Pointwise asymptotic stability of steady fluid motions

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Abstract

We are interested in establishing some sufficient conditions for the pointwise asymptotic stability of steady solutions of the Navier-Stokes equations. According to the Liapunov stability theory for solutions to ordinary differential equations, as stability property we mean an arbitrary measure in norm of the perturbation, which is uniform from the initial instant; as asymptotic stability we mean a stability property jointly with attractivity property for large t, with respect to the same norm; as pointwise asymptotic stability we mean the asymptotic stability with respect the uniform norm ($C(\overline{\Omega})$). It is known (cf. [1, 2]) that the perturbation (u, π) satisfies the following initial boundary value problem:

$$\begin{aligned} & u_t + u \cdot \nabla u + v \cdot \nabla u + u \cdot \nabla v + \nabla \pi = \frac{1}{R} \Delta u, \\ & \nabla \cdot u = 0, \quad \text{in } \Omega \times (0, T), \\ & u(x,t) = 0, \text{ on } \partial\Omega \times (0,T), \ u(x,0) = u_0(x) \text{ in } \Omega. \end{aligned} \tag{1}$$

The symbol u_t denotes $\dfrac{\partial}{\partial t} u$ and, for any pair of vectors (a, b), by $a \cdot \nabla b$ we mean the term $(a \cdot \nabla) b$. By R we indicate the Reynolds number associated to the unperturbed motion $(v, \tilde{\pi})$ and by u_0 the initial value of the perturbation. The domain $\Omega \subset \mathbb{R}^3$ is assumed bounded and $C^{2,\alpha}$-smooth ($\alpha \in (0, 1)$). Usually, the nonlinear stability of a steady motion is studied with respect to the L^2−norm of the perturbations and the motion is said "stable in energy". A very interesting approach in studying the energy stability is the one based on a variational formulation. It was introduced by Serrin in [3] and developed by several authors. The advantage of such a formulation essentially consists in the possibility of stating a critical Reynolds number R_c. If $R < R_c$, then the steady motion is unconditionally stable in the L^2-norm and asymptotically stable also. The condition $R < R_c$ means that the result of stability is related

to a family of motions, that is any motion with Reynolds number R ($< R_c$) is stable. However, as stressed in [1], from a physical view point it is interesting to evaluate the stability of a motion with respect to the uniform norm also. A coherent way to attack the question is to consider a continuous distribution of velocity of the perturbation at the initial instant and to assume finite its maximum modulus value. Then, one establishes the evolution of the perturbation. *A priori* no other regularity requirement is plausible. Of course, the above assumption implies that the perturbation to the initial instant is in $L^2(\Omega)$, that is it has finite energy. However, the energy stability does not imply pointwise stability. Actually, we have the following implication: the energy stability implies attractivity of the basic motion $(v, \tilde{\pi})$ with respect to the uniform norm. That is there exists an instant $T_0 = T_0(|u_\circ|_2, R, \Omega)$ such that $u(x,t) \in C(\overline{\Omega})$ and $|u(x,t)| \leq C(|u_0|_2, R, w, \Omega)e^{-\gamma t}$ for any $t \geq T_0$. If we assume that $u_0(x) \in C(\overline{\Omega}) \cap L^2(\Omega)$, even if we make an assumption of smallness as $\max_{\overline{\Omega}} |u_0(x)| + |u_0|_2 << \varepsilon$, we do not know if the L^2-theory ensures that $u(x,t)$ exists as a classical solution (smooth in the ordinary sense) for $t > 0$ and, in particular, if $|u(x,t)| < \infty$ for any $(x,t) \in \Omega \times [0, T_0)$. If we assume $u_0(x) \in C(\overline{\Omega}) \cap L^3(\Omega)$, then we have a quite analogous statement in a neighborhood of $t = 0$.
As far as the pointwise asymptotic stability is concerned we prove the following result:

Theorem 1 *Let us assume $v \in C^{1,\alpha}(\overline{\Omega})$ in system (1), for some $\alpha \in (0,1)$. There exist two positive numbers $\mathscr{R}_c$ and μ such that if $R < \mathscr{R}_c$ and $u_0 \in C(\overline{\Omega})$ with $u_{0|\partial\Omega} = 0$, $\nabla \cdot u_0 = 0$ and $|u_0|_0 < \mu^{-1}$, then system (1) admits a unique classical solution (u, π), defined for any $t > 0$, and*

$$|u(t)|_0 \leq c(|u_0|_0)e^{-\overline{\gamma}t}, t > 0,$$

where $c(|u_0|_0) = \dfrac{\mu^{\frac{1}{2}}|u_0|_0}{1 + (1 - \mu|u_0|_0)^{\frac{1}{2}}}$ and $\overline{\gamma} = \frac{5}{8}\gamma$, where γ is the constant of the Poincaré inequality.

References

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