathbf{m}}(R'_1, \overrightarrow{v}'_1)$. We want to check $(R_2, \overrightarrow{v}_2) = (R'_2, \overrightarrow{v}'_2)$.

Since Ψ does not change the matrices and the changes of the matrices under the morphisms $\hat{T}$ do not depend on the choice of the array of values, it is obvious that $R_2 = R'_2$.

$$
\begin{aligned}
\overrightarrow{v}_2 &= B_{n,k} * \overrightarrow{v}_1 \\
&= B_{n,k} * [R * (e_1 * \mathbf{m} \oplus \overrightarrow{v}_1)] \\
&= B_{n,k} R * (e_1 * \mathbf{m} \oplus \overrightarrow{v}_1) \\
&= B_{n,k} R B^t_{n,k} B_{n,k} * (e_1 * \mathbf{m} \oplus \overrightarrow{v}_1) \\
&= [B_{n,k} R B^t_{n,k} B_{n,k} * (e_1 * \mathbf{m} \oplus \overrightarrow{v}_1)] \vee [D_{n+2,k} B_{n,k} * (e_1 * \mathbf{m} \oplus \overrightarrow{v}_1)] \\
&= (B_{n,k} R B^t_{n,k} + D_{n+2,k}) * [B_{n,k} * (e_1 * \mathbf{m} \oplus \overrightarrow{v}_1)] \\
&= R'_1 * [(B_{n,k} e_1 * \mathbf{m}) \oplus (B_{n,k} * \overrightarrow{v}_1)] \\
&= R'_1 * [(B_{n,k} e_1 * \mathbf{m}) \oplus \overrightarrow{v}'_1]
\end{aligned}
$$

On the other hand,

$$
\overrightarrow{v}'_2 = R'_1 * (e_1 * \mathbf{m} \oplus \overrightarrow{v}'_1)
$$

If $k > 2$ then $B_{n,k} e_1 = e_1$ and therefore $\overrightarrow{v}_2 = \overrightarrow{v}'_2$.
If $k = 2$ then

$$
\begin{aligned}
\overrightarrow{v}_2 &= R'_1 * [(B_{n,k} e_1 * \mathbf{m}) \oplus \overrightarrow{v}'_1] \\
&= R'_1 * [(e_1 * \mathbf{m} \vee e_3 * \mathbf{m}) \oplus \overrightarrow{v}'_1] \\
&= R'_1 * (e_1 * \mathbf{m} \oplus \overrightarrow{v}'_1) \vee R'_1 * (e_3 * \mathbf{m} \oplus \overrightarrow{v}'_1) \\
&= R'_1 * (e_1 * \mathbf{m} \oplus \overrightarrow{v}'_1) \\
&= \overrightarrow{v}'_2
\end{aligned}
$$

because $R'_1 * (e_3 * \mathbf{m} \oplus \overrightarrow{v}'_1) = R'_1 * (e_1 * \mathbf{m} \oplus \overrightarrow{v}'_1)$ as we will prove next:

$$
\begin{aligned}
R'_1 e_1 &= (B_{n,2} R B^t_{n,2} + D_{n+2,2}) e_1 &&= B_{n,2} R B^t_{n,2} e_1 \vee D_{n+2,2} e_1 &&= B_{n,2} R e_1 \\
R'_1 e_3 &= (B_{n,2} R B^t_{n,2} + D_{n+2,2}) e_3 &&= B_{n,2} R B^t_{n,2} e_3 \vee D_{n+2,2} e_3 &&= B_{n,2} R e_1
\end{aligned}
$$

thus $R'_1 e_1 * \mathbf{m} = R'_1 e_3 * \mathbf{m}$, and using the next lemma we conclude $R'_1 * (e_3 * \mathbf{m} \oplus \overrightarrow{v}'_1) = R'_1 * (e_1 * \mathbf{m} \oplus \overrightarrow{v}'_1)$.

Lemma 16 *If $R = R^t$ and $R * \overrightarrow{v} = \overrightarrow{v}$ then $R * (\overrightarrow{u} \oplus \overrightarrow{v}) = R * \overrightarrow{u} \oplus \overrightarrow{v}$.*

Proof. Let $\overrightarrow{x} := R * (\overrightarrow{u} \oplus \overrightarrow{v})$ and $\overrightarrow{y} := R * \overrightarrow{u} \oplus \overrightarrow{v}$, then

$$
x_i = \bigvee_j r_{i,j} * (u_j \oplus v_j) \quad \text{and} \quad y_i = (\bigvee_j r_{i,j} * u_j) \oplus v_i
$$

$$\begin{array}{rcl} x_i & = & \bigvee_j r_{i,j} * (u_j \oplus v_j) \\ & = & \bigvee_{j:r_{i,j}=1}(u_j \oplus v_j) \\ & = & \bigvee_{j:r_{i,j}=1}(u_j \oplus v_i) \\ & = & (\bigvee_{j:r_{i,j}=1} u_j) \oplus v_i \\ & = & (\bigvee_j r_{i,j} * u_j) \oplus v_i \\ & = & y_i \end{array}$$

Note that, since $R * \overrightarrow{v} = \overrightarrow{v}$, we have that $v_j = v_i$ whenever $r_{i,j} = 1$. ■

Second equation $\check{T}_{n,k} \circ \Psi_{n+2,\mathbf{m}} = \Psi_{n,\mathbf{m}} \circ \check{T}_{n,k}$.

For an arbitrary $(R, \overrightarrow{v}) \in \mathcal{O}_n$, let $(R_1, \overrightarrow{a}) = \Psi_{n+2,\mathbf{m}}(R, \overrightarrow{v})$, $(R_2, \overrightarrow{b}) = \check{T}_{n,k}(R_1, \overrightarrow{a})$, $(R_1', \overrightarrow{a}') = \check{T}_{n,k}(R, \overrightarrow{v})$ and $(R_2', \overrightarrow{b}') = \Psi_{n,\mathbf{m}}(R_1', \overrightarrow{a}')$. We want to check $(R_2, \overrightarrow{b}) = (R_2', \overrightarrow{b}')$.

Since Ψ does not change the matrices and the changes of the matrices under the morphisms $\check{T}$ do not depend on the choice of the array of values, it is obvious that $R_2 = R_2'$.

$$\overrightarrow{b} = R_2 * [(B_k^t * \overrightarrow{a}) \oplus e_{k-1} * \tilde{x}_k]$$

with

$$\overrightarrow{a} = R * (e_1 * \mathbf{m} \oplus \overrightarrow{v}) = Re_1 * \mathbf{m} \oplus \overrightarrow{v}$$

and

$$\tilde{x}_k = \begin{cases} a_{k-1} \wedge a_{k+1} & \text{if} \quad r_{k-1,k+1} = 0 \\ \varphi(a_k) & \text{if} \quad r_{k-1,k+1} = 1 \end{cases}$$

On the other hand

$$\overrightarrow{b}' = R_2' * (e_1 * \mathbf{m} \oplus \overrightarrow{a}') = R_2' e_1 * \mathbf{m} \oplus \overrightarrow{a}'$$

We want to check $\overrightarrow{b} = \overrightarrow{b}'$.

A) $b_{k-1} = b_{k-1}'$.

By lemma 11, we have

$$b_{k-1} = (a_{k-1} \vee a_{k+1}) \oplus \tilde{x}_k = \begin{cases} a_{k-1} \oplus a_{k+1} & \text{if} \quad r_{k-1,k+1} = 0 \\ a_{k-1} \oplus \varphi(a_k) & \text{if} \quad r_{k-1,k+1} = 1 \end{cases}$$

Bearing in mind that

$$\begin{aligned} a_{k-1} &= e_{k-1}^t * (Re_1 * \mathbf{m} \oplus \overrightarrow{v}) = r_{k-1,1} * \mathbf{m} \oplus v_{k-1} \\ a_{k+1} &= e_{k+1}^t * (Re_1 * \mathbf{m} \oplus \overrightarrow{v}) = r_{k+1,1} * \mathbf{m} \oplus v_{k+1} \\ a_k &= e_k^t * (Re_1 * \mathbf{m} \oplus \overrightarrow{v}) = r_{k,1} * \mathbf{m} \oplus v_k \end{aligned}$$

we have that

$$b_{k-1} = \begin{cases} (r_{k-1,1} * \mathbf{m}) \oplus v_{k-1} \oplus (r_{k+1,1} * \mathbf{m}) \oplus v_{k+1} & \text{if} \quad r_{k-1,k+1} = 0 \\ (r_{k-1,1} * \mathbf{m}) \oplus v_{k-1} \oplus \varphi(r_{k,1} * \mathbf{m} \oplus v_k) & \text{if} \quad r_{k-1,k+1} = 1 \end{cases}$$

Now we note that $r_{k-1,k+1} = 0$ implies that $r_{k-1,1} = 0$ or $r_{k+1,1} = 0$ and therefore $(r_{k-1,1} * \mathbf{m}) \oplus (r_{k+1,1} * \mathbf{m}) = (r_{k-1,1} + r_{k+1,1}) * \mathbf{m}$. On the other hand $r_{k-1,k+1} = 1$ implies that $r_{k,1} = 0$ and $r_{k-1,1} = r_{k+1,1} = r_{k-1,1} + r_{k+1,1}$.
Thus

$$b_{k-1} = \begin{cases} (r_{k-1,1} + r_{k+1,1}) * \mathbf{m} \oplus v_{k-1} \oplus v_{k+1} & \text{if} \quad r_{k-1,k+1} = 0 \\ (r_{k-1,1} + r_{k+1,1}) * \mathbf{m} \oplus v_{k-1} \oplus \varphi(v_k) & \text{if} \quad r_{k-1,k+1} = 1 \end{cases}$$

That means

$$b_{k-1} = (r_{k-1,1} + r_{k+1,1}) * \mathbf{m} \oplus a'_{k-1}$$

by lemma 11. Now

$$\overrightarrow{b}' = R'_1 * (e_1 * \mathbf{m} \oplus \overrightarrow{a}') = R'_1 e_1 * \mathbf{m} \oplus \overrightarrow{a}'$$

i.e.

$$\begin{aligned} b'_{k-1} &= e_{k-1}^t * (R'_1 e_1 * \mathbf{m} \oplus \overrightarrow{a}') \\ &= e_{k-1}^t R'_1 e_1 * \mathbf{m} \oplus a'_{k-1} \\ &= r'_{k-1,1} * \mathbf{m} \oplus a'_{k-1} \end{aligned}$$

and using lemma 12 we have

$$\begin{aligned} b'_{k-1} &= (r_{k-1,1} + r_{k+1,1}) * \mathbf{m} \oplus a'_{k-1} \\ &= b_{k-1} \end{aligned}$$

B) $b_i = b'_i$ with $r'_{i,k-1} = 0$.

By lemma 11, $b_i = e_i^t B_k^t * \overrightarrow{a}$.

$$\begin{aligned} b_i &= e_i^t B_k^t * (Re_1 * \mathbf{m} \oplus \overrightarrow{v}) \\ &= e_i^t B_k^t Re_1 * \mathbf{m} \oplus e_i^t B_k^t \overrightarrow{v} \\ &= r_{\check{k}(i),1} * \mathbf{m} \oplus e_i^t B_k^t \overrightarrow{v} \end{aligned}$$

On the other hand,

$$\begin{aligned} b'_i &= e_i^t (R'_1 e_1 * \mathbf{m} \oplus \overrightarrow{a}') \\ &= e_i^t R'_1 e_1 * \mathbf{m} \oplus e_1^t \overrightarrow{a}') \\ &= r'_{i,1} * \mathbf{m} \oplus e_i^t B_k^t \overrightarrow{v} \end{aligned}$$

Thus we have $b_i = b'_i$ if $r'_{i,1} = r_{\check{k}(i),1}$.

If $k-1 \neq 1$ then, by lemma 12, $r'_{i,1} = r_{\check{k}(i),\check{k}(1)} = r_{\check{k}(i),1}$.

If $k-1 = 1$ then, by lemma 12, $r'_{i,1} = r'_{i,k-1} \geq r_{\check{k}(i),k-1} = r_{\check{k}(i),1}$ and since $r'_{i,1} = r'_{i,k-1} = 0$ we have $r'_{i,1} = r_{\check{k}(i),1}$.

C) $b_i = b'_i$ with $r'_{i,k-1} = 1$.

Since, by lemma 11, $b_i = b_{k-1}$ and $b'_i = b'_{k-1}$ this shows that $b_i = b'_i$.

■

Corollary 17 **1.** *For any* $T \in \hom(\mathcal{O}_m, \mathcal{O}_n)$*,* $T \circ \Psi_{m,\mathbf{m}} = \Psi_{n,\mathbf{m}} \circ T$*;*

2. *For any* $T \in \hom(\mathcal{O}_1, \mathcal{O}_1)$ *(corresponding to a system of non-singular planar curves)* $T([1], \emptyset) = ([1], \mathbf{x}) \quad \Rightarrow \quad T([1], \mathbf{m}) = ([1], \mathbf{x} \oplus \mathbf{m})$*;*

3. *For any* $T_1, T_2 \in \hom(\mathcal{O}_1, \mathcal{O}_1)$ $T_2 \circ T_1([1], \emptyset) = ([1], \mathbf{m_2} \oplus \mathbf{m_1})$ *where* $([1], \mathbf{m_i}) = T_i([1], \emptyset)$*,* $i = 1, 2$*.*

Proof.

1. It is obvious that if $\{\Psi_{n,\mathbf{m}}\}_{n \in \mathbb{N}}$ commute with the generators $\hat{T}_{n,k}$ and $\check{T}_{n,k}$ then they commute with any morphism T.

2. $T([1], \mathbf{m}) = T \circ \Psi_{1,\mathbf{m}}([1], \emptyset) = \Psi_{1,\mathbf{m}} \circ T([1], \emptyset) = \Psi_{1,\mathbf{m}}([1], \mathbf{x}) = ([1], \mathbf{x} \oplus \mathbf{m})$.

3. $T_2 \circ T_1([1], \emptyset) = T_2([1], \mathbf{m_1}) = ([1], \mathbf{m_2} \oplus \mathbf{m_1})$.

■

This result shows that all information about a morphism $T \in \hom(\mathcal{O}_1, \mathcal{O}_1)$ is contained in a single value $\mathbf{v} \in \mathbb{M}$ (which is obtained evaluating the morphism T on $([1], \emptyset)$). Thus, we can associate a value in $\mathbb{M}$ to each system of non-singular planar curves. Moreover this association is a monoid homomorphism.

Next we will look at what value is obtained when a collection of planar curves is encircled by another curve.

Consider, for each n, the following subset of $\mathcal{O}_n$

$$\mathcal{U}_n = \{(R, \overrightarrow{v}) \in \mathcal{O}_n : e_1^t R e_n = 1\}$$

By the property T1 we have that $\mathcal{U}_n = \emptyset$ for any even number n. It is clear that $\mathcal{U}_1 = \mathcal{O}_1$ and therefore it easy to see that $\mathcal{U}_n \neq \emptyset$ for any odd number n, using the following result.

Proposition 18 *For any n and k, if $(R, \overrightarrow{v}) \in \mathcal{U}_n$ then $\hat{T}_k(R, \overrightarrow{v}) \in \mathcal{U}_{n+2}$ and $\check{T}_k(R, \overrightarrow{v}) \in \mathcal{U}_{n-2}$.*

Proof. Let $(R', \overrightarrow{v}') = \hat{T}_k(R, \overrightarrow{v})$. We want to see that if $e_1^t R e_n = 1$ then $e_1^t R' e_{n+2} = 1$.

$$\begin{aligned} e_1^t R' e_{n+2} &= e_1^t (B_k R B_k^t + D_k) e_{n+2} \\ &= e_1^t B_k R B_k^t e_{n+2} \\ &= e_1^t R e_n = 1 \end{aligned}$$

Let $(R', \overrightarrow{v}') = \check{T}_k(R, \overrightarrow{v})$. We want to see that if $e_1^t R e_n = 1$ then $e_1^t R' e_{n-2} = 1$.

$$\begin{aligned} e_1^t R' e_{n-2} &= e_1^t (B_k^t R B_k)^2 e_{n-2} \\ &\geq e_1^t B_k^t R B_k e_{n-2} \\ &\geq e_1^t R e_n = 1 \end{aligned}$$

■

Now we consider, for each odd natural number n, the following morphism:

$$\begin{array}{rccc} \varepsilon_n : & \mathcal{U}_n & \longrightarrow & \mathcal{U}_{n+2} \\ & (R, \overrightarrow{v}) & \longmapsto & (\tilde{R}, \overrightarrow{\tilde{v}}) \end{array}$$

with

$$\tilde{R} = E_n R E_n^t + F_{n+2}$$

and

$$\overrightarrow{\tilde{v}} = E_n * \overrightarrow{v}$$

where $E_n = [e_{i,j}]$ is an $(n+2) \times n$ matrix defined by $e_{i,j} = 1 \Leftrightarrow i-1 = j$ and $F_{n+2} = [f_{i,j}]$ is an $(n+2)\times(n+2)$ matrix defined by $f_{i,j} = 1 \Leftrightarrow i, j \in \{1, n+2\}$.

Proposition 19 *For any odd natural number n, ε_n is well defined.*

Proof. We need to see that if $(R, \overrightarrow{v}) \in \mathcal{U}_n$ then $\varepsilon_n(R, \overrightarrow{v}) \in \mathcal{U}_{n+2}$.

Let $(\tilde{R}, \overrightarrow{\tilde{v}}) = \varepsilon_n(R, \overrightarrow{v})$. We are going to check that $R * \overrightarrow{v} = \overrightarrow{v}$ implies $\tilde{R} * \overrightarrow{\tilde{v}} = \overrightarrow{\tilde{v}}$, $\tilde{R}$ satisfies the properties E1, E2, E3, T1, T2 and T3 and the $(1, n+2)$ entry of $\tilde{R}$ is 1.

For this purpose, we are going to use the identities: $E_n^t E_n = I$, $E_n^t F_{n+2} = O$ and $F_{n+2} E_n = O$, which we will leave to the reader to check.

$$\begin{aligned} R * \overrightarrow{v} = \overrightarrow{v} \Rightarrow \tilde{R} * \overrightarrow{\tilde{v}} &= (E_n R E_n^t + F_{n+2}) * (E_n * \overrightarrow{v}) \\ &= (E_n R E_n^t E_n) * \overrightarrow{v} \vee (F_{n+2} E_n) * \overrightarrow{v} \\ &= E_n R * \overrightarrow{v} \\ &= E_n * \overrightarrow{v} \\ &= \overrightarrow{\tilde{v}} \end{aligned}$$

The properties E1, E2 and E3 of $\tilde{R}$ are very easy to verify, so we leave them as an exercise.

For the properties T1, T2 and T3, we observe that the (i, j) entry of $\tilde{R}$ is the $(i-1, j-1)$ entry of R if $2 \leq i, j \leq n+1$, 1 if $i, j \in \{1, n+2\}$ and 0 in all other cases, i.e.:

$$\tilde{r}_{i,j} = e_i^t (E_n R E_n^t + F_{n+2}) e_j = \begin{cases} r_{i-1,j-1} & \text{if} \quad 2 \leq i, j \leq n+1 \\ 1 & \text{if} \quad i, j \in \{1, n+2\} \\ 0 & \text{otherwise} \end{cases}$$

In particular, we have $\tilde{r}_{1,n+2} = 1$.

Property T1: $\tilde{r}_{i,j} = 1 \Rightarrow j - i \in 2\mathbb{Z}$.

We have $\tilde{r}_{i,j} = 1 \Rightarrow i, j \in \{1, n+2\}$ or $r_{i-1,j-1} = 1$. Since we have taken n to be odd and R satisfies T1 we have $j - i \in 2\mathbb{Z}$.

Property T2: $\forall_{\alpha \leq \beta \leq \gamma \leq \delta} \quad \tilde{r}_{\alpha,\gamma} = \tilde{r}_{\beta,\delta} = 1 \Rightarrow \tilde{r}_{\alpha,\beta} = \tilde{r}_{\beta,\gamma} = \tilde{r}_{\gamma,\delta} = 1$.

We only need consider the case $\alpha < \beta < \gamma < \delta$, and then $\tilde{r}_{\alpha,\gamma} = \tilde{r}_{\beta,\delta} = 1 \Rightarrow 2 \leq \alpha < \beta < \gamma < \delta \leq n+1$. Thus $\tilde{r}_{\alpha,\gamma} = r_{\alpha-1,\gamma-1}$ and $\tilde{r}_{\beta,\delta} = r_{\beta-1,\delta-1}$, hence, since R satisfies T2, we have $r_{\alpha-1,\beta-1} = r_{\beta-1,\gamma-1} = r_{\gamma-1,\delta-1} = 1$, that is $\tilde{r}_{\alpha,\beta} = \tilde{r}_{\beta,\gamma} = \tilde{r}_{\gamma,\delta} = 1$.

Property T3: $\forall_{\alpha<\beta} \quad \tilde{r}_{\alpha,\beta} = 1 \Rightarrow \tilde{r}_{\alpha+1,\beta-1} = 1$ or $\exists_{\alpha<\gamma<\beta} : \tilde{r}_{\alpha,\gamma} = 1$.

If $\alpha = 1$ then $\tilde{r}_{\alpha,\beta} = 1 \Rightarrow \beta = n+2$. Thus $\tilde{r}_{\alpha+1,\beta-1} = \tilde{r}_{2,n+1} = r_{1,n} = 1$ since $R \in \mathcal{U}_n$[8].

If $\alpha > 1$ then $\tilde{r}_{\alpha,\beta} = 1 \Rightarrow \beta < n+2$. Thus

$$\begin{array}{rcll} \tilde{r}_{\alpha,\beta} = r_{\alpha-1,\beta-1} & \Rightarrow & r_{\alpha,\beta-2} = 1 & \text{or} \quad \exists_{\alpha-1<\gamma-1<\beta-1} : r_{\alpha-1,\gamma-1} = 1 \\ & \Rightarrow & \tilde{r}_{\alpha+1,\beta-1} = 1 & \text{or} \quad \exists_{\alpha<\gamma<\beta} : \tilde{r}_{\alpha,\gamma} = 1 \end{array}$$

■

Lemma 20 *For any $2 \leq k \leq n+1$, we have the identities $\hat{T}_{n+2,k+1} \circ \varepsilon_n = \varepsilon_{n+2} \circ \hat{T}_{n,k}$ and $\check{T}_{n,k+1} \circ \varepsilon_n = \varepsilon_{n-2} \circ \check{T}_{n-2,k}$*

Proof.

First identity: $\hat{T}_{n+2,k+1} \circ \varepsilon_n = \varepsilon_{n+2} \circ \hat{T}_{n,k}$.

For an arbitrary $(R, \overrightarrow{v}) \in \mathcal{U}_n$, let $(R_1, \overrightarrow{v}_1) = \varepsilon_n(R, \overrightarrow{v})$, $(R_2, \overrightarrow{v}_2) = \hat{T}_{n+2,k+1}(R_1, \overrightarrow{v}_1)$, $(R'_1, \overrightarrow{v}'_1) = \hat{T}_{n,k}(R, \overrightarrow{v})$ and $(R'_2, \overrightarrow{v}'_2) = \varepsilon_{n+2}(R'_1, \overrightarrow{v}'_1)$. We want to check $(R_2, \overrightarrow{v}_2) = (R'_2, \overrightarrow{v}'_2)$.

$$\begin{array}{rcl} R_2 & = & B_{n+2,k+1} R_1 B^t_{n+2,k+1} + D_{n+4,k+1} \\ & = & B_{n+2,k+1}(E_n R E^t_n + F_{n+2}) B^t_{n+2,k+1} + D_{n+4,k+1} \\ & = & B_{n+2,k+1} E_n R E^t_n B^t_{n+2,k+1} + B_{n+2,k+1} F_{n+2} B^t_{n+2,k+1} + D_{n+4,k+1} \\ & = & E_{n+2} B_{n,k} R B^t_{n,k} E^t_{n+2} + F_{n+4} + E_{n+2} D_{n+2,k} E^t_{n+2} \\ & = & E_{n+2}(B_{n,k} R B^t_{n,k} + D_{n+2,k}) E^t_{n+2} + F_{n+4} \\ & = & E_{n+2} R'_1 E^t_{n+2} + F_{n+4} \\ & = & R'_2 \end{array}$$

$$\begin{array}{rcl} \overrightarrow{v}_2 & = & B_{n+2,k+1} * \overrightarrow{v}_1 \\ & = & B_{n+2,k+1} * (E_n * \overrightarrow{v}) \\ & = & (B_{n+2,k+1} E_n) * \overrightarrow{v} \\ & = & (E_{n+2} B_{n,k}) * \overrightarrow{v} \\ & = & E_{n+2} * (B_{n,k} * \overrightarrow{v}) \\ & = & E_{n+2} * \overrightarrow{v}'_1 \\ & = & \overrightarrow{v}'_2 \end{array}$$

[8]This is an abuse of notation, since we should write $(R, \overrightarrow{v}) \in \mathcal{U}_n$.

Second identity $\check{T}_{n,k+1} \circ \varepsilon_n = \varepsilon_{n-2} \circ \check{T}_{n-2,k}$.

For an arbitrary $(R, \overrightarrow{v}) \in \mathcal{U}_n$, let $(R_1, \overrightarrow{a}) = \varepsilon_n(R, \overrightarrow{v})$, $(R_2, \overrightarrow{b}) = \check{T}_{n,k+1}(R_1, \overrightarrow{a})$, $(R_1', \overrightarrow{a}') = \check{T}_{n-2,k}(R, \overrightarrow{v})$ and $(R_2', \overrightarrow{b}') = \varepsilon_{n-2}(R_1', \overrightarrow{a}')$. We want to check $(R_2, \overrightarrow{b}) = (R_2', \overrightarrow{b}')$.

$$
\begin{aligned}
R_2 &= [B^t_{n,k+1} R_1 B_{n,k+1}]^2 \\
&= [B^t_{n,k+1}(E_n R E_n^t + F_{n+2}) B_{n,k+1}]^2 \\
&= [B^t_{n,k+1} E_n R E_n^t B_{n,k+1} + B^t_{n,k+1} F_{n+2} B_{n,k+1}]^2 \\
&= (E_{n-2} B^t_{n-2,k} R B_{n-2,k} E^t_{n-2} + F_n)^2 \\
&= E_{n-2} B^t_{n-2,k} R B_{n-2,k} E^t_{n-2} E_{n-2} B^t_{n-2,k} R B_{n-2,k} E^t_{n-2} \\
&\quad + E_{n-2} B^t_{n-2,k} R B_{n-2,k} E^t_{n-2} F_n + F_n E_{n-2} B^t_{n-2,k} R B_{n-2,k} E^t_{n-2} + F_n^2 \\
&= E_{n-2} R_1' E^t_{n-2} + F_n \\
&= R_2'
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{b} &= R_2 * [(B^t_{n,k+1} * \overrightarrow{a}) \oplus (e_k * \tilde{x}_{k+1})] \\
&= (E_{n-2} R_1' E^t_{n-2} + F_n) * \{[B^t_{n,k+1} * (E_n * \overrightarrow{v})] \oplus (e_k * \tilde{x}_{k+1})\} \\
&= E_{n-2} R_1' E^t_{n-2} * [(B^t_{n,k+1} E_n * \overrightarrow{v}) \oplus (e_k * \tilde{x}_{k+1})] \\
&\quad \vee F_n * [(B^t_{n,k+1} E_n * \overrightarrow{v}) \oplus (e_k * \tilde{x}_{k+1})] \\
&= E_{n-2} R_1' * \{E^t_{n-2} * (E_{n-2} B^t_{n-2,k} * \overrightarrow{v}) \oplus (E^t_{n-2} e_k * \tilde{x}_{k+1})\} \\
&\quad \vee (F_n E_{n-2} B^t_{n-2,k} * \overrightarrow{v}) \oplus (F_n e_k * \tilde{x}_{k+1})] \\
&= E_{n-2} R_1' * [(E^t_{n-2} E_{n-2} B^t_{n-2,k} * \overrightarrow{v}) \oplus (e_{k-1} * \tilde{x}_{k+1})] \\
&= E_{n-2} R_1' * [(B^t_{n-2,k} * \overrightarrow{v}) \oplus (e_{k-1} * x_k)] \\
&= E_{n-2} * \overrightarrow{a}' \\
&= \overrightarrow{b}'
\end{aligned}
$$

Here $\tilde{x}_{k+1} = \neg \tilde{r}_{k,k+2} * (a_k \wedge a_{k+2}) \oplus \tilde{r}_{k,k+2} * \varphi(a_{k+1})$ where $\tilde{r}_{k,k+2} = e_k^t R_1 e_{k+2}$.

Since $1 < k, k+2 < n+2$ we have that $\tilde{r}_{k,k+2} = r_{k-1,k+1}$ and $a_k = v_{k-1}$, $a_{k+1} = v_k$ and $a_{k+2} = v_{k+1}$. Thus $\tilde{x}_{k+1} = x_k$.

■

Corollary 21 *Let $F : \mathbf{PI}_{\mathbb{M}} \longrightarrow \mathbf{PI}_{\mathbb{M}}$ be the functor which sends $\hat{T}_{n,k}$ and $\check{T}_{n,k}$ to $\hat{T}_{n+2,k+1}$ and $\check{T}_{n+2,k+1}$ (respectively). Then*

1. *For any* $T \in \hom(\mathcal{U}_n, \mathcal{U}_m)$*,* $\varepsilon_m \circ T = F(T) \circ \varepsilon_n$*.*

2. *For any* $T \in \hom(\mathcal{O}_1, \mathcal{O}_1)$ *(note that* $\mathcal{U}_1 = \mathcal{O}_1$*),* $T([1], \emptyset) = ([1], \mathbf{x}) \Rightarrow \check{T}_{1,2} \circ F(T) \circ \hat{T}_{1,2}([1], \emptyset) = ([1], \varphi(\mathbf{x}))$*.*

Proof.

1. This follows immediately from the previous lemma and the definition of the functor F.

2.

$$\begin{array}{rcl}
\check{T}_{1,2} \circ F(T) \circ \hat{T}_{1,2}([1], \emptyset) & = & \check{T}_{1,2} \circ F(T) \left(\left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right], \left(\begin{array}{c} \emptyset \\ \emptyset \\ \emptyset \end{array} \right) \right) \\
& = & \check{T}_{1,2} \circ F(T) \circ \varepsilon_1([1], \emptyset) \\
& = & \check{T}_{1,2} \circ \varepsilon_1 \circ T([1], \emptyset) \\
& = & \check{T}_{1,2} \circ \varepsilon_1([1], \mathbf{x}) \\
& = & \check{T}_{1,2} \left(\left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right], \left(\begin{array}{c} \emptyset \\ \mathbf{x} \\ \emptyset \end{array} \right) \right) \\
& = & ([1], \varphi(\mathbf{x}))
\end{array}$$

■

Note that $\check{T}_{1,2} \circ F(T) \circ \hat{T}_{1,2}$ is the representation of $\check{t}_{1,2} \circ F(t) \circ \hat{t}_{1,2}$[9] which corresponds to the encirclement of the morphism t.

In summary we see that the representation of a system of non-singular planar curves is a morphism in $\hom(\mathcal{O}_1, \mathcal{O}_1)$, but this is determined by a single value. This means that the representation gives a map from the monoid of systems of non-singular planar curves $\hom(\emptyset, \emptyset)$ to the monoid of the representation $\mathbb{M}$. To a morphism $t \in \hom(\emptyset, \emptyset)$ we associate the value $v(t) \in \mathbb{M}$ such that $([1], v(t)) = T([1], \emptyset)$ where $T \in \hom(\mathcal{O}_1, \mathcal{O}_1)$ is the representation of t. This map is a monoid morphism ($v(t_1 \circ t_2) = v(t_1) \oplus v(t_2)$) as we have seen (see corollary 17), and furthermore, the function φ is what corresponds in $\mathbb{M}$ to the operation of encircling a system of curves by another curve (see corollary 21), i.e. $v(\langle t \rangle) = \varphi(v(t))$ where $\langle t \rangle$ denotes the encirclement of the morphism t.

[9] Here the functor F is defined in the same way for **PT** as it was for $\mathbf{PI}_{\mathbb{M}}$.

We are going to study this map in the following two particular cases.

First case: the monoid is the natural numbers with the usual multiplication as the operation of the monoid and with the division order giving the lattice structure; the function φ is the function that sends a number n to the n^{th} prime number.

We are going to see that, for systems of non-singular planar curves, the map is a monoid isomorphism (and thus is a complete invariant for such systems). This means that two systems of curves with the same value are equivalent (isotopic). Let us prove this by induction on the value $v(s) = n$ of the system s. Note that it only remains to prove that the map is bijective.

First we observe that if a morphism $s \in \hom(\emptyset, \emptyset)$ is irreducible (and thus is encircled) then $v(s)$ is a prime number.

If $v(s) = 1$ then s is the empty system of curves (the identity in $\hom(\emptyset, \emptyset)$) because a non-empty system of curves is a non-empty composition of irreducible morphisms therefore it has a non-empty product of prime numbers as value v (i.e. $v(s) > 1$). This implies that if a morphism has a prime number as value then it is irreducible. Thus an encircled morphism is irreducible.

Now let $v(s_1) = v(s_2) = n$, and suppose by the induction hypothesis that, for $k < n$, $v(s_1) = v(s_2) = k$ implies $s_1 = s_2$.

If n is a prime number (say the k^{th} prime) then s_1 and s_2 are irreducible (encircled), that is $s_1 = \langle s_3 \rangle$ and $s_2 = \langle s_4 \rangle$ and $v(s_3) = v(s_4) = k$. Thus, by the induction hypothesis, $s_3 = s_4$ and therefore $s_1 = s_2$.

If n is a composite number then s_1 and s_2 factorize into the same irreducible morphisms because if an irreducible morphism s_3 is a factor of s_1 then $v(s_3)$ is a prime number that divides n and, by the induction hypothesis, s_3 is the unique morphism with value $v(s_3)$, therefore s_3 is also a factor of s_2. The same argument can be used to prove that each factor appears in s_1 and s_2 the same number of times. The conclusion that $s_1 = s_2$ comes from the following proposition.

Proposition 22 *The monoid* $\hom(\emptyset, \emptyset)$ *is commutative.*

Second case: the monoid is the non-negative integer numbers with the usual sum as the operation of the monoid and with the usual order giving the lattice structure; the function φ is the function that sends a number n to its successor $n + 1$.

We are going to see that, for a system of non-singular planar curves s, $v(s)$ is simply the number of curves of s.

This is very easy because, since the map v is uniquely determined by the relations $v(s_1 \circ s_2) = v(s_1) + v(s_2)$ and $v(\langle s \rangle) = \varphi(v(s)) = v(s) + 1$, we only need to observe that the number of curves $\nu(s)$ of a system of curves s satisfies the relation $\nu(s_1 \circ s_2) = \nu(s_1) + \nu(s_2)$ and $\nu(\langle s \rangle) = \nu(s) + 1$.

6 Temperley-Lieb algebras

A Temperley-Lieb algebra $\mathcal{A}_n$ is an algebra over $\mathbb{K}[\delta]$[10] generated by U_1, U_2, ... , U_{n-1} with the following relations:

$$\begin{array}{ll} U_iU_j = U_jU_i & \text{for } |i-j|>1 \\ U_iU_{i\pm 1}U_i = U_i & \text{for all } i=1,...,n-1 \\ U_i^2 = \delta U_i & \text{for all } i=1,...,n-1 \end{array}$$

Geometrically $\mathcal{A}_n$ is a subalgebra of the linearization of the monoid hom$(\{1, 2, ..., n\}, \{1, 2, ..., n\})$ quotiented by the relations $U_i^2 = \delta U_i$ for all $i = 1, ..., n-1$. Here each U_i corresponds to $\hat{t}_{n-1,i-1} \circ \check{t}_{n-1,i-1}$ and each relation $U_i^2 = \delta U_i$ can be substituted by $\check{t}_{n-1,i-1} \circ \hat{t}_{n-1,i-1} = \delta id_{n-1}$ (where id_{n-1} is the identity morphism on hom$(\{1, 2, ..., n-2\}, \{1, 2, ..., n-2\})$).

For the matrices the representation that we have constructed is compatible with the restriction $\check{t}_{n-1,i-1} \circ \hat{t}_{n-1,i-1} = \delta id_{n-1}$ as we will see shortly.

On the other hand this relation $\check{t}_{n-1,i-1} \circ \hat{t}_{n-1,i-1} = \delta id_{n-1}$ makes the array of monoid values unnecessary since each closed curve becomes a scalar value δ independent of the region where it appears. Besides we have a linear structure on the Temperley-Lieb algebra that we need to preserve in a representation.

So we will give a representation of the Temperley-Lieb algebra in the category $\mathbf{Vect}_{\mathbb{K}}$ of the linear spaces over $\mathbb{K}$ which is adapted from the previous representation.

For that we consider the Boolean matrix part of the previous representation.

Let $\mathcal{E}_n$ be the set of $n \times n$-Boolean matrices satisfying the equivalence relation conditions, E1-E3. We are going to associate to each partition of the line into n intervals the linear span $\mathbb{K}\mathcal{E}_n$ of the set $\mathcal{E}_n$.

We can consider the operators $\hat{T}_{n,k}$ and $\check{T}_{n,k}$ as functions between sets of Boolean matrices satisfying $E1$, $E2$ and $E3$, since the matricial parts of $\hat{T}_{n,k}(R, \overrightarrow{v})$ and $\check{T}_{n,k}(R, \overrightarrow{v})$ don't depend on $\overrightarrow{v}$.

We are going to represent the generator morphism $\hat{t}_{n,k}$ by the linear operator

$$\begin{array}{rccc} \hat{\tau}_{n,k}: & \mathbb{K}\mathcal{E}_n & \longrightarrow & \mathbb{K}\mathcal{E}_{n+2} \\ & \sum_{R\in\mathcal{E}_n} \alpha_R R & \longmapsto & \sum_{R\in\mathcal{E}_n} \alpha_R \hat{T}_{n,k}(R) \end{array}$$

[10] Here $\mathbb{K}[\delta]$ is the ring of polynomials in one variable δ over a field $\mathbb{K}$ (for convenience we can fix $\mathbb{K} = \mathbb{C}$).

and the generator morphism $\check{t}_{n,k}$ by the linear operator

$$\begin{array}{rccc} \check{\tau}_{n,k}: & \mathbb{K}\mathcal{E}_{n+2} & \longrightarrow & \mathbb{K}\mathcal{E}_n \\ & \sum_{R\in\mathcal{E}_{n+2}} \alpha_R R & \longmapsto & \sum_{R\in\mathcal{E}_{n+2}} \alpha_R (r_{k-1,k+1} * \delta)\check{T}_{n,k}(R) \end{array}$$

where $r_{k-1,k+1}$ is the $(k-1,k+1)$ entry of the matrix R.

Looking at the section 3 we can see that, for the matrices, the consistency proof of the relations

$\check{T}_{n,k+1} \circ \hat{T}_{n,k} = \check{T}_{n,k-1} \circ \hat{T}_{n,k} = id$;

$\hat{T}_{n+2,l} \circ \hat{T}_{n,k} = \hat{T}_{n+2,k} \circ \hat{T}_{n,l-2}$ for $l \geq k+2$;

$\hat{T}_{n-2,l-2} \circ \check{T}_{n-2,k} = \check{T}_{n,k} \circ \hat{T}_{n,l}$ for $l \geq k+2$;

$\check{T}_{n,l} \circ \hat{T}_{n,k} = \hat{T}_{n-2,k} \circ \check{T}_{n-2,l-2}$ for $l \geq k+2$ and

$\check{T}_{n-2,l-2} \circ \check{T}_{n,k} = \check{T}_{n-2,k} \circ \check{T}_{n,l}$ for $l \geq k+2$.

does not require the properties T1, T2 and T3. Thus, $\hat{T}_{n,k}$ and $\check{T}_{n,k}$ as functions on $\mathcal{E}_n$ (or $\mathcal{E}_{n+2}$) satisfy these relations.

Bearing this in mind, it is not difficult to prove that $\hat{\tau}_{n,k}$ and $\check{\tau}_{n,k}$ satisfy the following relations:

1. $\check{\tau}_{n,k+1}\hat{\tau}_{n,k} = \check{\tau}_{n,k-1}\hat{\tau}_{n,k} = id$;

2. $\hat{\tau}_{n+2,l}\hat{\tau}_{n,k} = \hat{\tau}_{n+2,k}\hat{\tau}_{n,l-2}$ for $l \geq k+2$;

3.1. $\hat{\tau}_{n-2,l-2}\check{\tau}_{n-2,k} = \check{\tau}_{n,k}\hat{\tau}_{n,l}$ for $l \geq k+2$;

3.2. $\check{\tau}_{n,l}\hat{\tau}_{n,k} = \hat{\tau}_{n-2,k}\check{\tau}_{n-2,l-2}$ for $l \geq k+2$ and

4. $\check{\tau}_{n-2,l-2}\check{\tau}_{n,k} = \check{\tau}_{n-2,k}\check{\tau}_{n,l}$ for $l \geq k+2$.

We are going to prove the last relation leaving the others as an exercise for the reader.

By linearity we only need to prove that $\check{\tau}_{n-2,l-2}\check{\tau}_{n,k} = \check{\tau}_{n-2,k}\check{\tau}_{n,l}$ for an arbitrary $R \in \mathcal{E}_n$.

Denoting by $(M)_{k-1,k+1}$ the $(k-1,k+1)$ entry of a matrix M we have to check the identity:

$$\begin{aligned} &((\check{T}_{n,k}(R))_{l-3,l-1} * \delta)((R)_{k-1,k+1} * \delta)\check{T}_{n-2,l-2} \circ \check{T}_{n,k}(R) = \\ &\quad = ((\check{T}_{n,l}(R))_{k-1,k+1} * \delta)((R)_{l-1,l+1} * \delta)\check{T}_{n-2,k} \circ \check{T}_{n,l}(R) \end{aligned}$$

Since we have just proved that $\check{T}_{n-2,l-2} \circ \check{T}_{n,k}(R) = \check{T}_{n-2,k} \circ \check{T}_{n,l}(R)$ the identity reduces to (dropping the index n on $\check{T}_{n,k}$ and $\check{T}_{n,l}$)

$$((\check{T}_k(R))_{l-3,l-1} * \delta)((R)_{k-1,k+1} * \delta) = ((\check{T}_l(R))_{k-1,k+1} * \delta)((R)_{l-1,l+1} * \delta) \quad (1)$$

By lemma 12 we know that $(\check{T}_k(R))_{l-3,l-1}$ is equal to $(R)_{l-1,l+1}$ if $(\check{T}_k(R))_{k-1,l-3}$ or $(\check{T}_k(R))_{k-1,l-1}$ are equal to zero, and if $(\check{T}_k(R))_{k-1,l-3}$ and $(\check{T}_k(R))_{k-1,l-1}$ are both equal to 1 then, by the transitive property E3, $(\check{T}_k(R))_{l-3,l-1}$ is equal to 1. Thus

$$(\check{T}_k(R))_{l-3,l-1} = (R)_{l-1,l+1} + (\check{T}_k(R))_{k-1,l-3}(\check{T}_k(R))_{k-1,l-1}$$

Also by lemma 12 we have that $(\check{T}_k(R))_{k-1,l-3} = (R)_{k-1,l-1} + (R)_{k+1,l-1}$ and $(\check{T}_k(R))_{k-1,l-1} = (R)_{k-1,l+1} + (R)_{k+1,l+1}$. Therefore

$$\begin{aligned}
(\check{T}_k(R))_{l-3,l-1} &= (R)_{l-1,l+1} + [(R)_{k-1,l-1} + (R)_{k+1,l-1}][(R)_{k-1,l+1} + (R)_{k+1,l+1}] \\
&= (R)_{l-1,l+1} + (R)_{k-1,l-1}(R)_{k-1,l+1} + (R)_{k-1,l-1}(R)_{k+1,l+1} \\
&\quad +(R)_{k+1,l-1}(R)_{k-1,l+1} + (R)_{k+1,l-1}(R)_{k+1,l+1} \\
&= (R)_{l-1,l+1} + (R)_{k-1,l-1}(R)_{k+1,l+1} + (R)_{k+1,l-1}(R)_{k-1,l+1}
\end{aligned}$$

using the transitive property.

By the same argument we get

$$(\check{T}_l(R))_{k-1,k+1} = (R)_{k-1,k+1} + (R)_{l-1,k-1}(R)_{l+1,k+1} + (R)_{l+1,k-1}(R)_{l-1,k+1}$$

If $(R)_{l-1,k-1}(R)_{l+1,k+1}+(R)_{l+1,k-1}(R)_{l-1,k+1} = 0$ then we have $(\check{T}_k(R))_{l-3,l-1} = (R)_{l-1,l+1}$ and $(\check{T}_l(R))_{k-1,k+1} = (R)_{k-1,k+1}$, and therefore the equality (1) holds. If $(R)_{l-1,k-1}(R)_{l+1,k+1}+(R)_{l+1,k-1}(R)_{l-1,k+1} = 1$ then we have $(R)_{l-1,l+1} = (R)_{k-1,k+1}$, by the equivalence relation properties, and $(\check{T}_k(R))_{l-3,l-1} = (\check{T}_l(R))_{k-1,k+1} = 1$, and therefore the equality (1) holds too.

We have also an extra identity

5. $\check{\tau}_{n,k}\hat{\tau}_{n,k} = \delta id$

which is very easy to verify, so that we leave it to the reader to do so.

7 The Kauffman bracket polynomial

The well known Kauffman bracket polynomial has a skein relation

$$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$$

which can be used to represent braids into Temperley-Lieb algebras. Each generator σ_i of the Artin braid group B_n is represented by $Aid + A^{-1}U_i$, where A is a non-vanishing variable. The inverse of σ_i is represented by $AU_i + A^{-1}id$. Fixing $\delta = -A^2 - A^{-2}$ the Artin relations[11] for B_n are preserved in this representation due to relations of the Temperley-Lieb algebra. Moreover, we can use the representation of the Temperley-Lieb algebras in the category $\mathbf{Vect}_{\mathbb{K}}$ that we gave to produce a representation of the braid group B_n in $\mathbf{Vect}_{\mathbb{K}}$. Indeed, this could be done in such a way as to extend this representation to the category of non-oriented tangles. This category (let us call it **Tang**) has the finite subsets of $\mathbb{R}$ as its objects and a morphism between two objects o_1 and o_2 is a 1-dimensional compact submanifold of $\mathbb{R}^2 \times [0,1]$ which has $o_1 \times \{(0,1)\} \cup o_2 \times \{(0,0)\}$ as boundary.

Here we consider two tangles with the same boundary to be the same if there exists an ambient isotopy changing one into the other without moving the boundary. For the composition of two tangles $t_1 \in \hom(o_1, o_2)$ and $t_2 \in \hom(o_2, o_3)$ we take the gluing of t_1 and t_2 in o_2 in the same way as for planar tangles.

We can give a presentation for this category taking as generators the same generators as for **PT** (the category of non-singular planar tangles): $\hat{t}_{n,k}$ and $\check{t}_{n,k}$ (for $n, k \in \mathbb{N}$ with $2 \leq k \leq n+1$) plus the crossings: $\sigma_{n,k}$ and $\sigma_{n,k}^{-1}$ (corresponding to the generators σ_{k-1} and σ_{k-1}^{-1} of B_{n-1}). As relations we take the relations of **PT**:

$$\check{t}_{n,k+1} \circ \hat{t}_{n,k} = \check{t}_{n,k-1} \circ \hat{t}_{n,k} = id$$

$$\hat{t}_{n+2,l} \circ \hat{t}_{n,k} = \hat{t}_{n+2,k} \circ \hat{t}_{n,l-2}$$

for $l \geq k+2$;

$$\check{t}_{n-2,l-2} \circ \check{t}_{n,k} = \check{t}_{n-2,k} \circ \check{t}_{n,l}$$

for $l \geq k+2$;

[11] $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$ and $\sigma_i\sigma_{i\pm1}\sigma_i = \sigma_{i\pm1}\sigma_i\sigma_{i\pm1}$

$$\hat{t}_{n-2,l-2} \circ \check{t}_{n-2,k} = \check{t}_{n,k} \circ \hat{t}_{n,l}$$

for $l \geq k+2$ and

$$\check{t}_{n,l} \circ \hat{t}_{n,k} = \hat{t}_{n-2,k} \circ \check{t}_{n-2,l-2}$$

for $l \geq k+2$,

plus the relations of the braid groups B_n:

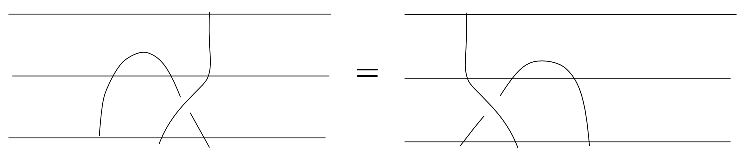
$$\sigma_{n,k} \circ \sigma_{n,k}^{-1} = \sigma_{n,k}^{-1} \circ \sigma_{n,k} = id_n$$

$$\sigma_{n,l} \circ \sigma_{n,k} = \sigma_{n,k} \circ \sigma_{n,l}$$

for $|l-k| > 1$;

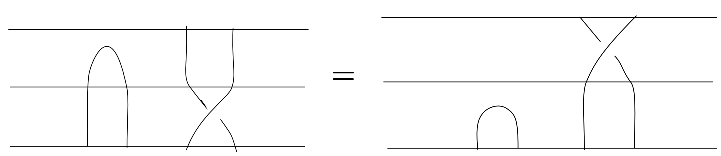
$$\sigma_{n,k}\sigma_{n,k\pm1}\sigma_{n,k} = \sigma_{n,k\pm1}\sigma_{n,k}\sigma_{n,k\pm1}$$

plus the following relations:



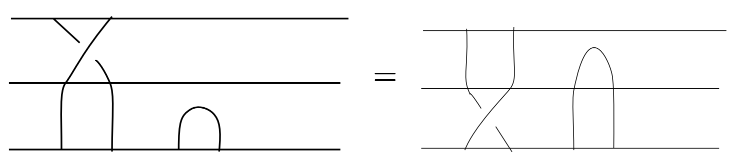
$$\sigma_{n+2,k+1}^{\epsilon} \circ \hat{t}_{n,k} = \sigma_{n+2,k}^{-\epsilon} \circ \hat{t}_{n,k+1}$$

with $\epsilon = \pm1$;



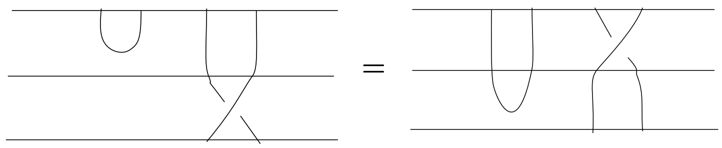
$$\sigma_{n+2,l} \circ \hat{t}_{n,k} = \hat{t}_{n,k} \circ \sigma_{n,l-2}$$

for $l \geq k+2$;



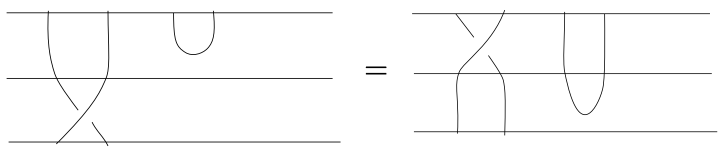
$$\sigma_{n+2,k} \circ \hat{t}_{n,l} = \hat{t}_{n,l} \circ \sigma_{n,k}$$

for $l \geq k+2$;



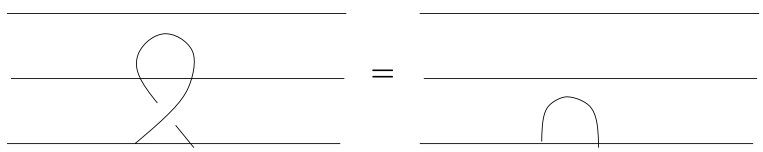
$$\sigma_{n-2,l-2} \circ \check{t}_{n,k} = \check{t}_{n-2,k} \circ \sigma_{n,l}$$

for $l \geq k+2$;



$$\sigma_{n-2,k} \circ \check{t}_{n,l} = \check{t}_{n-2,l} \circ \sigma_{n,k}$$

for $l \geq k+2$ and



$$\sigma^{\epsilon}_{n+2,k} \circ \hat{t}_{n,k} = \hat{t}_{n,k}$$

with $\epsilon = \pm 1$.

We have constructed (in the previous section) a linear representation $\hat{\tau}_{n,k}$ and $\check{\tau}_{n,k}$ for the generators $\hat{t}_{n,k}$ and $\check{t}_{n,k}$ which satisfies the relations:

1. $\check{\tau}_{n,k+1}\hat{\tau}_{n,k} = \check{\tau}_{n,k-1}\hat{\tau}_{n,k} = id$;

2. $\hat{\tau}_{n+2,l}\hat{\tau}_{n,k} = \hat{\tau}_{n+2,k}\hat{\tau}_{n,l-2}$ for $l \geq k+2$;

3.1. $\hat{\tau}_{n-2,l-2}\check{\tau}_{n-2,k} = \check{\tau}_{n,k}\hat{\tau}_{n,l}$ for $l \geq k+2$;

3.2. $\check{\tau}_{n,l}\hat{\tau}_{n,k} = \hat{\tau}_{n-2,k}\check{\tau}_{n-2,l-2}$ for $l \geq k+2$ and

4. $\check{\tau}_{n-2,l-2}\check{\tau}_{n,k} = \check{\tau}_{n-2,k}\check{\tau}_{n,l}$ for $l \geq k+2$.

and another relation:

5. $\check{\tau}_{n,k}\hat{\tau}_{n,k} = \delta id$

With the skein relation of the Kauffman-bracket polynomial we get a linear representation for the generator $\sigma_{n,k}$ (represented by $\Sigma_{n,k} := Aid_n + A^{-1}\hat{\tau}_{n-2,k}\check{\tau}_{n-2,k}$) and for its inverse $\sigma_{n,k}^{-1}$ (represented by $\Sigma_{n,k}^{-1} := A^{-1}id_n + A\hat{\tau}_{n-2,k}\check{\tau}_{n-2,k}$). Taking $\delta = -A^2 - A^{-2}$ it is easy to see that in this representation all the relations are satisfied except the relations $\sigma_{n+2,k} \circ \hat{t}_{n,k} = \hat{t}_{n,k}$ and $\sigma_{n+2,k}^{-1} \circ \hat{t}_{n,k} = \hat{t}_{n,k}$. Instead, this representation satisfies the relations $\Sigma_{n+2,k}\hat{\tau}_{n,k} = -A^3\hat{\tau}_{n,k}$ and $\Sigma_{n+2,k}^{-1}\hat{\tau}_{n,k} = -A^{-3}\hat{\tau}_{n,k}$.

Nevertheless, we can obtain, following the spirit of the Kauffman work [4] , a true representation of the category of oriented tangles **OTa** (see [10]) in the following way: given an oriented tangle t, take the corresponding non-oriented tangle $|t|$ and obtain its linear representation $\langle |t| \rangle$ described above. Then multiply $\langle |t| \rangle$ by the factor $(-A)^{-3w(t)}$ where $w(t)$ (called the *writhe* of t) is defined as follows: put a sign on each crossing: $sgn(\ ) = sgn(\ ) = sgn(\ ) = sgn(\ ) = 1$ and $sgn(\ ) = sgn(\ ) = sgn(\ ) = sgn(\ ) = -1$, and the writhe is the sum the signs of all crossings. In the end we get a representation $\mathcal{JK}(t) := (-A)^{-3w(t)}\langle |t| \rangle$ of the tangle t consistent with all relations of the category of oriented tangles **OTa**.

If l is a link (i.e. $l \in \hom(\emptyset, \emptyset)$) then $\mathcal{JK}(l)$ is an automorphism in a 1-dimensional $\mathbb{Z}[A, A^{-1}]$-module, thus it can be identified with a scalar value in $\mathbb{Z}[A, A^{-1}]$ (i.e. a Laurent polynomial $P_l(A)$ which, in this case, is just the Jones[12]-polynomial $V_l(A)$ multiplied by $(-A^2 - A^{-2})$, that means $\mathcal{JK}(l)(x) = (-A^2 - A^{-2})V_l(A)x$.

It is not difficult to see this, since the Jones polynomial is the unique polynomial invariant for links which satisfies:

1. $V_U(A) = 1$ for the unknot $U = \check{t}_{1,2}\hat{t}_{1,2}$;

2. $A^{-4}V_{\ } - A^4V_{\ } = (A^2 - A^{-2})V_{\ }$.

and $(-A^2 - A^{-2})^{-1}\mathcal{JK}(l)(1)$ also satisfies these relations.

Proposition 23 *If l is a long link (i.e. a tangle from a single point to a single point) then $\mathcal{JK}(l)$ is an automorphism in a 2-dimensional $\mathbb{Z}[A, A^{-1}]$-module, which is a scalar multiple $P_l(A)$ of the identity operator. This scalar*

[12]with the variables as in the Kauffman-bracket polynomial

$P_l(A)$ is just the Jones-polynomial $V_{\bar{l}}(A)$ of the closure $\bar{l}$ of the long link l. Note that the closure of the composition of two of these tangles is just a connected sum of their closures.

Proof. $\mathbb{K}\mathcal{E}_2$ is a 2-dimensional linear space generated by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. $\langle l \rangle : \mathbb{K}\mathcal{E}_2 \longrightarrow \mathbb{K}\mathcal{E}_2$ is a linear combination of products of $\hat{\tau}$ and $\check{\tau}$.

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unique matrix in $\mathbb{K}\mathcal{E}_2$ that satisfies the topological condition T1 and T1 is preserved by $\hat{\tau}$ and $\check{\tau}$, we have that $\langle l \rangle(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = P_l(A) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ where $P_l(A)$ is a Laurent polynomial in the variable A.

Looking at the proof of proposition 18 we can see that $\hat{\tau}$ and $\check{\tau}$ preserve the value 1 in the first row - last column entry, thus $\langle l \rangle(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = P'_l(A) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ where $P'_l(A)$ is a Laurent polynomial in the variable A.

Therefore, to see that $\langle l \rangle$ is a multiple of the identity operator we only need to check that $P_l(A) = P'_l(A)$. For that we are going to consider the subset $\mathcal{E}'_n \subseteq \mathcal{E}_n$ of the matrices that satisfy the condition T1, and we take the linear operators $\chi_n : \mathbb{K}\mathcal{E}'_n \longrightarrow \mathbb{K}\mathcal{E}_n$ defined by the formula $\chi_n(R) = \overline{R + T_n}$ (the transitive closure of $R + T_n$) where T_n is the matrix in which the (i,j)-entry is 1 if and only if $|i - j| = n - 1$ (i.e. has 1 only in the first row - last column and last row - first column entries). In the topological sense, it is as if the strip $\mathbb{R} \times [0,1]$ were transformed into a cylinder joining the exterior regions.

Lemma 24 *For even dimension, χ commutes with the operators $\hat{\tau}$ and $\check{\tau}$ in the following sense: $\chi_n \hat{\tau}_{n-2,k} = \hat{\tau}_{n-2,k} \chi_{n-2}$ and $\chi_n \check{\tau} = \check{\tau} \chi_{n+2}$, for any even natural number n.*

Proof. The first equation: $\chi_n \hat{\tau}_{n-2,k} = \hat{\tau}_{n-2,k} \chi_{n-2}$.

By linearity, all we need to see is that, for any matrix $R \in \mathcal{E}'_n$, $\chi_n \hat{\tau}_{n-2,k}(R) = \hat{\tau}_{n-2,k} \chi_{n-2}(R)$.

This corresponds to the identity:

$$\overline{B_k R B_k^t + D_k + T} = B_k \overline{R + T} B_k^t + D_k$$

First we prove

$$\overline{B_kRB_k^t + D_k + T} = \overline{B_k(R+T)B_k^t + D_k}$$

Since

$$\begin{array}{rcl}\overline{B_kRB_k^t + D_k + T} & \geq & \overline{B_kB_k^t + D_k + T} \\ & \geq & (B_kB_k^t + D_k + T)^3 \\ & \geq & B_kB_k^tTB_kB_k^t \\ & = & B_kTB_k^t\end{array}$$

we have that

$$\overline{B_kRB_k^t + D_k + T} = \overline{B_kRB_k^t + D_k + T + B_kTB_k^t}$$

Then, since $T \leq B_kTB_k^t$ we have

$$\overline{B_kRB_k^t + D_k + T} = \overline{B_k(R+T)B_k^t + D_k}$$

Now the identity

$$\overline{B_kRB_k^t + D_k + T} = B_k\overline{R+T}B_k^t + D_k$$

comes from the identity

$$(B_k(R+T)B_k^t + D_k)^n = B_k(R+T)^nB_k^t + D_k \quad \text{for all natural } n$$

Now let us prove the second equation: $\chi_n\check{\tau} = \check{\tau}\chi_{n+2}$.

By linearity, all we need to check is that, for any matrix $R \in \mathcal{E}'_n$, $\chi_n\check{\tau}(R) = \check{\tau}\chi_{n+2}(R)$.

$\chi_n\check{\tau}(R) = \alpha\overline{\overline{B_k^tRB_k} + T} = \alpha\overline{B_k^tRB_k + T}$ where $\alpha = (e_{k-1}^tRe_{k+1}) * \delta$.

On the other hand, $\check{\tau}\chi_{n+2}(R) = \beta\overline{B_k^t\overline{R+T}B_k}$ where $\beta = (e_{k-1}^t\overline{R+T}e_{k+1}) * \delta$.

Thus, all we need to check is that $\beta = \alpha$ and $\overline{B_k^t\overline{R+T}B_k} = \overline{B_k^tRB_k + T}$.

For that we observe that $\overline{R+T} = R + T + RT + TR + RTR$ because $R^2 = R$ and $TMT \leq I + T$ for any matrix M. Indeed, since $R \geq I$ (and therefore $T + RT + TR \leq RTR$), we have $\overline{R+T} = R + RTR$.

Since n is even and R satisfies T1, an (i,j)-entry of RTR is 1 only if $|i-j|$ is odd. Thus the $(k-1,k+1)$-entry of $\overline{R+T}$ is equal to the $(k-1,k+1)$-entry of R, therefore $\beta = \alpha$.

To see that $\overline{B_k^t\overline{R+T}B_k} = \overline{B_k^tRB_k + T}$ we observe that this equality is equivalent to $\overline{B_k^t\overline{R+T}B_k} = \overline{B_k^t(R+T)B_k}$ because $B_k^tTB_k = T$. To check this last equality it is sufficient, by corollary 10, to see that $(I-D_k)\overline{R+T}(I-D_k) \leq \overline{(I-D_k)(R+T)(I-D_k)}$.

$$
\begin{aligned}
&(I-D_k)\overline{R+T}(I-D_k) = \\
&\quad = (I-D_k)(R+RTR)(I-D_k) = \\
&\quad = (I-D_k)R(I-D_k) + (I-D_k)RTR(I-D_k) \\
&\quad = (I-D_k)R(I-D_k) + (I-D_k)R(I-D_k)T(I-D_k)R(I-D_k) \\
&\quad \leq \overline{(I-D_k)R(I-D_k)+T} \\
&\quad = \overline{(I-D_k)(R+T)(I-D_k)}
\end{aligned}
$$

■

From this lemma it follows that χ commutes with $\langle l\rangle$ for any long link. Then

$$
P_l'(A)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \langle l\rangle \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \langle l\rangle\chi_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =
$$

$$
= \chi_2\langle l\rangle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \chi_2 P_l(A)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = P_l(A)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

and therefore $P_l'(A) = P_l(A)$.

To see that $(-A)^{-3w(l)}P_l(A)$ is the Jones polynomial of the closure of l we proceed in the following way. The closure of a long link l can be given by the composition $\bar{l} = \check{t}_{1,2} \circ J(l) \circ \hat{t}_{1,2}$, where J is a functor that sends the object $\{1,2,...,n\}$ to the object $\{1,2,...,n+1\}$ and the generators $\hat{t}_{n,k}$, $\check{t}_{n,k}$ and $\sigma_{n,k}$ to $\hat{t}_{n+1,k}$, $\check{t}_{n+1,k}$ and $\sigma_{n+1,k}$ respectively. This is represented by the operator $\langle\bar{l}\rangle = \check{\tau}_{1,2}J(\langle l\rangle)\hat{\tau}_{1,2}$ (here J is a different functor, but is defined in an analogous way). Now we consider the operators $\iota_n : \mathbb{K}\mathcal{E}_n' \longrightarrow \mathbb{K}\mathcal{E}_{n+1}'$ which sends a matrix R to $R\oplus 1$ which is equal to R in the first n rows and columns, 1 in the $(n+1,n+1)$-entry and zero in the other entries. It is quite simple to see that $\chi_{n+2}\iota_{n+2}\hat{\tau}_{n,k} = \hat{\tau}_{n+1,k}\chi_n\iota_n$ and $\chi_n\iota_n\check{\tau}_{n,k} = \check{\tau}_{n+1,k}\chi_{n+2}\iota_{n+2}$ and therefore $\chi\iota\langle l\rangle = J(\langle l\rangle)\chi\iota$ for any long link (although lemma 24 is restricted to χ_n with n even, all the proof is adaptable to the general case except the identity $e_{k-1}^t(\overline{R+T})e_{k+1} = e_{k-1}^t Re_{k+1}$, but it is easy to see that if R is in the image of ι this identity holds).

Thus we have

$$
\langle\bar{l}\rangle([1]) = \check{\tau}_{1,2}J(\langle l\rangle)\hat{\tau}_{1,2}([1]) = \check{\tau}_{1,2}J(\langle l\rangle)\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right) =
$$

$$= \check{\tau}_{1,2} J(\langle l \rangle) \chi_3 \iota \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \check{\tau}_{1,2} \chi_3 \iota \langle l \rangle \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = P_l(A) \check{\tau}_{1,2} \chi_3 \iota \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) =$$

$$= P_l(A) \check{\tau}_{1,2} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right) = P_l(A)(-A^2 - A^{-2})([1])$$

And therefore, since $w(\bar{l}) = w(l)$, we have $(-A)^{-3w(\bar{l})} P_l(A)$ is the Jones polynomial. ■

8 The Kauffman-Vogel Polynomial

In the article [6] Kauffman and Vogel extend the Kauffman Polynomial[13] to 4-valent graphs embedded in $\mathbb{R}^3$, via the local relations:

$$[\slashoverback] = A[\,)(\,] + B[\asymp] + [\times] \tag{2}$$

and

$$[\curlyvee\!\!\circ] = a[\,)\,]$$

This polynomial is invariant under rigid vertex regular isotopies, and can be made invariant under the first Reidemeister move by multiplying by some power of a in the same way as for the Jones polynomial.

Since $[\backslashoverslash] = A[\asymp] + B[\,)(\,] + [\times]$, this polynomial satisfies the axioms of the Kauffman Polynomial:

$$[\slashoverback] - [\backslashoverslash] = z([\,)(\,] - [\asymp])$$

and

$$[\curlyvee\!\!\circ] = a[\,)\,]$$

with $z = A - B$.

Thus we can calculate the polynomial of a 4-valent graph eliminating each vertex by the formula

$$[\times] = A[\,)(\,] + B[\asymp] - [\slashoverback]$$

and calculate the polynomials of the resulting links by the axioms of the Kauffman Polynomial.

However, there exists a graphical calculus that allow us to calculate the polynomial of any 4-valent planar graph without resort to formula (2). Thus we can calculate the polynomial in the inverse way (eliminating the crossings by the formula (2)). This is very practicable in the specific case $B = A^{-1}$ and $a = A$ (see [2] and [3]) where the polynomial of any 4-valent planar graph G is:

$$[G] = 2^{c-1}(-A - A^{-1})^v$$

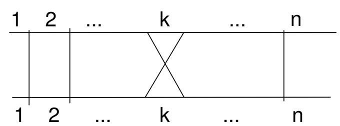
where c is the number of components of G and v is the number of vertices of G.

As we have done with the bracket polynomial we can use the formula (2) to decompose a tangle as a linear combination of planar tangles (in this case singular planar tangles).

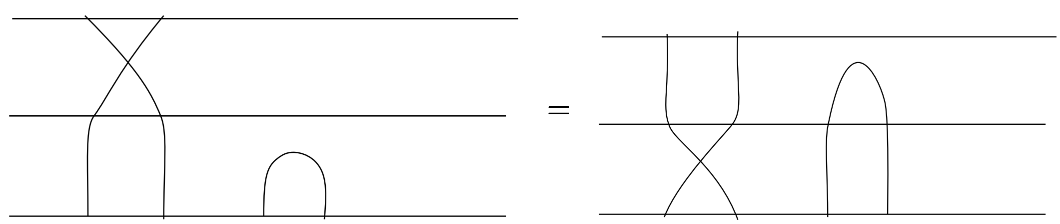
[13] Also known as the Dubrovnik polynomial.

8.1 Singular planar tangles

We consider a singular planar tangle to be a set of finitely many intervals and circles embedded in $\mathbb{R} \times [0, 1]$ which intersect one each other transversely in finitely many points (never on its boundaries) and whose boundaries are in $\mathbb{R} \times \{0, 1\}$. We can see singular planar tangles as an extension of the category of planar tangles **PT** (let us call it **SPT**) with the same objects. For a presentation of **SPT** we can take as the generators the same generators as those of **PT** and add vertex generators:

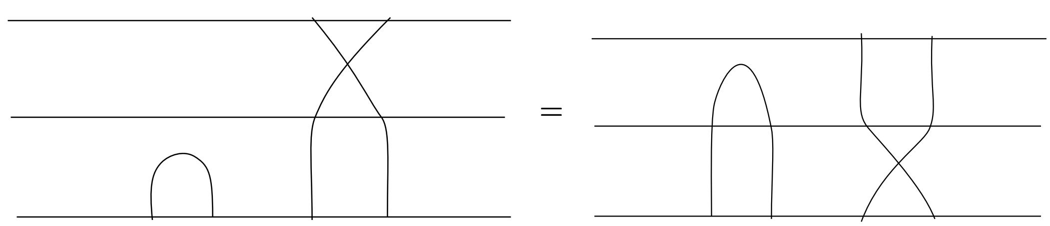
$v_{n,k} =$ 

(here $n \geq 3$ is the number of intervals and k is the position of the vertex $2 \leq k \leq n-1$). For the relations we consider all relations of **PT** plus the following relations:



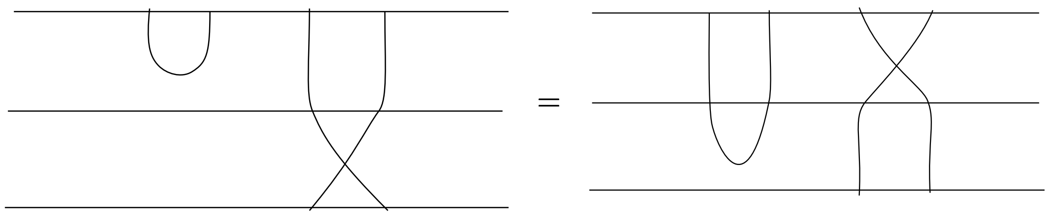
$$\hat{t}_{n,l} \circ v_{n,k} = v_{n+2,k} \circ \hat{t}_{n,l}$$

for $l \geq k+2$;



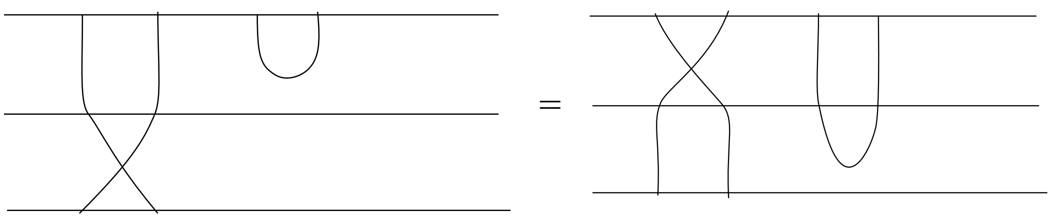
$$\hat{t}_{n,k} \circ v_{n,l} = v_{n+2,l+2} \circ \hat{t}_{n,k}$$

for $l \geq k+2$;



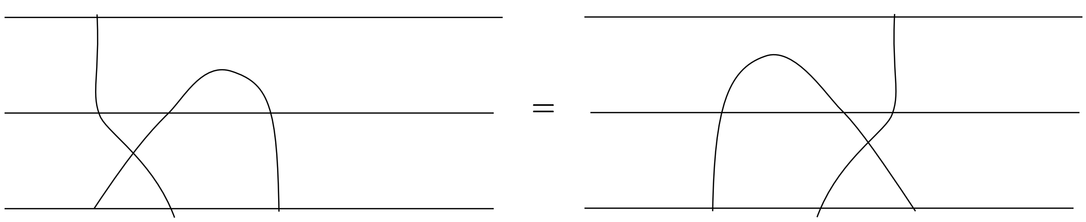
$$v_{n,l-2} \circ \check{t}_{n,k} = \check{t}_{n,k} \circ v_{n+2,l}$$

for $l \geq k+2$;



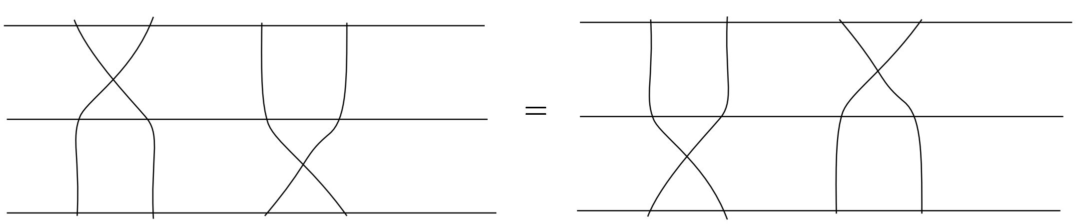
$$v_{n,k} \circ \check{t}_{n,l} = \check{t}_{n,l} \circ v_{n+2,k}$$

for $l \geq k+2$;



$$v_{n+2,k} \circ \hat{t}_{n,k+1} = v_{n+2,k+1} \circ \hat{t}_{n,k}$$

for $l \geq k+2$ and



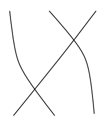
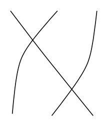
$$v_{n,l} \circ v_{n,k} = v_{n,k} \circ v_{n,l}$$

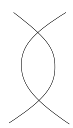
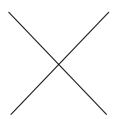
for $l \geq k+2$.

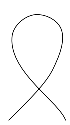
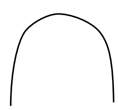
Following the spirit of the representation of $\hat{t}_{n,k}$ and $\check{t}_{n,k}$, we will represent $v_{n,k}$ by the operator $\nu_{n,k} : \mathbb{K}\mathcal{E}_n \longrightarrow \mathbb{K}\mathcal{E}_n$ defined by the formula $\nu_{n,k}(R) = \varepsilon[(I-D_k)R(I-D_k)+D_k]$ where ε is a constant in $\mathbb{K}$ (which plays the role of counting the number of vertices).

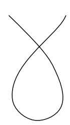
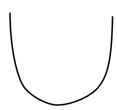
Making use of the identities $(I-D_k)B_l = B_l(I-D_k)$ and $(I-D_l)B_k = B_l(I-D_{k-2})$ for $l \geq k+2$, $(I-D_k)B_{k+1} = (I-D_{k+1})B_k$ and $(I-D_k)(I-D_l) = (I-D_l)(I-D_k)$ we can check easily that the representations $\hat{\tau}_{n,k}$, $\check{\tau}_{n,k}$ and $\nu_{n,k}$ of $\hat{t}_{n,k}$, $\check{t}_{n,k}$ and $v_{n,k}$ (resp.) satisfy all the relations of **SPT**.

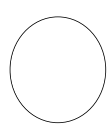
Moreover, it is easy to see that the following identities are satisfied:

$\nu_k \nu_{k+1} = \nu_{k+1}\nu_k$:  $=$  ;

$\nu_k^2 = \varepsilon \nu_k$:  $= \varepsilon$  ;

$\nu_k \hat{\tau}_k = \varepsilon \hat{\tau}_k$:  $= \varepsilon$  ;

$\check{\tau}_k \nu_k = \varepsilon \check{\tau}_k$:  $= \varepsilon$  ;

$\check{\tau}_k \hat{\tau}_k = \delta id$:  $= \delta$.

In this way, we have that for a 4-valent planar graph G its representation is given by the value $\delta^c \varepsilon^v$ where c is the number of connected components of G and v is the number of vertices of G.

Thus, fixing $\delta = 2$ and $\varepsilon = -A - A^{-1}$, we can extract the Kauffman-Vogel polynomial with variables $B = A^{-1}$ and $a = A$.

9 Conclusion, conjectures and developments

The theory contained in this thesis is very specific to the study of planar structures such as planar tangles. In fact, it uses the Jordan lemma implicitly in several aspects. In the final two chapters we applied it to ordinary tangles (spatial not planar) but making use of planar state models. It is left as an open problem to find other representations of spatial tangles within the same theory without decomposing them into planar tangles. Note that although this theory extends the Jones polynomial for tangles to operators it is very different from the Turaev theory.

It is important also to note that in the final chapters we didn't require that the Boolean matrices should satisfy the topological conditions T1, T2 and T3. This give us a higher-dimensional representation than if we had restricted to matrices satisfying the topological conditions. However, it is not clear whether we would lose any information with such a restriction. In other words, we don't know if, whenever two tangles have the same representation with such a restriction, they then have the same representation. We know that it is true for links (trivially) and long links (prop. 23) but we don't know if it is true in general.

Another open problem is to establish whether, subject to suitable choices, the representation of chapter 4 for planar tangles is faithful (i.e. is a complete invariant) or not. We have proved this only for the case of systems of non-singular planar curves. The following argument suggests that it is faithful. Given a representation of a non-singular planar tangle, if we evaluate it on the Boolean identity matrix we get a matrix that tells us which intervals at the bottom of the tangle are in the same region in the tangle. Thus we can draw the curves which have their boundary at the bottom of the tangle. By evaluating the representation on all Boolean matrices (satisfying E1-E3 and T1-T3), and looking at changes in the array of monoid values, we can find the curves which have their boundary at the top of the tangle. Now we can link the remaining ends at the top and the bottom in a unique way since the curves cannot cross each other, and finally, using the array of monoid values, we can draw the closed curves in their respective regions. Thus we can determine the tangle from its representation.

One possible way to develop this theory is to apply it to the theory of surfaces embedding in $\mathbb{R}^3$, where there is an analogous theorem to the Jordan lemma. Here, we would take an embedding of surfaces in $\mathbb{R}^3$ for which the coordinate on the vertical axis is a Morse function, and we would decompose such an embedding into a composition of 2-dimensional cobordisms with a single critical point in each one. Note that the complement of an embedding of surfaces in $\mathbb{R}^3$ is a 3-dimensional manifold which may be decomposed as a composition of 3-dimensional cobordisms between complements of embedding closed curves in $\mathbb{R}^2$. Here the Boolean matrices would give us information about whether regions in the source of the cobordism are in the same component of the cobordism. However, we don't have a natural order for regions in the plane in the same way as we have for intervals in the line. This small detail increases the difficulty in developing this theory for higher dimensions.

A Calculus with Boolean matrices

There are, for this paper, two useful ways to calculate the operations on Boolean matrices.

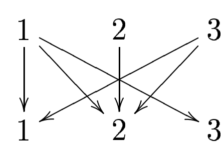
One is using propositional logic calculus. To each entry (i, j) of a Boolean matrix $M = [m_{i,j}]$ we associate a proposition $p_{i,j}(M)$ on variables i and j with logical value $m_{i,j}$. In this way we have the following correspondence:

> $p_{i,j}(A+B) = p_{i,j}(A) \vee p_{i,j}(B)$ for all i, j (here $\vee$ means the logical operation OR);
>
> $p_{i,j}(AB) = \exists_{k=1,\ldots,n} : p_{i,k}(A) \wedge p_{k,j}(B)$ for all i, j (here $\wedge$ means the logical operation AND);
>
> $A \leq B$ iff $p_{i,j}(A) \Rightarrow p_{i,j}(B)$ for all i, j.

This is useful, for example, to check the inequality $C_{l\times m}C_{m\times n} \leq C_{l\times n}$ which appears on page 22. Since $p_{i,j}(C) := (i - j) \in 2\mathbb{Z}$ the veracity of the inequality is the same as the veracity of the proposition $(\exists_{k=1,\ldots,m} : (i - k) \in 2\mathbb{Z} \wedge (k - j) \in 2\mathbb{Z}) \Rightarrow (i - j) \in 2\mathbb{Z}$ which is obvious.

The other way is more visual, and consists of associating to an $m \times n$ Boolean matrix $M = [m_{i,j}]$ a set of arrows beginning in a set of n ordered points and ending in a set of m ordered points such that there exists an arrow from the point j to the point i if and only if $m_{i,j} = 1$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightsquigarrow$$



In this way we have:

> The set of arrows corresponding to a sum of matrices $A+B$ is the union of the sets of arrows corresponding to A and B separately;
>
> An arrow from the point j to the point i is in the set corresponding to a product of matrices AB iff there exists a point k such that there exists an arrow from j to k in the set corresponding to B and an arrow from k to i in the set corresponding to A;

$A \leq B$ iff the set corresponding to A is contained in the set corresponding to B.

Some of the matrices used in the text have the following form:

$$
\begin{tikzcd}
D_{n,k} \rightsquigarrow & 1 & \cdots & k-1 & k \arrow[d] & k+1 & \cdots & n \\
& 1 & \cdots & k-1 & k & k+1 & \cdots & n
\end{tikzcd}
$$

$$
\begin{tikzcd}
I - D_{n,k} \rightsquigarrow & 1 \arrow[d] & \cdots & k-1 \arrow[d] & k & k+1 \arrow[d] & \cdots & n \arrow[d] \\
& 1 & \cdots & k-1 & k & k+1 & \cdots & n
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_{n,k} \rightsquigarrow & 1 \arrow[d] & \cdots & k-1 \arrow[d] \arrow[drr] & \cdots & n \arrow[drr] & & \\
& 1 & \cdots & k-1 & k & k+1 & \cdots & n+2
\end{tikzcd}
$$

$$
\begin{tikzcd}
B^t_{n,k} \rightsquigarrow & 1 \arrow[d] & \cdots & k-1 \arrow[d] & k & k+1 \arrow[dll] & \cdots & n+2 \arrow[dll] \\
& 1 & \cdots & k-1 & \cdots & n & &
\end{tikzcd}
$$

Thus we can check many of the identities (and inequalities) used in the text in this easier way. For example:

$$B_k B_k^t \geq I - D_k :$$

$$
\begin{tikzcd}
1 \arrow[d] & \cdots & k-1 \arrow[d] & k & k+1 \arrow[dll] & \cdots & n+2 \arrow[dll] \\
1 \arrow[d] & \cdots & k-1 \arrow[d] \arrow[drr] & \cdots & n \arrow[drr] & & \\
1 & \cdots & k-1 & k & k+1 & \cdots & n+2
\end{tikzcd}
$$

$$
= \begin{tikzcd}
1 \arrow[d] & \cdots & k-1 \arrow[d] \arrow[drr] & k & k+1 \arrow[d] \arrow[dll] & \cdots & n+2 \arrow[d] \\
1 & \cdots & k-1 & k & k+1 & \cdots & n+2
\end{tikzcd}
$$

$$
\geq \begin{tikzcd}
1 \arrow[d] & \cdots & k-1 \arrow[d] & k & k+1 \arrow[d] & \cdots & n+2 \arrow[d] \\
1 & \cdots & k-1 & k & k+1 & \cdots & n+2
\end{tikzcd}
$$

$$B_k^t B_k = I :$$

$$
\begin{tikzcd}
1 \arrow[d] & \cdots & k-1 \arrow[d] \arrow[drr] & \cdots & n \arrow[drr] & & \\
1 \arrow[d] & \cdots & k-1 \arrow[d] & k & k+1 \arrow[dll] & \cdots & n+2 \arrow[dll] \\
1 & \cdots & k-1 & \cdots & n & &
\end{tikzcd}
$$

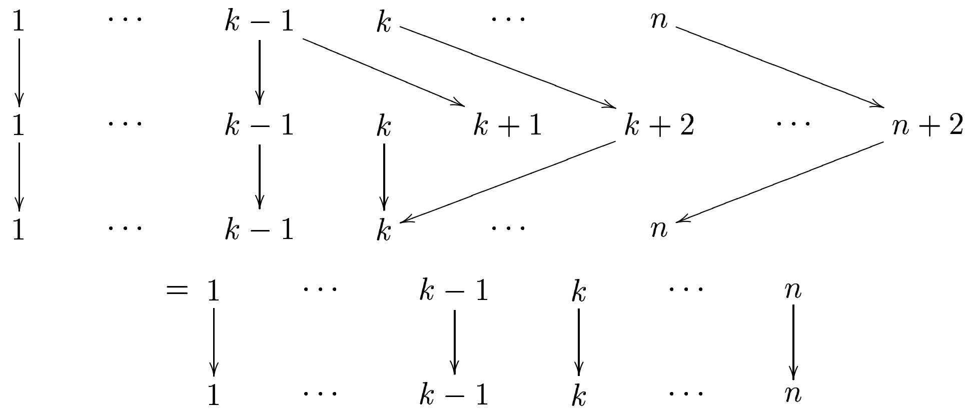
$$
= \begin{tikzcd}
1 \arrow[d] & \cdots & k-1 \arrow[d] & \cdots & n \arrow[d] \\
1 & \cdots & k-1 & \cdots & n
\end{tikzcd}
$$

$$B_k^t D_k = O :$$

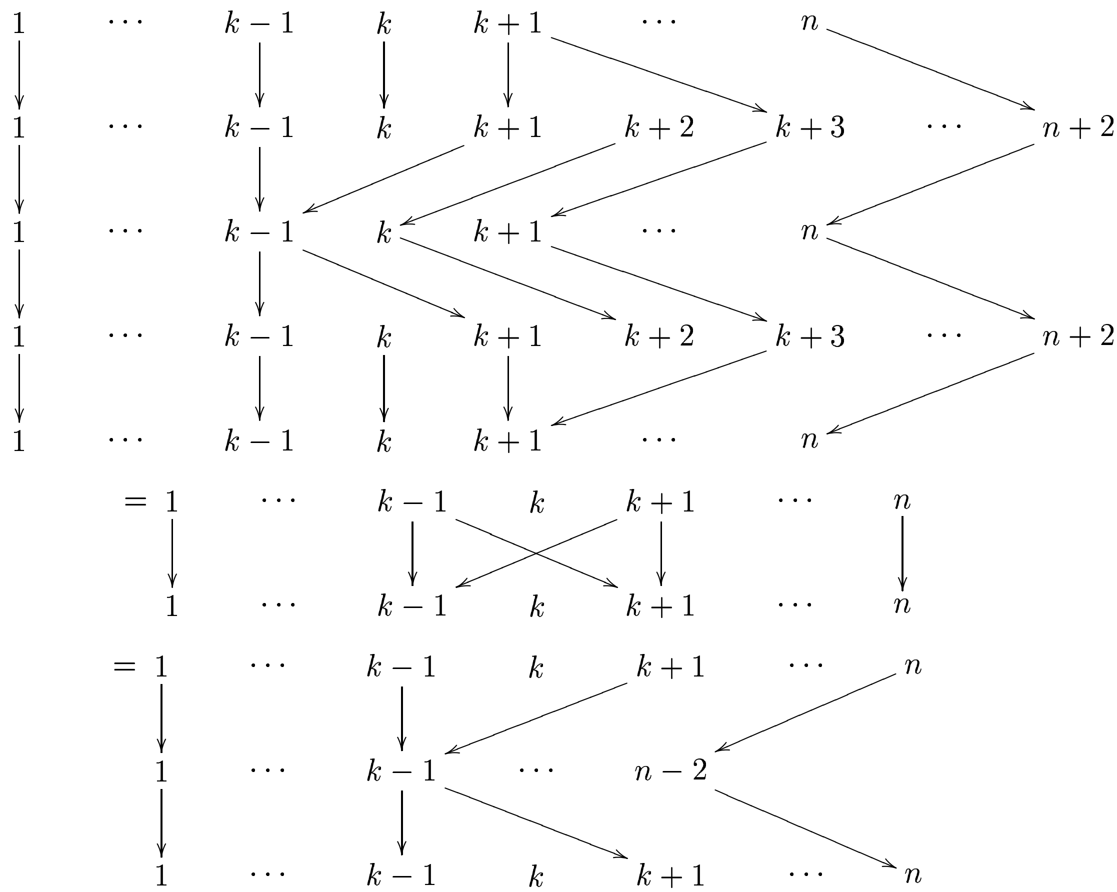
$$
\begin{tikzcd}
1 & \cdots & k-1 & k \arrow[d] & k+1 & \cdots & n+2 \\
1 \arrow[d] & \cdots & k-1 \arrow[d] & k & k+1 \arrow[dll] & \cdots & n+2 \arrow[dll] \\
1 & \cdots & k-1 & \cdots & n & &
\end{tikzcd}
$$

$$
= \begin{tikzcd}
1 & \cdots & k-1 & k & k+1 & \cdots & n \\
1 & \cdots & k-1 & k & k+1 & \cdots & n
\end{tikzcd}
$$

$$B_{k+1}^t B_k = I :$$

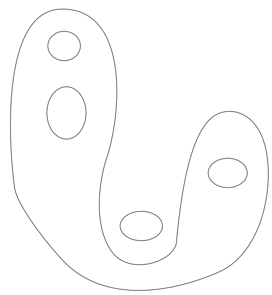


$$B^t_{k+2}B_kB^t_kB_{k+2} = B_kB^t_k :$$

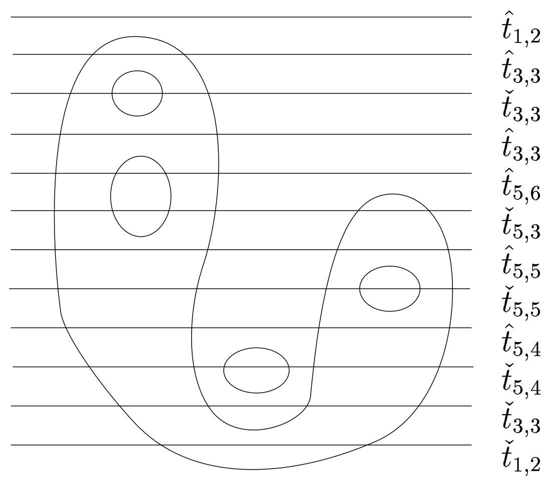


B An example of calculating the invariant for a SNPC

In this appendix we give an example of the calculation of the invariant for the following system of non-singular planar curves:



First we decompose the SNPC as a composition of generators ($\hat{t}_{n,k}$ and $\check{t}_{n,k}$):



Thus this SNPC $\check{t}_{1,2} \circ \check{t}_{3,3} \circ \check{t}_{5,4} \circ \hat{t}_{5,4} \circ \check{t}_{5,5} \circ \hat{t}_{5,5} \circ \check{t}_{5,3} \circ \hat{t}_{5,6} \circ \hat{t}_{3,3} \circ \check{t}_{3,3} \circ \hat{t}_{3,3} \circ \hat{t}_{1,2}$ is represented by the operator $\check{T}_{1,2} \circ \check{T}_{3,3} \circ \check{T}_{5,4} \circ \hat{T}_{5,4} \circ \check{T}_{5,5} \circ \hat{T}_{5,5} \circ \check{T}_{5,3} \circ \hat{T}_{5,6} \circ \hat{T}_{3,3} \circ \check{T}_{3,3} \circ \hat{T}_{3,3} \circ \hat{T}_{1,2}$. The calculation of the evaluation of this operator on $([1], \emptyset)$ is presented in the following sequence:

$$([1],\emptyset) \overset{\hat{T}_{1,2}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \end{pmatrix} \right) \overset{\hat{T}_{3,3}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \end{pmatrix} \right) \overset{\check{T}_{3,3}}{\mapsto}$$

$$\overset{\check{T}_{3,3}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \varphi(\emptyset) \\ \emptyset \end{pmatrix} \right) \overset{\hat{T}_{3,3}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \varphi(\emptyset) \\ \emptyset \\ \varphi(\emptyset) \\ \emptyset \end{pmatrix} \right) \overset{\hat{T}_{5,6}}{\mapsto}$$

$$\overset{\hat{T}_{5,6}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \varphi(\emptyset) \\ \emptyset \\ \varphi(\emptyset) \\ \emptyset \\ \emptyset \\ \emptyset \end{pmatrix} \right) \overset{\check{T}_{5,3}}{\mapsto}$$

$$\overset{\check{T}_{5,3}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \varphi(\emptyset) \oplus \varphi(\emptyset) \\ \emptyset \\ \emptyset \\ \emptyset \end{pmatrix} \right) \overset{\hat{T}_{5,5}}{\mapsto}$$

$$\overset{\hat{T}_{5,5}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \varphi(\emptyset) \oplus \varphi(\emptyset) \\ \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \\ \emptyset \end{pmatrix} \right) \overset{\check{T}_{5,5}}{\mapsto}$$

$$\overset{\check{T}_{5,5}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \varphi(\emptyset) \oplus \varphi(\emptyset) \\ \emptyset \\ \varphi(\emptyset) \\ \emptyset \end{pmatrix} \right) \overset{\hat{T}_{5,4}}{\mapsto}$$

$$\overset{\hat{T}_{5,4}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \emptyset \\ \varphi(\emptyset) \oplus \varphi(\emptyset) \\ \emptyset \\ \emptyset \\ \emptyset \\ \varphi(\emptyset) \\ \emptyset \end{pmatrix} \right) \overset{\check{T}_{5,4}}{\mapsto}$$

$$\overset{\check{T}_{5,4}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \varphi(\emptyset) \\ \varphi(\emptyset) \oplus \varphi(\emptyset) \\ \varphi(\emptyset) \\ \varphi(\emptyset) \\ \varphi(\emptyset) \end{pmatrix} \right) \overset{\check{T}_{3,3}}{\mapsto}$$

$$\overset{\check{T}_{3,3}}{\mapsto} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{pmatrix} \varphi(\emptyset) \\ \varphi(\emptyset) \oplus \varphi(\emptyset) \oplus \varphi(\emptyset) \\ \varphi(\emptyset) \end{pmatrix} \right) \overset{\check{T}_{3,3}}{\mapsto} ([1], \varphi(\emptyset) \oplus \varphi(\varphi(\emptyset) \oplus \varphi(\emptyset) \oplus \varphi(\emptyset)))$$

Thus we have that the value associated to this system of non-singular planar curves is $\varphi(\emptyset) \oplus \varphi(\varphi(\emptyset) \oplus \varphi(\emptyset) \oplus \varphi(\emptyset))$.

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