Random permutations, random matrices and integrable systems

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LECTURE I : Random permutations, words, longest increasing sequences

LECTURE II : General measure on partitions, Toeplitz and Fredholm determinants, the 2d-Toda lattice and Virasoro constraints

LECTURE III: Random matrices

LECTURE IV : Dyson's Brownian motion (1962) and Airy process

LECTURE I : Random permutations, words, longest increasing sequences

(i) **Permutations**

(ii) **Words**

(iii) **Growth processes**

(α) **Queuing problem**

(β) **Discrete polynuclear growth models**:

(i) **Permutations** π of $1, \ldots, n$: $(\# S_n = n!)$

$$S_n \ni \pi = \begin{pmatrix} 1, \ldots, n \\ j_1, \ldots, j_n \end{pmatrix}, \qquad \begin{array}{l} 1 \leq j_1, \ldots, j_n \leq n \\ \text{all distinct integers} \end{array}$$

An *increasing subsequence* of $\pi \in S_n$ is a sequence $1 \leq i_1 < ... < i_k \leq n$, such that

$$\pi(i_1) < ... < \pi(i_k).$$

$$L_n(\pi) := \left\{ \begin{array}{l} \text{length of the longest increasing} \\ \text{subsequence of } \pi \in S_n \ . \end{array} \right\}$$

Question: Given uniform probability on S_n,

$$P(L(\pi) \leq k, \ \pi \in S_n) \ ?$$

(Ulam 1961)

Example: for $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & \underline{1} & 4 & \underline{3} & 2 \end{pmatrix}$, we have $L(\pi_5) = 2$

The **Robinson-Schensted-Knuth algorithm** (RSK):

For $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & \underline{1} & 4 & \underline{3} & 2 \end{pmatrix}$

one computes:

$$P \Longrightarrow \quad \begin{array}{l} 5 \end{array} \qquad \begin{array}{l} 1 \\ 5 \end{array} \qquad \begin{array}{ll} 1 & 4 \\ 5 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 4 & \\ 5 & \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 3 & \\ 4 & \\ 5 & \end{array}$$

$$Q \Longrightarrow \quad \begin{array}{l} 1 \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \\ 4 & \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \\ 4 & \\ 5 & \end{array}$$

Hence (two ident. Young diagrams of size n)

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & \underline{1} & 4 & \underline{3} & 2 \end{pmatrix} \Longleftrightarrow \left(\underset{\text{standard tableau } P}{\left(\begin{array}{|c|c|} \multicolumn{2}{c}{\overbrace{\qquad}^{2}} \\ \hline 1 & \underline{2} \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \underline{5} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)}, \underset{\text{standard tableau } Q}{\left(\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right)} \right)$$

$$\Updownarrow$$

$$\overset{n}{\longleftrightarrow}$$

$$W = \begin{pmatrix} \textcircled{0} & & 0 & & 0 & & 0 & & 1 \\ \downarrow & & & & & & & & \\ \textcircled{1} & & 0 & & 0 & & 0 & & 0 \\ \downarrow & & & & & & & & \\ \textcircled{0} & \rightarrow & \textcircled{0} & \rightarrow & \textcircled{0} & \rightarrow & \textcircled{1} & \rightarrow & \textcircled{0} \\ & & & & & & & & \downarrow \\ 0 & & 0 & & 1 & & 0 & & \textcircled{0} \\ & & & & & & & & \downarrow \\ 0 & & 1 & & 0 & & 0 & & \textcircled{0} \end{pmatrix} \updownarrow \; n$$

(put 1 at entry (k, j_k) in matrix, if $\begin{pmatrix} k \\ j_k \end{pmatrix} \in \pi$)

Hence:

$$L(\pi) = \lambda_1 = \max\left\{ \sum w_{ij}, \begin{array}{l} \text{all right/down} \\ \text{paths starting from} \\ \text{entry } (1,1) \text{ to } (n,n) \end{array} \right\}$$

where

$$W = \left(w_{ij}\right)_{1\le i,j\le n} = \left\{ \begin{array}{l} (n\times n)\text{-matrix of 0 or 1's} \\ \text{with each row and column} \\ \text{containing exactly one 1} \end{array} \right\}$$

It follows:

$$\begin{aligned} P_n(L(\pi)\le k) &= \frac{1}{n!}\#\left\{ \begin{array}{l} \text{pairs of standard Young} \\ \text{tableaux } (P,Q)\text{, both of} \\ \text{same arbitrary shape } \lambda, \\ \text{with } |\lambda| = n \text{ and } \lambda_1 \le k \end{array} \right\} \\ &= \frac{1}{n!} \sum_{\substack{|\lambda|=n \\ \lambda_1\le k}} (f^\lambda)^2 \\ &= \frac{1}{n!} \sum_{\substack{|\lambda|=n \\ \lambda_1\le k}} |\lambda|!^2 \mathbf{s}_\lambda(1,0,\dots)^2 \\ &= n! \sum_{\substack{|\lambda|=n \\ \lambda_1\le k}} \mathbf{s}_\lambda(1,0,\dots)^2 \end{aligned}$$

where for $\lambda \vdash n$,

$$f^{\lambda} := \# \left\{ \begin{array}{l} \text{standard tableaux of} \\ \text{shape } \lambda \text{ with } |\lambda| = n, \\ \text{filled with integers} \\ 1, \ldots, n \end{array} \right\} = |\lambda|! \frac{\mathrm{s}_{\lambda}(u, 0, \ldots)}{u^{|\lambda|}}$$

The *Schur polynomial* s_{λ} associated with a Young diagram $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \vdash n$:

$$\mathrm{s}_{\lambda}(t_1, t_2, ...) = \det \left(\mathrm{s}_{\lambda_i - i + j}(t) \right)_{1 \leq i,j \leq n},$$

where $\mathrm{s}_i(t)$ are *elementary Schur Polynomials*:

$$e^{\sum_1^{\infty} t_i z^i} := \sum_{i \geq 0} \mathrm{s}_i(t_1, t_2, ...) z^i.$$

Generating function: (Gessel '91)

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} P(L(\pi_n) \leq k)$$

$$= \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{\substack{|\lambda| = n \\ \lambda_1 \leq k}} |\lambda|! \;\; \mathrm{s}_\lambda(1,0,0,\ldots)^2$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \leq k}} \mathrm{s}_\lambda(\sqrt{\xi},0,0,\ldots)^2$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \leq k}} \mathrm{s}_\lambda(t_1,t_2,\ldots)\; \mathrm{s}_\lambda(s_1,s_2,\ldots)\Big|_{t_i=s_i=\delta_{i1}\sqrt{\xi}}$$

$$\stackrel{*}{=} \det\left(\oint_{S^1} z^{\ell-\ell'} e^{-\sum_1^\infty (t_j z^j + s_j z^{-j})} \frac{dz}{2\pi i z}\right)_{1\leq \ell,\ell' \leq k}\Bigg|_{t_i=s_i=\delta_{i1}\sqrt{\xi}}$$

$$\stackrel{*}{=} \det\left(\oint_{S^1} z^{\ell-\ell'} e^{\sqrt{\xi}(z+z^{-1})} \frac{dz}{2\pi i z}\right)_{1\leq \ell,\ell' \leq k}$$

(Toeplitz determinant)

of Bessel functions

(Bessel functions: $e^{u(t-t^{-1})} = \sum_{-\infty}^{\infty} t^n J_n(2u)$)

Generating function: (Gessel '91)

$$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} P(L(\pi_n) \leq k)$$

$$= \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{\substack{|\lambda| = n \\ \lambda_1 \leq k}} |\lambda|! \;\; \mathrm{s}_\lambda(1,0,0,\ldots)^2$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \leq k}} \mathrm{s}_\lambda(\sqrt{\xi},0,0,\ldots)^2$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \leq k}} \mathrm{s}_\lambda(t_1,t_2,\ldots)\; \mathrm{s}_\lambda(s_1,s_2,\ldots)\Big|_{t_i=s_i=\delta_{i1}\sqrt{\xi}}$$

$$\stackrel{*}{=} \det\left(\oint_{S^1} z^{\ell-\ell'} e^{\sqrt{\xi}(z+z^{-1})} \frac{dz}{2\pi i z}\right)_{1\leq \ell,\ell' \leq k}$$

(Toeplitz determinant)

$$= \frac{1}{k!}\int_{(S^1)^k} |\Delta_k(z)|^2 \prod_{j=1}^{k}\left(e^{\sqrt{\xi}(z_j+\bar{z}_j)}\frac{dz_j}{2\pi i z_j}\right)$$

$$= \boxed{\int_{U(k)} e^{\sqrt{\xi}\ \mathrm{Tr}(M+\bar{M})} dM}$$

Haar measure on $U(k)$, expressed in spectral variables:

$$|\Delta(z)|^2 \prod_1^k \frac{dz_j}{2\pi i z_j}$$

Use:

$$\begin{aligned} &\det\,(a_{ik})_{1\le i,k\le n}\det\,(b_{ik})_{1\le i,k\le n} \\ &\quad = \sum_{\sigma\in S_n} \det\left(a_{i,\sigma(j)}\;\; b_{j,\sigma(j)}\right)_{1\le i,j\le n} \end{aligned}$$

Then we have for $z \in S^1$

$$\begin{aligned} |\Delta(z)|^2 &= \Delta_n(z)\Delta_n(\bar z) \\ &= \sum_{\sigma\in S_n} \det\left(z_{\sigma(\ell')}^{\ell-1}(\bar z_{\sigma(\ell')})^{\ell'-1}\right)_{1\le \ell,\ell'\le n} \\ &= \sum_{\sigma\in S_n} \det\left(z_{\sigma(\ell')}^{\ell-\ell'}\right)_{1\le \ell,\ell'\le n} \end{aligned}$$

Then

$$\int_{(S^1)^k} |\Delta_k(z)|^2 \prod_{j=1}^{k} \left(f(z_j) \frac{dz_j}{2\pi i z_j} \right)$$

$$= \int_{(S^1)^k} \sum_{\sigma \in S_k} \det \left(z_{\sigma(\ell')}^{\ell-\ell'} \right)_{1 \le \ell, \ell' \le k} \prod_{j=1}^{k} \left(f(z_j) \frac{dz_j}{2\pi i z_j} \right)$$

$$= \sum_{\sigma \in S_k} \det \left(\oint_{S^1} z_j^{\ell-\ell'} f(z_j) \frac{dz_j}{2\pi i z_j} \right)_{1 \le \ell, \ell' \le k}$$

$$= k! \det \left(\oint_{S^1} z^{\ell-\ell'} f(z) \frac{dz}{2\pi i z} \right)_{1 \le \ell, \ell' \le k}$$

(ii) **Words** π of length n from an alphabet $1, \ldots, p$: $(\#S_n^p = p^n)$

$$S_n^p \ni \pi = \begin{pmatrix} 1, \ldots, n \\ j_1, \ldots, j_n \end{pmatrix}, \qquad 1 \leq j_1, \ldots, j_n \leq p$$

An *increasing subsequence* of $\pi \in S_n^p$ is a sequence $1 \leq i_1 < ... < i_k \leq n$, such that $\pi(i_1) \leq ... \leq \pi(i_k)$.

$$L(\pi) = \left\{ \begin{array}{l} \text{length of the longest increasing} \\ \text{subsequence of } \pi \in S_n^p. \end{array} \right\}$$

Question: Given uniform probability on S_n^p,

$$P_n^p(L(\pi) \leq k, \ \pi \in S_n^p) \ ?$$

Example: for $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & \underline{1} & \underline{1} & 3 & \underline{2} \end{pmatrix} \in S_5^3$, we have $L(\pi) = 3$.

The **RSK algorithm** gives

For $\quad \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & \underline{1} & \underline{1} & 3 & \underline{2} \end{pmatrix} \in S_5^3$

one computes

$$\begin{array}{l} 2 \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array} \qquad \begin{array}{ll} 1 & 1 \\ 2 & \end{array} \qquad \begin{array}{lll} 1 & 1 & 3 \\ 2 & & \end{array} \qquad \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & \end{array}$$

$$\begin{array}{l} 1 \end{array} \qquad \begin{array}{l} 1 \\ 2 \end{array} \qquad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \qquad \begin{array}{lll} 1 & 3 & 4 \\ 2 & & \end{array} \qquad \begin{array}{lll} 1 & 3 & 4 \\ 2 & 5 & \end{array}$$

Hence (two ident. Young diagrams λ of size n)

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & \underline{1} & \underline{1} & 3 & \underline{2} \end{pmatrix} \Longleftrightarrow \left(\overset{\overset{3}{\longleftrightarrow}}{\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}}\ , \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \right)$$

semi-standard P with letters of alphabet $1, \ldots, p$

standard Q filled with $1, \ldots, n$

$$\Updownarrow$$

$$\begin{pmatrix} \textcircled{0} & & 1 & & 0 \\ \downarrow & & & & \\ \textcircled{1} & & 0 & & 0 \\ \downarrow & & & & \\ \textcircled{1} & \longrightarrow & \textcircled{0} & \longrightarrow & \textcircled{0} \\ & & & & \downarrow \\ 0 & & 0 & & \textcircled{1} \\ & & & & \downarrow \\ 0 & & 1 & & \textcircled{0} \end{pmatrix}$$

(put 1 at entry (k, j_k) in matrix, if $\begin{pmatrix} k \\ j_k \end{pmatrix} \in \pi$)

$$\boxed{L(\pi) = \lambda_1 = \max\left\{\sum w_{ij},\ \begin{array}{l}\text{all right/down paths}\\ \text{starting from}\\ \text{entry } (1,1) \text{ to } (n,p)\end{array}\right\}}$$

where

$$W = \Big(w_{ij}\Big)_{\substack{1 \le i \le n \\ 1 \le j \le p}} = \left\{\begin{array}{l}(n \times p)\text{-matrix of}\\ 0 \text{ or } 1\text{'s with each}\\ \text{row containing}\\ \text{exactly one } 1\end{array}\right\}$$

It follows:

$$\begin{aligned}P_n^p(L(\pi) \le k) &= \frac{1}{p^n}\#\left\{\begin{array}{l}\text{semi-standard and standard}\\ \text{Young tableaux } (P,Q) \text{ of}\\ \text{same shape } \lambda\text{, with } |\lambda| = n\\ \text{and } \lambda_1 \le k\text{, filled with}\\ (1,\dots,p) \text{ and } (1,\dots,n)\end{array}\right\}\\ &= \frac{1}{p^n}\sum_{\substack{|\lambda|=n\\ \lambda_1 \le k}} \mathbf{s}_\lambda\Big(p, \frac{p}{2}, \frac{p}{3}, \dots\Big)\mathbf{s}_\lambda(1,0,\dots)\end{aligned}$$

using

$$\#\left\{\begin{array}{l}\text{semi-standard tableaux}\\ \text{of shape } \lambda \text{ filled with}\\ \text{numbers from } 1,\ldots,p\end{array}\right\} = \mathbf{s}_\lambda\Big(p, \frac{p}{2}, \frac{p}{3}, \ldots\Big).$$

Generating function:

$$\sum_{n=0}^{\infty} \frac{(p\xi)^n}{n!} P_n^p(L(\pi) \le k)$$

$$= \sum_{n=0}^{\infty} \frac{(p\xi)^n}{n!} \sum_{\substack{|\lambda| = n \\ \lambda_1 \le k}} \frac{|\lambda|!}{p^n}$$

$$\mathbf{s}_\lambda(1,0,0,\ldots)\mathbf{s}_\lambda(p, \frac{p}{2}, \frac{p}{3}, \ldots)$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \le k}} \mathbf{s}_\lambda(t_1, t_2, \ldots)\ \mathbf{s}_\lambda(s_1, s_2, \ldots) \Big|_{\substack{t_i = \delta_{i1}\xi \\ is_i = p}}$$

$$\stackrel{*}{=} \det\left(\oint_{S^1} z^{\ell-\ell'} e^{-\sum_1^\infty (t_j z^j + s_j z^{-j})} \frac{dz}{2\pi i z} \right)_{1 \le \ell, \ell' \le k} \Bigg|_{\substack{t_i = \delta_{i1}\xi \\ is_i = p}}$$

$$= \det\left(\oint_{S^1} z^{\ell-\ell'} e^{-\xi z} (1 - z^{-1})^p \frac{dz}{2\pi i z} \right)_{1 \le \ell, \ell' \le k}$$

Use:

$$(1-z)^p = e^{-p\sum_1^\infty \frac{z^j}{j}}$$

Generating function:

$$\sum_{n=0}^{\infty} \frac{(p\xi)^n}{n!} P_n^p(L(\pi) \le k)$$

$$= \sum_{n=0}^{\infty} \frac{(p\xi)^n}{n!} \sum_{\substack{|\lambda| = n \\ \lambda_1 \le k}} \frac{|\lambda|!}{p^n}$$

$$\mathrm{s}_\lambda(1,0,0,\dots)\mathrm{s}_\lambda(p,\frac{p}{2},\frac{p}{3},\dots)$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \le k}} \mathrm{s}_\lambda(t_1,t_2,\dots)\ \mathrm{s}_\lambda(s_1,s_2,\dots)\Big|_{\substack{t_i = \delta_{i1}\xi \\ is_i = p}}$$

$$= \det\left(\oint_{S^1} z^{\ell-\ell'} e^{\xi z^{-1}} (1+z)^p \frac{dz}{2\pi i z}\right)_{1\le \ell,\ell' \le k}$$

(Toeplitz determinant)

$$= \frac{1}{k!}\int_{(S^1)^k} |\Delta_k(z)|^2 \prod_{j=1}^{k} e^{\xi \bar{z}_j} (1+z_j)^p \frac{dz_j}{2\pi i z_j}$$

$$= \boxed{\int_{U(k)} e^{\xi Tr \bar{M}} \det(I+M)^p dM}$$

(iii)**Growth processes** (Johansson '99, ...)

Consider the ensemble Mat_{qp} of $q \times p$ matrices

$$W = \begin{pmatrix} w_{11} & & w_{12} & \cdots & w_{1p} \\ \downarrow & & & & \\ w_{21} & & w_{22} & \cdots & w_{2p} \\ \downarrow & & & & \\ w_{31} & \longrightarrow & w_{32} & \cdots & w_{3p} \\ \vdots & & \vdots & & \vdots \\ w_{q1} & & w_{q2} & \cdots \longrightarrow & w_{qp} \end{pmatrix}$$

The entries $w_{ij} \in \mathbb{Z}_{\geq 0}$ are i.i.d and geometrically distributed:

$$P(w_{ij} = k) = (1 - \xi)\xi^k, \quad 0 < \xi < 1, \quad k \in \mathbb{Z}_{\geq 0}.$$

Then

$$L(W) = \max_{\substack{\text{all such} \\ \text{paths}}} \left\{ \sum w_{ij}, \begin{array}{l} \text{over all right/down} \\ \text{paths starting from} \\ \text{entry } (1,1) \text{ to } (q,p) \end{array} \right\}.$$

has the distribution:

$P(L(W) \le k)$

$$= \sum_{n=0}^{\infty} P\left(L(W) \le k \Big| \sum_{i,j} w_{ij} = n \right) P\left(\sum_{i,j} w_{ij} = n \right)$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \le k}} (1-\xi)^{pq} \xi^{|\lambda|} \mathbf{s}_\lambda(q, \frac{q}{2}, \ldots) \mathbf{s}_\lambda(p, \frac{p}{2}, \ldots)$$

$$= e^{-\sum_1^\infty k t_k s_k} \sum_{\substack{\lambda \\ \lambda_1 \le k}} \mathbf{s}_\lambda(t_1, t_2, \ldots) \mathbf{s}_\lambda(s_1, s_2, \ldots) \Big|_{\substack{kt_k = q\xi^{k/2} \\ ks_k = p\xi^{k/2}}}$$

$$\stackrel{*}{=} (1-\xi)^{pq} \det \left(\oint_{S^1} z^{\ell-\ell'} e^{-\sum_1^\infty (t_j z^j + s_j z^{-j})} \frac{dz}{2\pi i z} \right)_{1 \le \ell, \ell' \le k} \Bigg|_{\substack{kt_k = q\xi^{k/2} \\ ks_k = p\xi^{k/2}}}$$

$$= (1-\xi)^{pq} \det \left(\oint_{S^1} z^{\ell-\ell'} (1 - \sqrt{\xi} z)^q (1 - \sqrt{\xi} z^{-1})^p \frac{dz}{2\pi i z} \right)_{1 \le \ell, \ell' \le k}$$

(Toeplitz determinant)

$$P(L(W) \le k)$$

$$= \sum_{n=0}^{\infty} P\left(L(W) \le k \,\Big|\, \sum_{i,j} w_{ij} = n\right) P\left(\sum_{i,j} w_{ij} = n\right)$$

$$= \sum_{\substack{\lambda \\ \lambda_1 \le k}} (1-\xi)^{pq} \xi^{|\lambda|} \mathbf{s}_\lambda(q, \frac{q}{2}, \ldots) \mathbf{s}_\lambda(p, \frac{p}{2}, \ldots)$$

$$\stackrel{*}{=} (1-\xi)^{pq}$$

$$\det\left(\oint_{S^1} z^{\ell-\ell'} e^{-\sum_1^\infty (t_j z^j + s_j z^{-j})} \frac{dz}{2\pi i z}\right)_{1 \le \ell, \ell' \le k} \Bigg|_{\substack{k t_k = q\xi^{k/2} \\ k s_k = p\xi^{k/2}}}$$

$$= \frac{(1-\xi)^{pq}}{k!}$$

$$\int_{(S^1)^k} |\Delta_k(z)|^2 \prod_{j=1}^{k} (1+\sqrt{\xi} z_j)^p (1+\sqrt{\xi}\bar{z}_j)^p \frac{dz_j}{2\pi i z_j}$$

$$= \boxed{(1-\xi)^{pq} \int_{U(k)} \det(1+\sqrt{\xi} M)^q \det(1+\sqrt{\xi}\bar{M})^p dM}$$

(α) **Queuing problem** (Glynn-Whitt '91)

Servers $1, 2, \ldots, p$ waiting on customers $1, \ldots, q$:

servers:	1	2	3	4	$\ldots$	p
customers:	5	2		1		
	6	3				
	7	4				
	8					
	$\vdots$					
	q					

$$P(V_{k,\ell} = t) = P\left(\begin{array}{c} \text{service time} = t \text{ of} \\ \text{customer } \boxed{k} \text{ by server } \boxed{\ell} \end{array} \right)$$

$$= (1-\xi)\xi^t, \quad t = 0, 1, 2, \ldots \ .$$

(i.i.d and geometric distribution)

Question: distribution of

$$D(q,p) = \left\{ \begin{array}{l} \text{departure time for the last} \\ \text{customer } \boxed{q} \text{ at the last server } \boxed{p} \end{array} \right\} ?$$

Since

$$D(q,p) = \max\Big(D(q-1,p), D(q,p-1)\Big) + V(q,p).$$

we have

$$D(q,p) = \max\left\{ \sum V_{ij}, \quad \begin{array}{l} \text{over all right/down} \\ \text{paths starting from} \\ \text{entry } (1,1) \text{ to } (q,p) \end{array} \right\}$$

$$P(D(q,p) \le k)$$

$$= \boxed{(1-\xi)^{pq} \int_{U(k)} \det(1+\sqrt{\xi}M)^q \det(1+\sqrt{\xi}\bar{M})^p dM}$$

(β) **Discrete polynuclear growth models**:

Assume all $\omega(x,t)$ i.i.d and geometric $P(\omega(i,j)=k)=(1-\xi)\xi^k$,except

$$\omega(x,t)=0 \quad \text{if } t-x \text{ is even or } |x|>t$$

Define *height function*: (with $x\in\mathbb{Z}, t\in\mathbb{Z}_+$, $h(x,0)=0, x\in\mathbb{Z}$)

$$h(x,t+1)=\max\Big(h(x-1,t),h(x,t),h(x+1,t)\Big) + \omega(x,t+1)$$

Height curve at even sites $2x$ at times $2t-1$:

$$\boxed{h(2x,2t-1)=\max\left\{\sum\omega(i,j),\ \begin{array}{l}\text{over all right/down}\\ \text{paths starting from}\\ \text{entry } (1,1)\\ \text{to } (t+x,t-x)\end{array}\right\}}$$

and

$$P(h(2x,2t-1)\le k)=(1-\xi)^{t^2-x^2} \cdot\int_{U(k)}\det(1+\sqrt{\xi}M)^{t+x}\det(1+\sqrt{\xi}\bar{M})^{t-x}dM$$

Proof: Set

$$V(i,j) := \omega(i-j, i+j-1)$$

Then

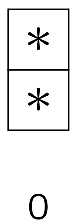
$$\begin{aligned} G(q,p) &:= h(q-p, q+p-1) \\ &= \max(G(q-1,p), G(q,p-1)) + V(q,p) \\ &= \max\left\{ \sum V_{ij}, \begin{array}{l} \text{over all right/down} \\ \text{paths starting from} \\ \text{entry } (1,1) \text{ to } (q,p) \end{array} \right\} \end{aligned}$$

So

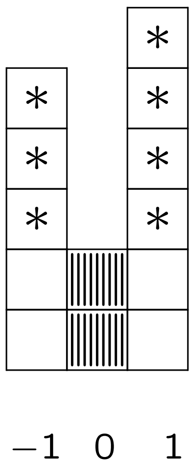
$$h(2x, 2t-1) = G(t+x, t-x)$$

(Krug-Spohn '92, Prähofer-Spohn '00, Johansson '00 and '02)

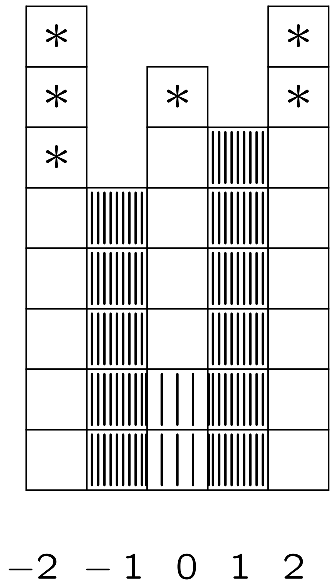
$t = 1$



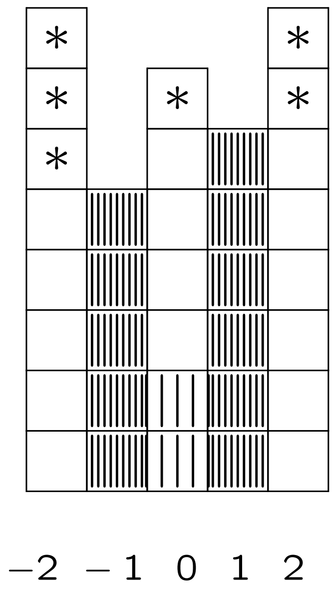
$t = 2$

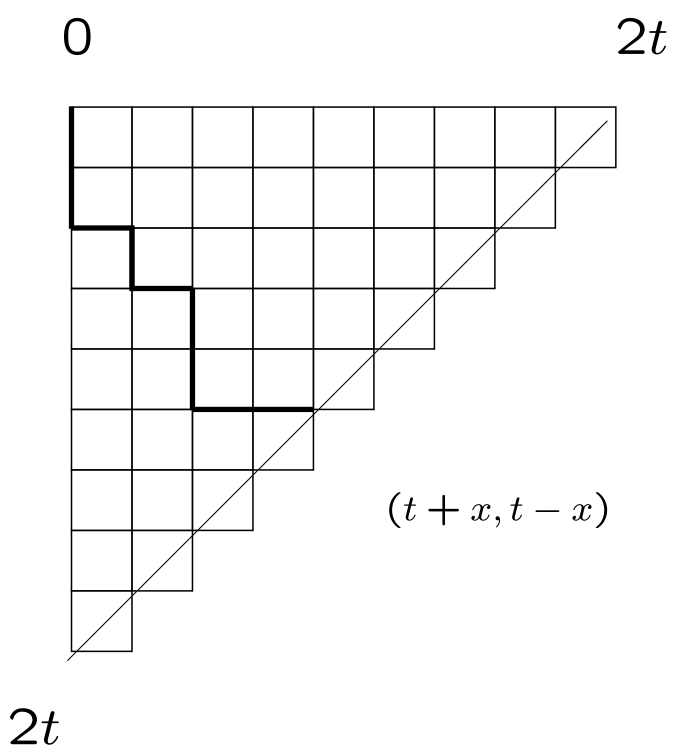


$t = 3$



$t = 3$





.

Interesting limit: $p \to \infty$

$$\lim_{p\to\infty} P\left(\frac{D(q,p) - \frac{\xi}{1-\xi}p}{\frac{\sqrt{\xi p}}{1-\xi}} \le y\right)$$

$$= \frac{\int_{(-\infty,y)^q} \Delta_q(x)^2 \prod_1^q e^{-x_i^2/2} dx_i}{\int_{\mathbb{R}^q} \Delta_q(x)^2 \prod_1^q e^{-x_i^2/2} dx_i}$$

Other limits later!

Other interesting matrix integrals:

$$\int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) e^{z\,\mathrm{Tr}(M+\bar{M})} dM$$

$$\int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) \det(I+M)^q e^{z\,\mathrm{Tr}\,\bar{M}} dM$$

$$\int_{U(n)} s_\lambda(M) s_\mu(\bar{M}) \det(I+zM)^p \det(I+z\bar{M})^q dM$$

$$\int_{Gr(p,\mathbb{C}^n)} e^{x\,\mathrm{Tr}(I+Z^\dagger Z)^{-1}} \det(Z^\dagger Z)^{-(q-p)} d\mu(Z)$$

leads to:

Random walks and other features of random permutations

Adler-PvM, Integrals over Grassmannians and Random permutations , Adv. in Math, 2003, 1-63

Adler-PvM, Virasoro action on Schur function expansions, skew Young tableaux and random walks , Comm. Pure Appl. Math. 2004

LECTURE II General measure on partitions, Toeplitz and Fredholm determinants, the 2d-Toda lattice and Virasoro constraints

"Probability measure" on

$$\lambda \in \mathbb{Y} = \{\text{all Young diagrams}\}$$

$$\boxed{\mathcal{P}(\lambda) = Z^{-1}\mathbf{s}_\lambda(t)\mathbf{s}_\lambda(s), \qquad Z = e^{\sum_{i\geq 1} it_i s_i},}$$

$$\sum_\lambda \mathcal{P}(\lambda) = Z^{-1}\sum_\lambda \mathbf{s}_\lambda(t)\mathbf{s}_\lambda(s) = 1 \text{ } \textbf{(Cauchy identity)}$$

(Borodin, Okounkov, Olshanskii '98-00)

Properties:

Statement I: (*Toeplitz determinant*)

$$\mathcal{P}\,(\lambda \text{ with } \lambda_1 \leq n)$$

$$\begin{aligned}
&= Z^{-1} \sum_{\substack{\lambda \\ \lambda_1 \leq n}} \mathbf{s}_\lambda(t)\mathbf{s}_\lambda(s) \\
&= Z^{-1} \det \left(\oint_{S^1} z^{\ell-\ell'} e^{-\sum_1^\infty (t_j z^j + s_j z^{-j})} \frac{dz}{2\pi i z} \right)_{1\leq \ell,\ell' \leq n} \\
&= Z^{-1} \int_{U(n)} e^{-\operatorname{Tr} \sum_1^\infty (t_j M^j + s_j \bar{M}^j)} dM
\end{aligned}$$

Statement II: (*Fredholm determinant*)

$$\mathcal{P}\,(\lambda \text{ with } \lambda_1 \leq n)$$

$$\begin{aligned}
&= \det \left(I - K(k,\ell) \Big|_{[n,n+1,\dots]} \right) \\
&= \sum_{m=0}^{\infty} (-1)^m \sum_{n \leq x_1 < \dots < x_m} \det \Big(K(x_i, x_j) \Big)_{1 \leq i,j \leq m}
\end{aligned}$$

where $K(k,\ell)$ is a kernel

$$(\ell - k)K(k,\ell) = \left(\frac{1}{2\pi i}\right)^2 \oint\oint_{\substack{|w| = \rho < 1 \\ |z| = \rho^{-1} > 1}} \frac{dz\ dw}{z^{k+1} w^{-\ell}}$$

$$\frac{z\frac{d}{dz}V(z) - w\frac{d}{dw}V(w)}{z - w} e^{V(z) - V(w)}.$$

with

$$V(z) = -\sum_{j \geq 1} (t_j z^{-j} - s_j z^j).$$

(Borodin, Okounkov '99)

Statement III: (*Toeplitz lattice*)

$$x_n^{\pm}(t,s) = (-1)^n \frac{\int_{U(n)} (\det M)^{\pm 1} e^{-\operatorname{Tr}\sum_1^\infty (t_i M^i + s_i \bar{M}^i)} dM}{\int_{U(n)} e^{-\operatorname{Tr}\sum_1^\infty (t_i M^i + s_i \bar{M}^i)} dM}$$

satisfy

$$\begin{aligned} \frac{\partial x_n^{\pm}}{\partial t_i} &= \mp(1 - x_n^+ x_n^-) \frac{\partial G_i}{\partial x_n^{\mp}} \\ \frac{\partial x_n^{\pm}}{\partial s_i} &= \pm(1 - x_n^+ x_n^-) \frac{\partial H_i}{\partial x_n^{\mp}} \end{aligned}$$

(Toeplitz lattice)

with Hamiltonians

$$G_i = -\frac{1}{i} \operatorname{Tr}\ L_1^i, \quad H_i = -\frac{1}{i} \operatorname{Tr}\ L_2^i, \quad i = 1, 2, 3, \ldots$$

(Adler-PvM, CPAM '91, CMP '93)

L_1 and L_2 are semi-infinite matrices (who try to be "rank 2" !)

$$L_1 = \begin{pmatrix} -x_1^+ x_0^- & 1 - x_1^+ x_1^- & 0 & 0 & \\ -x_2^+ x_0^- & -x_2^+ x_1^- & 1 - x_2^+ x_2^- & 0 & \\ -x_3^+ x_0^- & -x_3^+ x_1^- & -x_3^+ x_2^- & 1 - x_3^+ x_3^- & \\ -x_4^+ x_0^- & -x_4^+ x_1^- & -x_4^+ x_2^- & -x_4^+ x_3^- & \\ & & & & \ddots \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} -x_0^+ x_1^- & -x_0^+ x_2^- & -x_0^+ x_3^- & -x_0^+ x_4^- & \\ 1 - x_1^+ x_1^- & -x_1^+ x_2^- & -x_1^+ x_3^- & -x_1^+ x_4^- & \\ 0 & 1 - x_2^+ x_2^- & -x_2^+ x_3^- & -x_2^+ x_4^- & \\ 0 & 0 & 1 - x_3^+ x_3^- & -x_3^+ x_4^- & \\ & & & & \ddots \end{pmatrix}$$

$$\Updownarrow$$

2d-Toda Lattice $\quad (i = 1, 2$ and $n = 1, 2, \dots)$

$$\frac{\partial \hat{L}_i}{\partial t_n} = \left[\left(\hat{L}_1^n\right)_+, \hat{L}_i\right] \quad \text{and} \quad \frac{\partial \hat{L}_i}{\partial s_n} = \left[\left(\hat{L}_2^n\right)_-, \hat{L}_i\right]$$

with $\hat{L}_1 := h L_1 h^{-1}$ and $\hat{L}_2 := L_2$.

From 2d-Toda:

$$\frac{\displaystyle\int_{U(n)} e^{-\operatorname{Tr}\sum_1^\infty (t_j M^j + s_j \bar{M}^j)} dM}{\left(\displaystyle\int_{S^1} e^{-\sum_1^\infty (t_j z^j + s_j \bar{z}^j)} \frac{dz}{2\pi i z}\right)^n} = \prod_1^{n-1} (1 - x_i^+ x_i^-)^{n-i}$$

Statement IV: (*Virasoro constraints*)

$$\mathbb{V}_k(t,s,n)\ \int_{U(n)} e^{-\sum_1^\infty \mathrm{Tr}(t_j M^j + s_j \bar{M}^j)} dM = 0$$

(for $k=-1,0,1$)

where

$$\begin{aligned}
\mathbb{V}_{-1} &= \sum_{i\geq 1}(i+1)t_{i+1}\frac{\partial}{\partial t_i} - \sum_{i\geq 2}(i-1)s_{i-1}\frac{\partial}{\partial s_i} \\
&\qquad + n\left(\frac{\partial}{\partial s_1} - t_1\right) \\
\mathbb{V}_0 &= \sum_{i\geq 1}\left(it_i\frac{\partial}{\partial t_i} - is_i\frac{\partial}{\partial s_i}\right) \\
\mathbb{V}_1 &= -\sum_{i\geq 1}(i+1)s_{i+1}\frac{\partial}{\partial s_i} + \sum_{i\geq 2}(i-1)t_{i-1}\frac{\partial}{\partial t_i} \\
&\qquad + n\left(s_1 - \frac{\partial}{\partial t_1}\right).
\end{aligned}$$

(Adler-PvM, CPAM '91, CMP '93)

Proof of statement I: Moment matrix *(Toeplitz)*

$$m_n(t,s) = (\mu_{k\ell})_{1\le k,\ell\le n} = \left(\oint_{S^1} z^{k-\ell} e^{\sum_1^\infty (t_\alpha z^\alpha + s_\alpha z^{-\alpha})} \frac{dz}{2\pi i z}\right)_{1\le k,\ell\le n}$$

satisfies

$$\frac{\partial \mu_{k\ell}}{\partial t_j} = \oint_{S^1} z^{k-\ell+j} e^{\sum_1^\infty (t_\alpha z^\alpha + s_\alpha z^{-\alpha})} \frac{dz}{2\pi i z} = \mu_{k+j,\ell}$$

$$\frac{\partial \mu_{k\ell}}{\partial s_j} = \oint_{S^1} z^{k-\ell-j} e^{\sum (t_\alpha z^\alpha + s_\alpha z^{-\alpha})} \frac{dz}{2\pi i z} = \mu_{k,\ell+j}$$

with initial condition

$$\mu_{k\ell}(0,0) = \delta_{k\ell}$$

$$\Updownarrow$$

$$\frac{\partial m_\infty}{\partial t_j} = \Lambda^j m_\infty \text{ and } \frac{\partial m_\infty}{\partial s_j} = m_\infty (\Lambda^\top)^j$$

with

$$m_\infty(0,0) = I_\infty \quad \text{and} \quad \Lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Solution to this initial value problem:

(i) $m_\infty(t,s) = (\mu_{k\ell}(t,s))_{1\le k,\ell<\infty}$

(ii) $m_\infty(t,s) = e^{\sum_1^\infty t_j\Lambda^j} m_\infty(0,0) e^{\sum_1^\infty s_j\Lambda^{\top j}},$

where

$$e^{\sum_1^\infty t_j\Lambda^j} = \sum_0^\infty \mathrm{s}_i(t)\Lambda^i = \begin{pmatrix} 1 & \mathrm{s}_1(t) & \mathrm{s}_2(t) & \mathrm{s}_3(t) & \dots \\ 0 & 1 & \mathrm{s}_1(t) & \mathrm{s}_2(t) & \dots \\ 0 & 0 & 1 & \mathrm{s}_1(t) & \dots \\ 0 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Hence by uniqueness of solutions of ode's:

$$m_n(t,s) = \left. e^{\sum_1^\infty t_j\Lambda^j}\right|_{n\times\infty} m_\infty(0,0) \left. e^{\sum_1^\infty s_j\Lambda^{\top j}}\right|_{\infty\times n}$$

where

$$\left. e^{\sum_1^\infty t_j\Lambda^j}\right|_{n\times\infty} = \begin{pmatrix} 1 & \mathrm{s}_1(t) & \mathrm{s}_2(t) & \dots & \mathrm{s}_{n-1}(t) & \dots \\ 0 & 1 & \mathrm{s}_1(t) & \dots & \mathrm{s}_{n-2}(t) & \dots \\ \vdots & & & & \vdots & \\ 0 & & & 1 & \mathrm{s}_1(t) & \dots \\ 0 & \dots & & 0 & 1 & \dots \end{pmatrix}$$

Therefore the determinants coincide:

$$\det\left(\oint_{S^1} z^{k-\ell} e^{\sum_1^\infty (t_\alpha z^\alpha + s_\alpha z^{-\alpha})} \frac{dz}{2\pi i z}\right)_{1\le k,\ell\le n}$$

$$= \det\left(e^{\sum_1^\infty t_j \Lambda^j}\Big|_{n\times\infty} m_\infty(0,0)\ e^{\sum_1^\infty s_j \Lambda^{\top j}}\Big|_{\infty\times n}\right).$$

$$\stackrel{*}{=} \sum_{\substack{\lambda,\ \nu \\ \lambda_1^\top,\ \nu_1^\top \le n}} \det(m^{\lambda,\nu}(0,0))\mathbf{s}_\lambda(t)\mathbf{s}_\nu(s)$$

$$= \sum_{\substack{\lambda \\ \lambda_1^\top \le n}} \mathbf{s}_\lambda(t)\mathbf{s}_\lambda(s)$$

$$= \sum_{\substack{\lambda \\ \lambda_1^\top \le n}} \mathbf{s}_{\lambda^\top}(-t)\mathbf{s}_{\lambda^\top}(-s) = \sum_{\substack{\lambda \\ \lambda_1 \le n}} \mathbf{s}_\lambda(-t)\mathbf{s}_\lambda(-s)$$

Using **Cauchy-Binet formula**: (m large $\ge n$)

$$\det(\underset{(n,m)}{A}\quad \underset{(m,n)}{B})$$

$$= \det\left(\sum_i a_{\ell i} b_{ik}\right)_{1\le k,\ell\le n}$$

$$= \sum_{1\le i_1<\ldots<i_n\le m} \det(a_{k,i_\ell})_{1\le k,\ell\le n} \det(b_{i_k,\ell})_{1\le k,\ell\le n}.$$

Application 1: DISTRIBUTION of THE LENGTH of LONGEST INCREASING SEQUENCES in PERMUTATIONS & WORDS:

- $$\sum_{n=0}^{\infty} \frac{\xi^n}{n!} P(L(\pi) \le k, \quad \pi \in S_n) \qquad \textbf{(permutations)}$$

$$= \det \left(\oint_{S^1} z^{\ell-\ell'} e^{\sqrt{\xi}(z+z^{-1})} \frac{dz}{2\pi i z} \right)_{1 \le \ell, \ell' \le k}$$

$$= \det \left(\oint_{S^1} z^{\ell-\ell'} e^{-\sum_1^{\infty}(t_\alpha z^\alpha + s_\alpha z^{-\alpha})} \frac{dz}{2\pi i z} \right) \Bigg|_{\substack{t_i = \delta_{i1}\sqrt{\xi} \\ s_i = \delta_{i1}\sqrt{\xi}}} \quad (*)$$

$$= \boxed{\exp \int_0^{\xi} \log \left(\frac{\xi}{u} \right) g_k(u) du}$$

with g_k the unique solution to **Painlevé V**:

$$\begin{cases} g'' - \dfrac{g'^2}{2} \left(\dfrac{1}{g-1} + \dfrac{1}{g} \right) + \dfrac{g'}{u} + \dfrac{2}{u} g(g-1) - \dfrac{k^2}{2u^2} \dfrac{g-1}{g} = 0 \\ \text{with} \quad g_k(u) = 1 - \dfrac{u^k}{(k!)^2} + O(u^{k+1}), \ \text{near } u = 0. \end{cases}$$

(*) satisfies Virasoro relations + Toeplitz lattice.

(Tracy-Widom '99 Adler-PvM CPAM '01)

- $\displaystyle\sum_{n=0}^{\infty} \frac{(p\xi)^n}{n!} P(L(\pi_n) \leq k, \quad \pi \in S_n^p)$ **(words)**

$$= \det\left(\oint_{S^1} z^{\ell-\ell'} e^{-\xi z^{-1}} (1+z)^p \frac{dz}{2\pi i z}\right)_{1\leq \ell,\ell' \leq k}$$

$$= \det\left(\oint_{S^1} z^{\ell-\ell'} e^{-\sum_1^\infty (t_\alpha z^\alpha + s_\alpha z^{-\alpha})} \frac{dz}{2\pi i z}\right)\Bigg|_{\substack{t_\alpha = \delta_{\alpha 1}\xi \\ s_\alpha = \frac{p}{\alpha}}} (*)$$

$$= \boxed{\exp\left(xk + (k+p)\int_0^x \frac{h_k(u)}{u} du\right)}$$

with h_k the unique solution to **Painlevé V**:

$$\begin{cases} h''' - \dfrac{h''^2}{2}\left(\dfrac{1}{h'+1} + \dfrac{1}{h'}\right) + \dfrac{h''}{u} + \dfrac{2(k+p)}{u} h'(h'+1) \\ -\dfrac{(u-k)h' - h - k}{2u^2 h'(h'+1)}\Big((2h+u+k)h' + h + k\Big) = 0 \\ \\ \text{with} \quad h_k(u) = u\dfrac{p-k}{p+k} - \dfrac{u^{k+1}}{(k+1)!}\dbinom{p+k-1}{\ell} + O(u^{k+2}) \\ \qquad\qquad \text{near } u = 0. \end{cases}$$

(*) satisfies Virasoro relations + Toeplitz lattice.

Application 2: LENGTH of LONGEST INCREASING SEQUENCES for LARGE PERMUTATIONS:

$$\lim_{n\to\infty} P\left(2n^{1/3}\left(\frac{L(\pi_n)}{2\sqrt{n}}-1\right)<x\right)=F_2(x)$$

where $F_2(x)$ is the TW-distribution

$$F_2(x) := \exp\left(-\int_x^{\infty}(\alpha-x)g^2(\alpha)d\alpha\right),$$

with $g(\alpha)$ the solution of

$$g'' = \alpha g + 2g^3 \qquad \textbf{Painlevé II}$$

boundary condition

$$g(\alpha) \cong \begin{cases} -\dfrac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} & \text{for } \alpha \nearrow \infty \\ \sqrt{-\alpha/2} & \text{for } \alpha \searrow -\infty \end{cases}$$

(Ulam '61, Erdös-Szekeres '35, Hammersley '72, Vershik-Kerov '77, Logan-Shepp '77, Baik, Deift, Johansson, '98)

Proof: (Borodin-Okounkov-Olshanskii)

$$e^{-\xi}\sum_{n=0}^{\infty}\frac{\xi^n}{n!}P(L(\pi_n)\le k)$$

$$= \det\left(I - K(\ell,\ell')\Big|_{[k,k+1,\ldots]}\right)\Bigg|_{\substack{t_i=\delta_{i1}\sqrt{\xi}\\ s_i=\delta_{i1}\sqrt{\xi}}}$$

One checks that

$$(\ell-\ell')K(\ell,\ell')\Big|_{\substack{t_i=\delta_{i1}\sqrt{\xi}\\ s_i=\delta_{i1}\sqrt{\xi}}}$$

$$= \sqrt{\xi}\left(J_\ell(2\sqrt{\xi})J_{\ell'+1}(2\sqrt{\xi}) - J_{\ell+1}(2\sqrt{\xi})J_{\ell'}(2\sqrt{\xi})\right)$$

$$= \sum_{k=1}^{\infty} J_{\ell+k}(2\sqrt{\xi})J_{\ell'+k}(2\sqrt{\xi})$$

where $J_i(z)$ is the Bessel function.

Proof: (Borodin-Okounkov-Olshanskii)

$$e^{-\xi}\sum_{n=0}^{\infty}\frac{\xi^n}{n!}P(L(\pi_n)\le k)$$

$$= \det\left(I - K(\ell,\ell')\Big|_{[k,k+1,\ldots]}\right)\Bigg|_{\substack{t_i=\delta_{i1}\sqrt{\xi}\\ s_i=\delta_{i1}\sqrt{\xi}}}$$

One checks that

$$(\ell-\ell')K(\ell,\ell')\Big|_{\substack{t_i=\delta_{i1}\sqrt{\xi}\\ s_i=\delta_{i1}\sqrt{\xi}}}$$

$$= \sqrt{\xi}\left(J_\ell(2\sqrt{\xi})J_{\ell'+1}(2\sqrt{\xi}) - J_{\ell+1}(2\sqrt{\xi})J_{\ell'}(2\sqrt{\xi})\right)$$

$$= \sum_{k=1}^{\infty} J_{\ell+k}(2\sqrt{\xi})J_{\ell'+k}(2\sqrt{\xi})$$

where $J_i(z)$ is the Bessel function.

Asymptotics:

$\left|z^{1/3}J_{2z+xz^{1/3}}(2z) - \mathrm{Ai}(x)\right| = O(z^{-1/3})$, for $z\to\infty$ uniformly in $x\in$ compact $K\subset\mathbb{R}$.

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{u^3}{3}+xu\right)du$$

Then, for $\xi \nearrow \infty$,

$$\xi^{1/6} K(2\xi^{1/2} + x\xi^{1/6}, 2\xi^{1/2} + y\xi^{1/6})$$

$$\longrightarrow \int_0^\infty du \mathrm{Ai}(x+u)\mathrm{Ai}(y+u)$$

$$= \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$$

Hence:

$$\lim_{\xi\to\infty} e^{-\xi} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} P\left(L(\pi_n) \leq 2\xi^{1/2} + x\xi^{1/6}\right)$$

$$= 1 + \sum_{k=1}^{\infty} \int_{x \leq z_1 \leq \ldots \leq z_k} \det\Big(K_A(z_i, z_j)\Big)_{1\leq i,j\leq k} \prod_1^k dz_i$$

$$= \det\left(I - K_A \chi_{[x,\infty)}\right)$$

Finally, using *de-Poissonization Lemma* (Johansson)

$$\lim_{n\to\infty} P\left(L(\pi_n) \leq 2n^{1/2} + xn^{1/6}\right) = \det\left(I - K_A \chi_{[x,\infty)}\right)$$

Application 3: Recursion relations for U(n)-Integrals: (Borodin '03, Adler-PvM '03)

From Toeplitz lattice and Virasoro constraints:

(i) Permutations in S_n: $(t_\alpha = s_\alpha = \delta_{\alpha 1}\sqrt{\xi})$: $x_i := x_i^+ = x_i^-$

$$\int_{U(n)} e^{\sqrt{\xi}\ \mathrm{Tr}(M+\bar{M})} dM = c(\xi)^n \prod_{1}^{n-1} (1 - x_i^2)^{n-i}$$

with x_i satisfying a 3-step rational relation, (area preserving equation, with an invariant!)

$$\sqrt{\xi}(x_{i+1} + x_{i-1}) = \frac{i x_i}{x_i^2 - 1} \quad \textbf{(MacMillan equation)}$$

$^* c(\xi) = J_0(2\sqrt{-\xi})$

(ii) Words in S_n^p: $(t_\alpha = \delta_{\alpha 1}\xi$ and $s_\alpha = \frac{p}{\alpha})$

$$\int_{\mathrm{U(n)}} e^{\xi Tr \bar{M}} \det(I+M)^p dM = c(\xi)^n \prod_1^{n-1} (1 - x_i^+ x_i^-)^{n-i}$$

with x_i^+ and x_i^- satisfy 3-step and a 4-step relations, linear in x_{i+1}^+ and x_{i+1}^-:

$$(x_{i+1}^+ x_i^- - x_i^+ x_{i-1}^-) + \frac{\xi}{i+p+1}(x_i^+ x_{i+1}^- - x_{i-1}^+ x_i^-) = 0$$

$$\begin{aligned} & v_i((i+p+1)x_{i+1}^+ x_{i-1}^- - \xi) \\ & - v_{i-1}((i+p-2)x_i^+ x_{i-2}^- - \xi) \\ & - x_i^+ x_{i-1}^- (x_i^+ x_{i-1}^- - 1) \\ & = -v_1(\xi - (2+p)x_2^+) - x_1^+(x_1^+ - 1). \end{aligned}$$

setting $v_i = 1 - x_i^+ x_i^-$

*$c(\xi)$ = confluent hypergeometric function

LECTURE III: Random matrices

(1) Gaussian Hermitian ensemble (GUE)

(2) Laguerre Hermitian ensemble

(3) Jacobi Hermitian ensemble

Let

$$\mathcal{H}_n = \{n \times n \text{ matrices } X \text{ satisfying } X^\top = \bar{X}\}.$$

be the Hermitian ensemble, with probability

$$\begin{aligned} P(X \in dM) &= Z^{-1} e^{-\operatorname{Tr} V(M)} dM \\ &\qquad\qquad (\textit{Haar measure } dM \text{ on } \mathcal{H}_n) \\ &= Z^{-1} \Delta_n^2(z) \prod_{j=1}^{n} e^{-V(z_j)} dz_j dU \\ &= \frac{1}{n!} \det(K_n(z_k, z_\ell))_{1 \le k,\ell \le n} \prod_{j=1}^{n} dz_j dU \end{aligned}$$

$(z_1, \ldots, z_n)$ are the eigenvalues of M:

(Wigner, Dyson, Mehta '60's)

Explanations: **(i)** Map : $M \mapsto \Big(\text{diag}\,(z_1, \ldots, z_n), U\Big)$ such that

$M = U\text{diag}(z_1, ..., z_n)U^{-1}$, with $U = e^A \in SU(n)$,

Then

$$\begin{aligned} dM\Big|_{A=0} &= d(e^A\text{diag}(z_1, ..., z_n)e^{-A})\Big|_{A=0} \\ &= \prod_{1\le j<k\le n} (z_j - z_k)^2 \prod_1^n dz_i dU \end{aligned}$$

(ii) Christoffel-Darboux kernel

$$\begin{aligned} K_n(y,z) &= \sqrt{\frac{h_n}{h_{n-1}}} e^{-\frac{1}{2}(V(y)+V(z))} \\ &\qquad \frac{p_n(y)p_{n-1}(z) - p_{n-1}(y)p_n(z)}{y-z} \\ &= \sum_{j=0}^{n-1} p_j(y)p_j(z)e^{-\frac{1}{2}(V(y)+V(z))} \end{aligned}$$

where $p_n = \frac{1}{\sqrt{h_n}} z^n + \ldots$ are orthonormal polynomials with regard to the weight $e^{-V(y)}$.

Hence:

- $P(\text{all eigenvalues} \in A^c)$
$$= \frac{1}{n!}\int_{\mathbb{R}^n} \det\Big(K_n(z_\alpha, z_\beta)\Big)_{1\le\alpha,\beta\le n} \prod_1^n (1-\chi_A(z_i))dz_i$$
$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!}\int_{A^k} \det\Big(K_n(z_\alpha, z_\beta)\Big)_{1\le\alpha,\beta\le k} \prod_1^k dz_i$$
$$= \det(I - K_n\chi_A)$$
(Fredholm determinant)

- $E(\#\text{ of eigenvalues} \in A) = \int_A K_n(z,z)dz$

(1) Gaussian Hermitian ensemble (GUE)

All independent and Gaussian entries:

$$P(X_{ii} \in du_{ii}) = \frac{1}{\sqrt{\pi}} e^{-u_{ii}^2} du_{ii} \qquad 1 \leq i \leq n$$

$$P(\Re\, X_{jk} \in dv_{jk}) = \sqrt{\frac{2}{\pi}} e^{-2v_{jk}^2} dv_{jk} \qquad 1 \leq j < k \leq n$$

$$P(\Im\, X_{jk} \in dw_{jk}) = \sqrt{\frac{2}{\pi}} e^{-2w_{jk}^2} dw_{jk} \qquad 1 \leq j < k \leq n.$$

Hence

$$
\begin{aligned}
P(X \in dM), &\qquad M \in \mathcal{H}_n, \\
= & \prod_1^n P(X_{ii} \in dM_{ii}) \\
& \prod_{1 \le j < k \le n} P(\Re X_{jk} \in d(\Re M_{jk}))P(\Im X_{jk} \in d(\Im M_{jk})) \\
= & c_n \prod_1^n e^{-M_{ii}^2} \prod_{1 \le j < k \le n} e^{-2((\Re\ M_{jk})^2 + (\Im\ M_{jk})^2)} \\
& \prod_1^n dM_{ii} \prod_{1 \le j < k \le n} d(\Re\ M_{jk})d(\Im\ M_{jk}) \\
= & c_n e^{-\operatorname{Tr} M^2} dM \\
= & c_n \Delta_n^2(z) \prod_1^n e^{-z_i^2} dz_i dU
\end{aligned}
$$

Questions:

$$P\left(\begin{array}{l}\text{matrices } M \in \mathcal{H}_n \text{ with all} \\ \text{eigenvalues} \in A = \bigcup_{i=1}^{r}[a_{2i-1}, a_{2i}]\end{array}\right)$$

$$= \frac{\int_{A^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{-z_k^2} dz_k}{\int_{\mathbb{R}^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{-z_k^2} dz_k}$$

$$= \det(I - K_n \chi_{A^c})$$

(1) Can one write down PDE's for these probabilities?

(2) What happens when the size $n \to \infty$?

• For $A = \bigcup_{i=1}^{r}[a_{2i-1}, a_{2i}] \subset \mathbb{R}$, define differential operators:

$$\mathcal{D}_m := \sum_{1}^{2r} a_i^{m+1} \frac{\partial}{\partial a_i}$$

Then

$$F := \log P \begin{pmatrix} \text{matrices } M \in \mathcal{H}_n \text{ with all} \\ \text{eigenvalues} \in \bigcup_{i=1}^{r}[a_{2i-1}, a_{2i}] \end{pmatrix}$$

satisfies (Adler-PvM '95)

$$(\mathcal{D}_{-1}^4 + 12\mathcal{D}_0^2 - 16\mathcal{D}_{-1}\mathcal{D}_1 + 24\mathcal{D}_0 + 8n\mathcal{D}_{-1}^2)F + 6(\mathcal{D}_{-1}^2 F)^2 = 0$$

• For $A = (-\infty, a)$,

$$f = \frac{\partial}{\partial a} \log P(\text{largest eigenvalue} \leq a)$$

satisfies (Tracy-Widom '94)

$$f''' + 4(2n - a^2)f' + 4af + 6f'^2 = 0 \quad (\textbf{Painlevé IV}).$$

For **large size matrices**:

- *Wigner's semi-circle*: (non-universal)

$$\lim_{n\to\infty} E\left(\frac{1}{n}\#\{\text{eigenvalues in } A\}\right) \to \frac{2}{\pi}\int_A (\sqrt{1-t^2})_+ dt$$

- *Gaussian fluctuations about Wigner's semi-circle*: (Johansson '98) (universal)

$$\lim_{n\to\infty} P(a < \sum_1^n f(z_k) - \frac{2n}{\pi}\int_{-1}^1 \sqrt{1-t^2} f(t)dt < b) = \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$$

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$$= \int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx$$

- *Soft edge scaling limit*: (Forrester, Tracy-Widom,...) (universal) .

$$\lim_{n\nearrow\infty} P\left(2n^{\frac{2}{3}}\left(\frac{z_{\max}}{\sqrt{2n}}-1\right)\leq u\right)$$
$$= \exp\left(-\int_u^\infty (\alpha-u)g^2(\alpha)d\alpha\right) =: F_2(u),$$

with $g(\alpha)$ a solution of

$$\begin{cases} g'' = \alpha g + 2g^3 \\ g(\alpha) \cong -\frac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} \text{ for } \alpha \nearrow \infty. \end{cases} \quad \textbf{(Painlevé II)}$$

- *Bulk scaling limit*:

$$P(\text{exactly } k \text{ eigenvalues } \in [-u,u]) = \frac{(-1)^k}{k!}\left(\frac{\partial}{\partial\lambda}\right)^k \exp \int_0^{\pi u} \frac{f(\alpha;\lambda)}{\alpha} d\alpha \Big|_{\lambda=1}$$

where $f(\alpha,\lambda)$ satisfies $(' = \partial/\partial x)$

$$(\alpha f'')^2 = 4(\alpha f' - f)(-f'^2 - \alpha f' + f)$$

$$\text{with } f(\alpha;\lambda) \cong -\frac{\lambda}{\pi}\alpha \text{ for } \alpha \simeq 0.$$

(**Painlevé V**)

Jimbo, Miwa, Mori, Sato

Kamien,Mahoux,Nagao,Pastur

(2) Laguerre Hermitian ensemble

$(x_1, \ldots, x_p)^\top$ are p i.i.d, Gaussian complex variables: (i.e. covariance matrix $\Sigma = \lambda I$)

$$\begin{aligned} P(\Re x_i \in du_i) &= c\ e^{-u_i^2/\lambda} du_i \\ P(\Im x_i \in dv_i) &= c\ e^{-v_i^2/\lambda} dv_i \end{aligned}$$

n samples

$$n \text{ samples of } (x_1, \ldots, x_p)^\top$$

$$\Updownarrow$$

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pn} \end{pmatrix}$$

Then

$$\left\{ \begin{matrix} \text{sample covar.} \\ \text{matrix} \end{matrix} \right\} = X = x\bar{x}^\top = \left(\sum_{i=1}^{n} x_{ki} \bar{x}_{\ell i} \right)_{1 \le k, \ell \le p}$$

(unbiased estimator of the covariance matrix Σ)

The eigenvalues $z_1, \ldots, z_p \geq 0$ have the (Laguerre) distribution: (Hotelling, James, Constantine, Muirhead)

$$\begin{aligned} P_{n,p}(X \in dM) &= c_{np}\Delta_p^2(z)\prod_1^p e^{-z_i/\lambda} z_i^{n-p} dz_i dU \\ &= e^{-\frac{1}{\lambda}\operatorname{Tr} M}(\det M)^{n-p} dM \end{aligned}$$

• Then (normalize $\lambda = 1$)

$$F := \log P_{n,p}(\text{all eigenvalues} \in \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}])$$

satisfies ($\mathcal{D}_m := \sum_1^{2r} a_i^{m+1} \frac{\partial}{\partial a_i}$)

$$\left(\begin{array}{l} \mathcal{D}_0^4 - 2\mathcal{D}_0^3 + (1-(n-p)^2)\mathcal{D}_0^2 + 3\mathcal{D}_1^2 \\ -(n+p)\mathcal{D}_1 - 2\mathcal{D}_2 \\ -4\mathcal{D}_2\mathcal{D}_0 + 2(n+p)\mathcal{D}_1\mathcal{D}_0 \end{array} \right) F$$

$$+6(\mathcal{D}_0^2 F)^2 - 4(\mathcal{D}_0^2 F)(\mathcal{D}_0 F) = 0$$

(Adler-Shiota-PvM '95)

- For $A = (-\infty, a)$, function

$$f(a) = a \, \frac{\partial}{\partial a} \log P_{n,p}(z_{\mathsf{max}} \le a)$$

satisfies

$$-a^2 f^{'''} - a f^{''} + ((n-p-a)^2 - 4pa) f^{'}$$
$$+(n+p-a) f - 6a f'^2 + 4 f f^{'} = 0$$

(Painlevé V)

(Tracy-Widom ’94)

For **large size matrices**:

- *Soft edge scaling limit*: (Johansson '99)

. (Johnstone '00)

n samples: $$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pn} \end{pmatrix}$$

Largest principal component

$$= \left\{ \begin{array}{l} \text{largest eigenvalue of the} \\ \text{ sample cov. matrix } X = x\bar{x}^\top \end{array} \right\}$$

$$= (\text{singular value of } x)^2$$

$$= z_{\text{max}}$$

Pick

population size $\simeq$ sample size

Then:

$$\lim_{\substack{n,p \nearrow \infty \\ n/p \to \gamma \geq 1}} P\left(\frac{\sqrt{n} + \sqrt{p}}{(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{p}})^{1/3}} \Big(\frac{z_{\text{max}}}{(\sqrt{n} + \sqrt{p})^2} - 1 \Big) \leq u \right)$$

$$= F_2(u)$$

• *Hard edge scaling limit*: $\rho(dz) = z^{\nu/2}e^{-z/2}dz$.

$$\lim_{n\nearrow\infty} \frac{1}{4n} K_n^{(\nu)}\left(\frac{u}{4n}, \frac{v}{4n}\right) = K^{(\nu)}(u,v),$$

with *Bessel kernel* $K^{(\nu)}(u,v)$

$$\begin{aligned} K^{(\nu)}(u,v) &= \frac{1}{2}\int_0^1 xJ_\nu(xu)J_\nu(xv)dx \\ &= \frac{J_\nu(u)\sqrt{u}J_\nu'(v) - J_\nu(\sqrt{v})\sqrt{v}J_\nu'(\sqrt{u})}{2(u-v)}. \end{aligned}$$

Then

$$P(\text{ no eigenvalues } \in [0,x]) = \exp\left(-\int_0^x \frac{f(u)}{u}du\right),$$

with f satisfying

$$(xf'')^2 - 4\Big(xf' - f\Big)f'^2 + \Big((x-\nu^2)f' - f\Big)f' = 0$$

(Painlevé V)

Nagao, Forrester

Tracy-Widom

Adler-Shiota-van Moerbeke

(3) Jacobi Hermitian ensemble: $p+q$ normally distributed complex variables

$$(X_1,...,X_p,Y_1,...,Y_q)^\top$$

with mean zero and covariance matrix

$$\Sigma = \begin{pmatrix} \overset{\overset{p}{\longleftrightarrow}}{\Sigma_{11}} & \overset{\overset{q}{\longleftrightarrow}}{\Sigma_{12}} \\ \bar{\Sigma}_{12}^\top & \Sigma_{22} \end{pmatrix} \begin{matrix} \updownarrow p \\ \updownarrow q \end{matrix}$$

$$\curvearrowright \Sigma_{can} = \begin{pmatrix} I_p & P \\ \bar{P}^\top & I_q \end{pmatrix},$$

where

$$\begin{cases} P = \left(\begin{array}{cccccc|c} \rho_1 & & & & & & \\ & \ddots & & O & & & \\ & & \rho_k & & & & O \\ & & & \rho_{k+1} & & & \\ & O & & & \ddots & & \\ & & & & & \rho_p & \end{array}\right) \updownarrow p \quad (\text{width } p \text{ left block}, \; q \text{ total}) \\[2ex] 1 \ge \rho_1 \ge \rho_2 \ge \ldots \ge \rho_k > 0, \\[2ex] \rho_i \text{ roots of } \det(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}_{12}^\top - \rho^2 I) = 0. \end{cases}$$

n independent samples:

$$x = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pn} \\ y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \vdots & & \vdots \\ y_{q1} & y_{q2} & \cdots & y_{qn} \end{pmatrix}$$

$$\left\{\begin{matrix}\text{sample covar.} \\ \text{matrix}\end{matrix}\right\} = X := \begin{pmatrix} x\bar{x}^\top & x\bar{y}^\top \\ y\bar{x}^\top & y\bar{y}^\top \end{pmatrix},$$

and

$$\left\{\begin{matrix}\text{maximum likelihood} \\ \text{estimates of } \rho_i^2\end{matrix}\right\}$$
$$= \left\{\begin{matrix}\text{eigenvalues of} \\ X := x\bar{y}^\top (x\bar{y}^\top)^{-1} y\bar{x}^\top (x\bar{x}^\top)^{-1}\end{matrix}\right\}$$

If $\rho_1^2 = \ldots = \rho_p^2 = 0$, then the joint density of the eigenvalues is given by

$$
\begin{aligned}
&P(X \in dM) \\
&= (\det M)^{q-p-1} \det(I-M)^{n-q-p-1} dM \\
&= c_{n,p,q} \Delta_p^2(z) \prod_{i=1}^{p} z_i^{q-p-1}(1-z_i)^{n-q-p-1} dz_i dU \\
&= c'_{n,p,q} \Delta_p^2(x) \prod_{i=1}^{p} (1-x_i)^{q-p-1}(1+x_i)^{n-q-p-1} dx_i dU
\end{aligned}
$$

Then $f_n(a) = (1-a^2)\frac{d}{da}\log P_n(x_{\max} \le a)$ satisfies:

$$
\begin{aligned}
(a^2-1)^2 f''' + 2(a^2-1)\left(af'' - 3f'^2\right) \\
+ \left(8af - q(a^2-1) - 2sa - 2r\right) f' \\
- f\,(2f - qa - s) = 0.
\end{aligned}
$$

(Painlevé VI)

$$
\begin{aligned}
r &= \alpha^2 + \beta^2, \quad s = \alpha^2 - \beta^2, \quad q = (2p + \alpha + \beta)^2 \\
\alpha &= q - p - 1, \qquad \beta = n - q - p - 1
\end{aligned}
$$

(Adler-PvM, Ann. of Math '01 , Haine-Semengue '00)

Proof for soft edge for GUE: **Step I**: From asymptotics of Hermite polynomials:

$$e^{-x^2/2}\frac{H_k(x)}{2^{k/2}\sqrt{k!}\pi^{1/4}}\bigg|_{x=\sqrt{2n+1}+\frac{u}{n^{1/6}\sqrt{2}}}$$
$$=\frac{2^{1/4}}{n^{1/12}}(\ \mathrm{Ai}(u)+O(n^{-2/3})),$$

where ($\mathrm{Ai}''(y)=y\mathrm{Ai}(y)$)

$$\mathrm{Ai}(x)=\frac{1}{\pi}\int_0^\infty \cos\left(\frac{u^3}{3}+xu\right)du,$$

one finds

$$\lim_{n\nearrow\infty}\frac{1}{\sqrt{2}n^{1/6}}K_n\left(\sqrt{2n}+\frac{u}{\sqrt{2}n^{1/6}},\sqrt{2n}+\frac{v}{\sqrt{2}n^{1/6}}\right),$$
$$=\ K_A(u,v):=\int_0^\infty \mathrm{Ai}(x+u)\mathrm{Ai}(x+v)dx$$

Then

$$\lim_{n\nearrow\infty}P\left(v\leq 2n^{\frac{2}{3}}\left(\frac{\lambda_{\max}}{\sqrt{2n}}-1\right)\leq u\right)$$
$$=\ \lim_{n\nearrow\infty}P\left(v\leq\sqrt{2}n^{\frac{1}{6}}\left(\lambda_{\max}-\sqrt{2n}\right)\leq u\right)$$
$$=\ \det(I-K_A\ \chi_{[v,u]^c})$$

Step II: Differential equation for this Fredholm determinant in terms of u:

$$\Psi(x;z) = \sqrt{\frac{z}{2\pi}}\mathrm{Ai}(x+z^2)$$

satisfies

$$\begin{cases} (D_x^2 - x)\Psi(x;z) = z^2\Psi(x;z) \\ \Psi(x;z) = e^{xz+\frac{2}{3}z^3}\left(1+O(z^{-1})\right) \end{cases}$$

$$\Big\Downarrow \quad \text{Add } (t_1, t_2, \ldots)\text{-variables}$$

$$\begin{cases} (D_x^2 - q(x,t))\Psi(x,t;z) = z^2\Psi(x,t;z) \\ \Psi(x,t;z) = e^{xz+\sum_1^\infty t_i z^i}\left(1+O(z^{-1})\right) \\ \frac{\partial}{\partial t_n}\Psi = \left((D_x^2 - q(x,t))^{n/2}\right)_+\Psi \\ q(x;t)\Big|_{t=(0,0,\frac{2}{3},0,\ldots)} = x \end{cases}$$

<u>Solution</u>: *Kontsevich's integral*

$$\Psi(x,t;z) = e^{xz+\sum_1^\infty t_i z^i} \frac{\tau(t_1 - z^{-1}, t_2 - \frac{z^{-2}}{2}, t_3 - \frac{z^{-3}}{3}, \ldots)}{\tau(t_1, t_2, \ldots)},$$

$$q(x,t) = 2\frac{\partial^2}{\partial t_1^2} \log \tau(t),$$

$$\tau(t) = \frac{\int_{\mathcal{H}_N} dX e^{-\operatorname{Tr}(X^3/3 + X^2 Z)}}{\int_{\mathcal{H}_N} dX e^{-\operatorname{Tr}(X^2 Z)}}$$

$$\text{with } t_n := -\frac{1}{n} \operatorname{Tr} Z^{-n} + \frac{2}{3}\delta_{n3}$$

Kernels:

$$K(z^2, z'^2) = \int_0^\infty \mathrm{Ai}(x+z^2)\mathrm{Ai}(x+z'^2)dx$$

$$\Downarrow \quad \text{Add } (t_1, t_2, \ldots)\text{-variables}$$

$$K_t(z^2, z'^2) := \frac{1}{2z^{\frac{1}{2}} z'^{\frac{1}{2}}} \int_0^\infty \Psi(x,t;z)\Psi(x,t;z')dx$$

For $A = \cup_{i=1}^{r} [a_{2i-1}, a_{2i}] \subset \mathbb{R}$, define:

$$\mathcal{D}_m := \sum_{1}^{2r} a_i^{m+1} \frac{\partial}{\partial a_i}$$

Then

$$\begin{cases} \tau(t) = \text{Kontsevich integral} \\ \tau(t,a) := \tau(t) \det\Big(I - K_t(\lambda,\lambda')\chi_A(\lambda')\Big) \end{cases}$$

satisfy:

• The same **Virasoro constraints** (decoupling of the a- and t-variables):

$$\begin{aligned} \mathcal{D}_{-1}\tau &= \left(\frac{\partial}{\partial t_1} + \frac{1}{2}\sum_{i\geq 3} i t_i \frac{\partial}{\partial t_{i-2}} + \frac{t_1^2}{4}\right)\tau \\ \mathcal{D}_0\tau &= \left(\frac{\partial}{\partial t_3} + \frac{1}{2}\sum_{i\geq 1} i t_i \frac{\partial}{\partial t_i} + \frac{1}{16}\right)\tau \\ &\dots \end{aligned}$$

• **The Korteweg-de Vries** equation

$$\left(\left(\frac{\partial}{\partial t_1}\right)^4 - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right)\log\tau + 6\left(\frac{\partial^2}{\partial t_1^2}\log\tau\right)^2 = 0$$

From Virasoro, setting all $t_i = 0$,

$$\begin{aligned}
\mathcal{D}_{-1}\tau|_{t=0} &= \left.\frac{\partial}{\partial t_1}\tau\right|_{t=0} \\
\mathcal{D}_{-1}\mathcal{D}_{-1}\tau|_{t=0} &= \left.\frac{\partial^2}{\partial t_1^2}\tau\right|_{t=0} \\
\left.\mathcal{D}_{-1}\mathcal{D}_{0}\tau\right|_{t=0} &= \left.\left(\frac{\partial^2}{\partial t_1 \partial t_3} + \frac{1}{2}\frac{\partial}{\partial t_1}\right)\tau\right|_{t=0} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

- Setting t-partials in KdV, one finds the PDE:

$$R := \sum_1^{2r} \frac{\partial}{\partial a_i} P(\text{no eigenvalues } \in A)$$

satisfies

$$\left(\mathcal{D}_{-1}^3 - 4(\mathcal{D}_0 - \frac{1}{2})\right) R + 6(\mathcal{D}_{-1}R)^2 = 0$$

- When $A = (a, \infty)$, the function $R = \partial F / \partial a$ satisfies

$$R''' - 4aR' + 2R + 6R'^2 = 0 \quad \textbf{(Chazy)}$$

$$R''' - 4aR' + 2R + 6R'^2 = 0 \quad \textbf{(Chazy)}$$

$$\Downarrow$$

$$(R'')^2 + 4R'(R'^2 - aR' + R) = 0$$

$$\Downarrow$$

$$\text{Setting } \left\{ \begin{array}{l} R' = -g^2 \\ R = g'^2 - ag^2 - g^4 \end{array} \right\} \text{ implies}$$

$$g'' = 2g^3 + ag$$

$$\Downarrow$$

$$-g(a)^2 = R'(a) = \frac{\partial^2}{\partial a^2} \log P(\text{all eigenvalues } \leq a)$$

$$\Downarrow$$

$$
\begin{aligned}
-g(a)^2 &= R'(a) \\
&= \frac{\partial^2}{\partial a^2} \log P(\text{no eigenvalues } \in (a,\infty)) \\
&= \frac{\partial^2}{\partial a^2} \log P(\lambda_{\text{max}} \leq a)
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\lim_{n \nearrow \infty} P\left(2n^{\frac{2}{3}}\left(\frac{\lambda_{\text{max}}}{\sqrt{2n}} - 1\right) \leq u\right) \\
&\qquad = \exp\left(-\int_u^\infty (\alpha - u) g^2(\alpha) d\alpha\right),
\end{aligned}
$$

with $g(\alpha)$ a solution of

$$
\begin{cases}
g'' = \alpha g + 2g^3 \\
g(\alpha) \cong -\frac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} \text{ for } x \nearrow \infty.
\end{cases}
\qquad \textbf{(Painlevé II)}
$$

LECTURE IV : Dyson's Brownian motion (1962) and Airy process

Coulomb gas, consisting of n point charges executing Brownian motions, subjected to mutual electrostatic repulsions

$$\Phi = Ce^{-\beta W}, \qquad \beta = (kT)^{-1} = 2$$

with

$$W(\lambda_1, \ldots, \lambda_n) = -\sum_{i<j} \ln |\lambda_i - \lambda_j| + \sum_i \frac{\lambda_i^2}{2a^2}$$

$$\Updownarrow$$

This gas gives an exact description of the behavior of the eigenvalues on a $n \times n$ Hermitian matrix, when the elements execute independent Brownian motions without mutual interaction.

Consider n^2 independent **Ornstein-Uhlenbeck diffusions** on the n^2 entries B_ν of the Hermitian matrix B:

$$\frac{\partial P}{\partial t} = \sum_{\nu=1}^{n^2} \left(\frac{1}{4}(1+\delta_\nu) \frac{\partial^2}{\partial B_\nu^2} + \frac{\partial}{\partial B_\nu} B_\nu \right) P$$

$$\Updownarrow$$

Then the n eigenvalues of the matrix B

$$\lambda_1(t) < \ldots < \lambda_n(t) \text{ on } \mathbb{R}$$

evolve according the following **diffusion** with transition density $p(t, \mu, \lambda)$:

$$\begin{aligned} \frac{\partial p}{\partial t} &= \frac{1}{2} \sum_1^n \frac{\partial}{\partial \lambda_i} \Phi(\lambda) \frac{\partial}{\partial \lambda_i} \frac{1}{\Phi(\lambda)} p, \\ &= \sum_1^n \left(\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} - \frac{\partial}{\partial \lambda_i} \frac{\partial \log \sqrt{\Phi(\lambda)}}{\partial \lambda_i} \right) p \end{aligned}$$

with

$$\Phi(\lambda) = \Delta_n^2(\lambda) \prod_1^n e^{-\lambda_i^2}.$$

Transition density of the O-U process ($r = e^{-t}$)

$$P(t, \bar{B}, B) \;=\; \frac{Z^{-1}}{(1-r^2)^{n^2/2}} e^{-\frac{1}{(1-r^2)} \operatorname{Tr}(B - r^2 \bar{B})^2},$$

• Equilibrium measure: (Haar $dB = \Delta_n^2(\lambda) \prod_1^n d\lambda_i dU$)

$$P(B(t) \in dB) \;=\; Z^{-1} e^{-\operatorname{Tr} B^2} dB$$

"GUE distribution"

• Joint distribution (assuming equilibrium measure at $t = 0$):

$$P(B(0) \in dB_1, B(t) \in dB_2)$$

$$= \; Z^{-1} \frac{dB_1 dB_2}{(1-r^2)^{n^2/2}} e^{-\frac{1}{(1-r^2)} \operatorname{Tr}(B_1^2 - 2rB_1B_2 + B_2^2)}$$

(2-matrix model)

• For $A = \bigcup_1^r [a_{2i-1}, a_{2i}] \subset \mathbb{R}$,

$$P_n(\text{Sp } B(t) \in A) = P_n(\text{all } \lambda_i(t) \in A)$$

$$= Z^{-1} \int_A \Delta_n^2(x) \prod_1^n e^{-x_i^2} dx_i$$

$$= \det(I - K^{(n)} \chi_{A^c})$$

satisfies $\quad (\mathcal{D}_m := \sum_1^{2r} a_i^{m+1} \frac{\partial}{\partial a_i})$

$$F := \log P_n \left(\begin{array}{l} \text{matrices } M \in \mathcal{H}_n \text{ with all} \\ \text{eigenvalues} \in \bigcup_{i=1}^r [a_{2i-1}, a_{2i}] \end{array} \right)$$

satisfies (Adler-PvM '95)

$$(\mathcal{D}_{-1}^4 + 12\mathcal{D}_0^2 - 16\mathcal{D}_{-1}\mathcal{D}_1 + 24\mathcal{D}_0 + 8n\mathcal{D}_{-1}^2)F + 6(\mathcal{D}_{-1}^2 F)^2 = 0$$

In particular, for $A = (-\infty, a)$

$$f_n(a) = \frac{d}{da} \log P_n(\max_i \lambda_i \leq a) = \frac{d}{da} \log P_n(\lambda_n \leq a)$$

satisfies **Painlevé IV**

$$f''' + 6 \, f'^2 + 4(2n - a^2) f' + 4af = 0$$

• For $A_1 := \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}], \quad A_2 := \bigcup_{i=1}^{s} [b_{2i-1}, b_{2i}]$

$$P_n(\text{Sp } B(t_1) \in A_1, \quad \text{Sp } B(t_2) \in A_2)$$

$$= P_n(\text{all } \lambda_i(t_1) \in A_1, \text{all } \lambda_i(t_2) \in A_2)$$

$$= Z'^{-1} \iint_{A_1 \times A_2} \frac{\Delta_n(x)\Delta_n(y)}{(1-r^2)^{n^2/2}} \prod_{i=1}^{n} e^{-\frac{1}{(1-r^2)}(x_i^2 - 2rx_iy_i + y_i^2)} dx_i dy_i$$

$$= \det\left(I - (\chi_{A_k^c} K_{t_k t_\ell}^{(n)} \chi_{A_\ell^c})_{1 \le k, \ell \le 2}\right)$$

satisfies ($G_n = \log P_n$, $r = e^{-(t_2 - t_1)}$)

$$\mathcal{A}_1 \frac{\mathcal{B}_2 \mathcal{A}_1 G_n}{\mathcal{B}_1 \mathcal{A}_1 G_n + 2nr} = \mathcal{B}_1 \frac{\mathcal{A}_2 \mathcal{B}_1 G_n}{\mathcal{A}_1 \mathcal{B}_1 G_n + 2nr}.$$

where ($r = e^{-t}$)

$$\begin{aligned}
\mathcal{A}_1 &= \sum_1^{2r} \frac{\partial}{\partial a_j} + r \sum_1^{2s} \frac{\partial}{\partial b_j} \\
\mathcal{B}_1 &= r \sum_1^{2r} \frac{\partial}{\partial a_j} + \sum_1^{2s} \frac{\partial}{\partial b_j} \\
\mathcal{A}_2 &= \sum_1^{2r} a_j \frac{\partial}{\partial a_j} + r^2 \sum_1^{2s} b_j \frac{\partial}{\partial b_j} + (1 - r^2) \frac{\partial}{\partial t} - r^2 \\
\mathcal{B}_2 &= r^2 \sum_1^{2r} a_j \frac{\partial}{\partial a_j} + \sum_1^{2s} b_j \frac{\partial}{\partial b_j} + (1 - r^2) \frac{\partial}{\partial t} - r^2.
\end{aligned}$$

(Adler-PvM, Ann. of Math '99, Ann. of Prob. '04)

Classical Christoffel-Darboux kernel:

$$K^{(n)}(x,y) := \sum_{k=1}^{\infty} \varphi_{n-k}(x)\varphi_{n-k}(y)$$

Extended Christoffel-Darboux kernel:

$$K^{(n)}_{t_i t_j}(x,y) :=$$

$$\begin{cases} \displaystyle\sum_{k=1}^{\infty} e^{-k(t_i-t_j)}\varphi_{n-k}(x)\varphi_{n-k}(y), & \text{if } t_i \geq t_j \\ \displaystyle -\sum_{k=-\infty}^{0} e^{k(t_j-t_i)}\varphi_{n-k}(x)\varphi_{n-k}(y), & \text{if } t_i < t_j \end{cases},$$

Set

$$\widehat{K}_{t_it_j}(x,y) := \chi_{E_i^c}(x) K_{t_it_j}(x,y) \chi_{E_j^c}(y).$$

Matrix Fredholm determinants:

$$P(x(t_1) \in E_1, x(t_2) \in E_2)$$
$$= \; 1 + \sum_{N=1}^{\infty} (-z)^N \sum_{\substack{0 \le r,s \le N \\ r+s=N}} \int_{\left\{ \begin{array}{c} -\infty < \alpha_1 \le \cdots \le \alpha_r < \infty \\ -\infty < \beta_1 \le \cdots \le \beta_s < \infty \end{array} \right\}}$$

$$\prod_1^r d\alpha_i \prod_1^s d\beta_i$$

$$\det \left(\begin{array}{cc} \left(\widehat{K}_{t_1t_1}(\alpha_i, \alpha_j) \right)_{1 \le i,j \le r} & \left(\widehat{K}_{t_1t_2}(\alpha_i, \beta_j) \right)_{\substack{1 \le i \le r \\ 1 \le j \le s}} \\ \left(\widehat{K}_{t_2t_1}(\beta_i, \alpha_j) \right)_{\substack{1 \le i \le s \\ 1 \le j \le r}} & \left(\widehat{K}_{t_2t_2}(\beta_i, \beta_j) \right)_{1 \le i,j \le s} \end{array} \right|_{z=1}$$

Reminder: SPECTRUM of RANDOM MATRICES NEAR THE EDGE:

$$\lim_{n\to\infty} P\left(\sqrt{2}\; n^{1/6}(\lambda_n - \sqrt{2n}) \le x\right) = F_2(x)$$

where $F_2(x)$ is the TW-distribution

$$F_2(x) := \exp\left(-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha\right),$$

with $g(\alpha)$ a solution of

$$g'' = \alpha g + 2g^3 \qquad \textbf{Painlevé II}$$

boundary condition

$$g(\alpha) \cong \begin{cases} -\dfrac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} \text{ for } \alpha \nearrow \infty \\ \sqrt{-\alpha/2} \text{ for } \alpha \searrow -\infty \end{cases}$$

The AIRY PROCESS:

$$\boxed{A(t) = \lim_{n\to\infty} \sqrt{2} n^{1/6} \left(\lambda_n (n^{-1/3} t) - \sqrt{2n} \right).}$$

"Motion of the right most particle in the rescaled Dyson Brownian motion"

(Prähofer-Spohn '02)

Properties:

(i) continuous sample paths.

(ii) Stationary process, with $P(A(t) \leq u) = F_2(u)$.

$$f_n(a) \;\; := \;\; \frac{d}{da} \log P_n(\max_i \lambda_i(t) \leq a)$$

satisfies

$$f''' + 6 \; f'^2 + 4(2n - a^2)f' + 4af = 0 \qquad \textbf{P IV}$$

$$\Bigg\Downarrow \begin{array}{l} a = \sqrt{2n} + \frac{u}{\sqrt{2}n^{1/6}} \\ n \to \infty \end{array}$$

$$f(u) := \frac{d}{du} \log P(A(t) \leq u)$$

satisfies

$$f''' - 4uf' + 2f + 6f'^2 = 0. \qquad \textbf{P II}$$

(iii) Joint probability

The disjoint union of intervals

$E_1 := \cup_{i=1}^{r} [u_{2i-1}, u_{2i}], \quad E_2 := \cup_{i=1}^{s} [v_{2i-1}, v_{2i}]$

define linear operators

$$L_u := \sum_1^{2r} \frac{\partial}{\partial u_i}, \qquad L_v := \sum_1^{2s} \frac{\partial}{\partial v_i}$$

$$E_u := \sum_1^{2r} u_i \frac{\partial}{\partial u_i} + t\frac{\partial}{\partial t}, \qquad E_v := \sum_1^{2s} v_i \frac{\partial}{\partial v_i} + t\frac{\partial}{\partial t}.$$

Then

$$G(t; u, v) := \log P\left(A(t_1) \in E_1, A(t_2) \in E_2\right),$$
$$= \log \det \left(I - (\chi_{E_i^c} K^A_{t_i t_j} \chi_{E_j^c})_{1 \le i,j \le 2} \right)$$

satisfies the non-linear equation: $\quad (t = t_2 - t_1)$

$$\Big((L_u + L_v)(L_u E_v - L_v E_u) + t^2 (L_u - L_v) L_u L_v \Big) G$$
$$= \frac{1}{2} \Big\{ (L_u^2 - L_v^2) G \ , \ (L_u + L_v)^2 G \Big\}_{L_u + L_v}$$

Adler-PvM, Ann. of Prob. 2004

$$G(t; u, v) := \log P\left(A(t_1) \in E_1, A(t_2) \in E_2\right),$$

$$= \log\det\left(I - (\chi_{E_i^c} K^A_{t_i t_j} \chi_{E_j^c})_{1 \le i,j \le 2}\right)$$

where $K^A_{t_i t_j}(x, y)$ is the extended Airy kernel

$$K^A_{t_i t_j}(x, y) :=$$

$$\begin{cases} \int_0^\infty e^{-z(t_i - t_j)} \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)dz, & \text{if } t_i \ge t_j \\ -\int_{-\infty}^0 e^{z(t_j - t_i)} \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)dz, & \text{if } t_i < t_j \end{cases},$$

(iv) In particular, for $E_1 = (-\infty, \frac{y+x}{2})$ and $E_2 = (-\infty, \frac{y-x}{2})$

$$g(t;x,y) := \log P\left(A(0) < \frac{y+x}{2}, A(t) < \frac{y-x}{2}\right),$$

satisfies the non-linear equation:

$$2t\frac{\partial^3 g}{\partial t \partial x \partial y} = \left(t^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2}\right) + 8\left\{\frac{\partial^2 g}{\partial x \partial y}, \frac{\partial^2 g}{\partial y^2}\right\}_y,$$

with initial condition

$$\lim_{t \searrow 0} g(t;x,y) = \log F_2\left(\min(\frac{y+x}{2}, \frac{y-x}{2})\right).$$

Adler-PvM, Ann. of Prob. 2004

(v) Joint probability for $t \nearrow \infty$:

$$P(A(0) < u, A(t) < v)$$
$$= F_2(u)F_2(v) + \frac{F_2'(u)F_2'(v)}{t^2} + \frac{\Phi(u,v) + \Phi(v,u)}{t^4} + O\left(\frac{1}{t^6}\right).$$

(vi) Covariance:

$$E(A(t)A(0)) - E(A(t))E(A(0))$$
$$= \frac{1}{t^2} + \frac{2}{t^4} \iint_{\mathbb{R}^2} \Phi(u,v)du\ dv + \dots$$

with

$$\Phi(u,v) \;:=\; F_2(u)F_2(v)$$

$$\times \left(\begin{array}{l} \frac{1}{4}\left(\int_u^\infty q^2\right)^2 \left(\int_v^\infty q^2\right)^2 \\ \\ +q^2(u)\left(\frac{q^2(v)}{4} + \frac{1}{2}\int_v^\infty q^2\right) \\ \\ +\int_u^\infty q^2 \int_v^\infty \left(\begin{array}{l} 2(v-\alpha)q^2(\alpha) \\ +q'(\alpha)^2 - q^4(\alpha) \end{array} \right) d\alpha \end{array} \right).$$

$$F_2(u) := \exp\left(-\int_u^\infty (\alpha - u)q^2(\alpha)d\alpha\right),$$

with $q(\alpha)$ the solution of

$$q'' = \alpha q + 2q^3 \qquad \textbf{Painlevé II}$$

boundary condition $\quad q(\alpha) \cong \begin{cases} -\dfrac{e^{-\frac{2}{3}\alpha^{\frac{3}{2}}}}{2\sqrt{\pi}\alpha^{1/4}} \text{ for } \alpha \nearrow \infty \\ \\ \sqrt{-\alpha/2} \text{ for } \alpha \searrow -\infty \end{cases}$

Adler-PvM, Ann. of Prob. 2004

Application: Discrete polynuclear growth models: (Prähofer-Spohn '02, Johansson '02)

Assume all $\omega(x,t)$ i.i.d and geometric
$P(\omega(i,j)=k)=(1-\xi)\xi^k$,except

$$\omega(x,t)=0 \quad \text{if } T-x \text{ is even or } |x|>T$$

Define *height function*: (with $x\in\mathbb{Z}, T\in\mathbb{Z}_+$, $h(x,0)=0, x\in\mathbb{Z}$)

$$h(x,T+1)=\max\Big(h(x-1,T),h(x,T),h(x+1,T)\Big) + \omega(x,T+1)$$

Height curve at even sites $2u$ at times $2T-1$:

$$h(2u,2T-1)=\max\left\{\sum\omega(i,j),\ \begin{array}{l}\text{over all right/down}\\ \text{paths starting from}\\ \text{entry } (1,1)\\ \text{to } (T+u,T-u)\end{array}\right\}$$

Height curve at even sites $2u$ at times $2T-1$:

$$h(2u, 2T\text{–}1) = \max\left\{ \sum \omega(i,j), \begin{array}{l}\text{over all right/down} \\ \text{paths starting from} \\ \text{entry } (1,1) \\ \text{to } (T+u, T-u)\end{array} \right\}$$

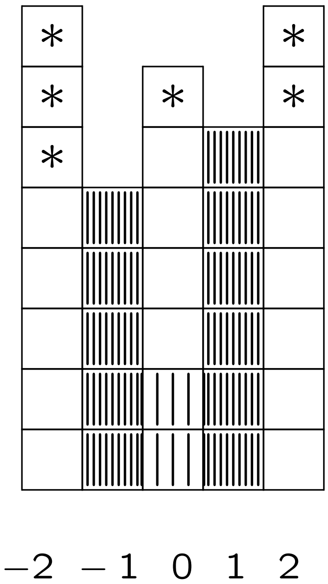
For appropriate constants a, b,

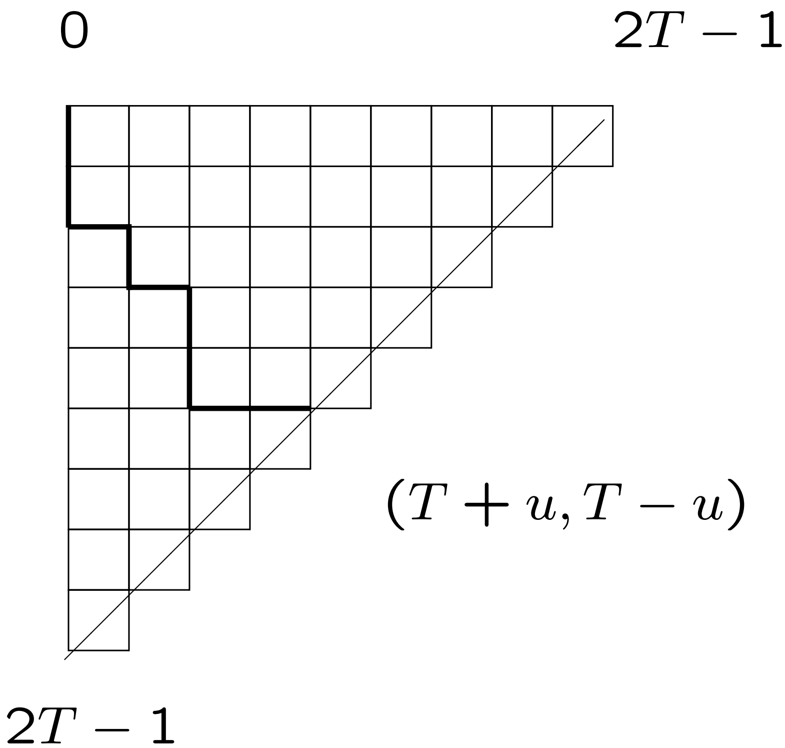
$$\begin{aligned} H_T(aT^{-2/3}u) &:= bT^{2/3}\Big(\frac{G(T+u, T-u)}{cT} - 1\Big) \\ &:= bT^{2/3}\Big(\frac{h(2u, 2T-1)}{cT} - 1\Big) \end{aligned}$$

For $u, T \nearrow \infty$ such that $uT^{-2/3} \to t$,

$$\boxed{H_T(t) \to A(t) - t^2}$$

$t = 3$





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The SINE PROCESS:

$$S_i(t) := \lim_{n\to\infty} \frac{\sqrt{2n}}{\pi} \lambda_{\frac{n}{2}+i}\left(\frac{\pi^2 t}{2n}\right) \quad \text{for } -\infty < i < \infty$$

"Motion of the particles in the bulk of the rescaled Dyson Brownian motion"

For compact E_1 and $E_2 \subset \mathbb{R}$ define (as before) linear operators

$$L_u := \sum_1^{2r} \frac{\partial}{\partial u_i}, \qquad L_v := \sum_1^{2s} \frac{\partial}{\partial v_i}$$

$$E_u := \sum_1^{2r} u_i \frac{\partial}{\partial u_i} + t\frac{\partial}{\partial t}, \qquad E_v := \sum_1^{2s} v_i \frac{\partial}{\partial v_i} + t\frac{\partial}{\partial t}.$$

Then

$$G(t;u,v) := \log P\left(\text{all } S_i(t_1) \in E_1^c, \text{all } S_i(t_2) \in E_2^c\right),$$
$$= \det\left(I - (\chi_{E_k} K^S_{t_k t_\ell} \chi_{E_\ell})_{1\leq k,\ell\leq 2}\right)$$

satisfies the 3^{rd}-order non-linear PDE,

$$L_u \frac{\left(2E_v L_u + (E_v - E_u - 1)L_v\right)G}{(L_u + L_v)^2 G + \pi^2}$$
$$= L_v \frac{\left(2E_u L_v + (E_u - E_v - 1)L_u\right)G}{(L_u + L_v)^2 G + \pi^2}.$$

Extended sine kernel:

$$K^S_{t_i t_j}(x,y) :=$$
$$\begin{cases} \frac{1}{\pi}\int_0^\pi e^{-z^2(t_i-t_j)/2} \cos z(x-y)\,dz, & \text{if } t_i \geq t_j \\ -\frac{1}{\pi}\int_\pi^\infty e^{-z^2(t_j-t_i)/2} \cos z(x-y)\,dz, & \text{if } t_i < t_j \end{cases},$$