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CHARACTERIZATION OF FRACTIONALLY DIFFERENTIABLE FUNCTIONS

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We study the existence of the Riemann-Liouville, Caputo a.o. fractional derivatives of a given function. Let us formulate a result about Caputo differentiability. Denote by $\mathcal{H}_0^\alpha[0,T]$, $0<\alpha<1$, the closed subspace of the standard Hölder space $\mathcal{H}^\alpha[0,T]$:

$$v \in \mathcal{H}_0^\alpha[0,T] \text{iff} \sup_{0\le s<t\le T,\ t-s\le\varepsilon} |v(t)-v(s)|\,(t-s)^{-\alpha} \to 0 \text{ as } \varepsilon \to 0$$

.

Theorem 1. *For $v \in C^m[0,T]$, $m \in \mathbb{N}_0 = \{0,1,2,...\}$, $m<\alpha<m+1$, the following conditions (i),(ii),(iii) are equivalent:*

(i) the fractional derivative $D^\alpha_{Cap} v \in C[0,T]$ exists;

(ii) a finite limit $\lim_{t\to 0} t^{m-\alpha}\big(v^{(m)}(t)-v^{(m)}(0)\big) =: \gamma_m$ exists, and the Riemann improper integrals $\int_{\theta t}^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds$, $0<t<T$, equiconverge:

$$\sup_{0<t\le T}\left|\int_{\theta t}^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds\right| \to 0 \ \text{ as } \theta\uparrow 1;$$

(iii) $v^{(m)}$ has the structure $v^{(m)}-v^{(m)}(0) = \gamma_m t^{\alpha-m}+v_m$ where γ_m is a constant, $v_m \in \mathcal{H}_0^{\alpha-m}[0,T]$, and $\int_0^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds =: w_m(t)$ converges for every $t\in(0,T]$ defining a function $w_m \in C(0,T]$ with a finite limit $\lim_{t\to 0} w_m(t) =: w_m(0)$.

For $v\in C^m[0,T]$ with $D^\alpha_{Cap} v \in C[0,T]$, it holds $(D^\alpha_{\mathrm{Cap}} v)(0) = \Gamma(\alpha+1-m)\gamma_m$,

$$(D^\alpha_{\mathrm{Cap}} v)(t) = \frac{1}{\Gamma(m+1-\alpha)}\Big(t^{m-\alpha}\big(v^{(m)}(t)-v^{(m)}(0)\big)$$

$$+(\alpha-m)\int_0^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds\Big),\ \ 0<t\le T.$$

The results are used in treating the Abel equation (with a coefficient function) and in numerical methods for it.