Numerical determination of the resonance frequencies in a bounded domain using the MFS

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Abstract: *In this work we present a numerical algorithm for the determination of the eigenvalues and eigenfunctions associated to the Dirichlet problem for the Laplacian, in a bounded domain. The determination of higher eigenfrequencies is a well known numerical problem that has been addressed with other numerical methods. Here we propose to use the method of fundamental solutions. Since the MFS produces highly ill conditioned matrices, a particular technique was derived to overcome the difficulty of determining accurately those eigenfrequencies. Extensive numerical simulations will be presented.*

Keywords: eigenfrequencies, resonance, acoustic waves, method of fundamental solutions.

1 Introduction

The determination of the resonance frequencies associated to the Laplace operator is an old mathematical problem with applications in several scientific areas (eg. [6, 7]). Several classical numerical methods for PDE's have been used to determine both the eigenvalues and the eigenfunctions for arbitrary domains. More recently, meshfree methods using radial basis functions (eg. [5]) have been considered. Here we propose to consider an algorithm for the determination of eigenvalues and eigenfrequencies based on the method of fundamental solutions (MFS). In particular, we present several numerical experiments that show some interesting nodal domains (eg. [1]) for high Dirichlet eigenvalues associated to non trivial 2D domains. The application of this type of algorithm might be made for other boundary conditions and also for the exterior problem, but for simplicity we will consider here only the 2D Dirichlet problem for bounded domains.

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2 The Method of Fundamental Solutions

Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with regular boundary $\partial\Omega$. Consider the Dirichlet problem for the Helmholtz equation,

$$\begin{cases} \Delta u + \kappa^2 u = 0 & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{1}$$

A fundamental solution Φ_κ of the Helmholtz equation verifies $(\Delta + \kappa^2)\Phi = -\delta$, where δ is Dirac delta distribution. In the 2D case, we take

$$\Phi_\kappa(x) = \frac{i}{4} H_0^{(1)}(\kappa|x|)$$

where $H_0^{(1)}$ is the first Hänkel function. A density result (cf. [2]) shows that

$$L^2(\partial\Omega) = \overline{\text{span}\left\{\Phi_\kappa(x-y)|_{x\in\partial\Omega} : y \in \hat{\Gamma}\right\}},$$

where $\hat{\Gamma}$ is an admissible source set, for instance, the boundary of an open set in $\mathbb{R}^2\backslash\bar{\Omega}$. In particular we will consider $\hat{\Gamma}$ surrounding $\partial\Omega$. This allows to justify the approximation of a L^2 function, with complex values, defined on $\partial\Omega$, using a sequence of functions

$$u_n(x) = \sum_{j=1}^{n} \alpha_{n,j} \Phi_\kappa(x - y_{n,j}) \tag{2}$$

that converges to $u|_\Gamma = g$ in $L^2(\partial\Omega)$. This is a partial justification to the convergence of the Method of Fundamental Solution (MFS) based on density results. It is similar to Bogomolny's approach (in [4]), but it avoids the use of boundary layer potentials (e.g. [3, 8]). As pointed out in [2] (or in [4]), the convergence of the MFS, in a general case, is not completely related to the discretization of a single layer potential, although there is a straightforward relation. A single layer potential defined on $\hat{\Gamma}$ implies that the restriction to $\partial\Omega$ is an analytic function, and therefore such an approach would only be appropriate for functions g that are analytic functions on $\partial\Omega$.

We will be interested in the problem of finding the eigenfrequencies and eigenfunctions for the Dirichlet problem associated to the Laplace operator, ie. we will be interested in finding the values $-\kappa^2$ for which there exists a non null function u verifying the problem (1) with $g \equiv 0$. As an application, this corresponds to recovering the resonance frequencies $\kappa > 0$ associated to a particular shape of a drum Ω.

Since $g \equiv 0$ is an analytic function, it makes sense to consider the approach of the MSF as being related to the discretization of the single layer potential. Thus, we consider the operator

$$\begin{array}{rccc} \mathcal{H}_\kappa : & H^{-1/2}(\hat{\Gamma}) & \to & C^\infty(\partial\Omega) \\ & \varphi & \to & \displaystyle\int_{\hat{\Gamma}} \Phi_\kappa(x-y)\varphi(y)\, ds_y. \end{array} \tag{3}$$

Suppose that for a given κ there exists a $\varphi \neq 0$ such that $\mathcal{H}_\kappa\varphi = 0$, ie. $dim(Ker(\mathcal{H}_\kappa)) > 0$. Then the natural analytic extension of $\mathcal{H}_\kappa\varphi$ to Ω, given by the single layer potential, that we will call $\tilde{\mathcal{H}}_\kappa\varphi$, would be an eigenfunction associated to the eigenvalue $-\kappa^2$.

Theorem. If $dim(Ker(\mathcal{H}_\kappa)) > 0$ then $-\kappa^2$ is an eigenfrequency associated to the Dirichlet problem for the Laplace operator, and any non null $\varphi \in Ker(\mathcal{H}_\kappa)$ is an associated eigenfunction.

Proof. The hypothesis imply there exists $\varphi \not\equiv 0 : \mathcal{H}_\kappa \varphi = 0$. It suffices to show now that $\tilde{\mathcal{H}}_\kappa \varphi \not\equiv 0$ in Ω. Suppose that $\tilde{\mathcal{H}}_\kappa \varphi \equiv 0$, then by analytic extension the internal trace of $\tilde{\mathcal{H}}_\kappa \varphi$ on $\hat{\Gamma}$ would be null. Since the exterior trace of the single layer is equal to the interior, we would have $\tilde{\mathcal{H}}_\kappa \varphi$ to be a solution of the exterior problem with null exterior trace. Then, the well posedness of the exterior problem with Sommerfeld radiation condition (which is verified by the single layer potential) implies $\varphi \equiv 0$, contradicting the hypothesis. □

We skip here the part of showing that an eigenfunction may be represented by a single layer potential in some $\hat{\Gamma}$. For a regular domain (verifying the cone condition) we may consider any neighborhood of $\partial\Omega$, denoted by $\Gamma_\varepsilon \subset \Omega$ and the approximation of $u|_{\Gamma_\varepsilon}$ can be made using $\tilde{\mathcal{H}}_\kappa \varphi|_{\Gamma_\varepsilon}$.

We also note that by increasing the number of points on a piecewise analytic boundary the approximation made by collocation is appropriate, since it will lead in the limit to the only possible analytic solution, $g \equiv 0$.

3 Numerical Algorithm

From the previous considerations we may sketch a procedure for finding the eigenvalues of the Laplace operator by checking the κ for which $dim(Ker(\mathcal{H}_\kappa)) > 0$, which corresponds in the MFS to find the κ for which

$$\sum_{j=1}^{\infty} \alpha_j \Phi_\kappa(x - y_j) = 0, \quad (x \in \partial\Omega).$$

The simplest way to get those values is to find the values κ for which the $m \times m$ matrix

$$A(\kappa) = \left[\Phi_\kappa(x_i - y_j)\right]_{m\times m} \tag{4}$$

has a null determinant. We will choose the points $x_1, ..., x_m \in \partial\Omega$ and $y_1, ..., y_m \in \hat{\Gamma}$ in a particular way.

Given the m points x_i on $\partial\Omega$, we take

$$y_i = x_i + \tilde{n}$$

where $\tilde{n}$ is an approximation of a vector which is normal to the boundary $\partial\Omega$ on x_i. To obtain the vector $\tilde{n}$ we just consider $v^- = x_i - x_{i-1}$, $v^+ = x_i - x_{i+1}$ (index addition modulus m) and calculate n^-, n^+ which are normal to v^- and v^+ (respectively) pointing outwards Ω. Then we take $\tilde{n} = \frac{n^+ + n^-}{2}$. By some experimental criteria, we will use $|n^-| = |n^+| = \beta \approx 1/3$.

The components of the matrix $A(\kappa)$ are complex numbers, so the determinant is a complex number too. We consider the function $g(\kappa) = |Det[A(\kappa)]|$ and we will make use of the rough approximation $g'(w) \approx \frac{g(w+\varepsilon) - g(w)}{\varepsilon}$ for small $\varepsilon > 0$. This will allow us to get the points where there is a clear change of sign of the derivative. It is clear that the function g will be very small in any case, since the MFS is highly ill conditioned, and

the determinant is quite small. To avoid machine precision problems the code was built in *Mathematica.* While approximating an eigenfrequency, the value of g will be much smaller, producing a clear change on the sign of the derivative. The search of those points is made using the simple bissection method, which revealed to be quite accurate for the search of high eigenfrequencies near to each other.

Once we have an eigenfrequency determined, we may get the eigenfunctions just by considering extra collocation points inside the domain. Depending on the multiplicity of the eigenvalue, we add one or more collocation points to make the linear system well determined.

4 Numerical Simulations

Since the values of the eigenfrequencies for the unit disc are well known, given by a Bessel function, we will first test the results of this method for the three first resonance frequencies considering $\beta = 0.3$

m	abs. error		m	abs. error		m	abs. error
30	0.0000263383		30	0.0000291533		30	0.0000145574
40	1.49154×10^{-6}		40	1.54666×10^{-6}		40	8.1613×10^{-7}
50	8.21724×10^{-8}		50	7.57298×10^{-8}		50	3.31955×10^{-8}
60	4.98448×10^{-9}		60	4.68111×10^{-9}		60	2.08940×10^{-9}

To obtain an eigenfunction associated to the resonance frequencies κ_1, κ_2,... we use a collocation method on $m+1$ points, with $x_1, \cdots, x_m$ on $\partial\Omega$ and a point $x_{m+1} \in \Omega$. Then, the approximation of the eigenfunction is given by

$$\tilde{u}(x) = \sum_{j=1}^{m+1} \alpha_j \Phi_\kappa(x - y_j) \tag{5}$$

and, to exclude the solution $\tilde{u}(x) \equiv 0$, the coefficients α_k are determined solving the system

$$\begin{cases} \tilde{u}(x_i) = 0 \qquad i = 1, \ldots, m, \\ \tilde{u}(x_{m+1}) = 1 \end{cases} \tag{6}$$

In Fig. 1 we show the plots of eigenfunctions associated to the 21th, ... , 24th eigenvalues for the domain Ω_1 with boundary given by the parametrization

$$t \mapsto \left(\cos(t) - \frac{\cos(t)\sin(2t)}{2}, \sin(t) + \frac{\cos(4t)}{6} \right)$$

In Fig. 2 we present the respective nodal domains (ie. the domains where the real eigenfunction keeps the same sign)

In Fig. 3 and Fig. 4 we present the same plots (associated the 21th, ... , 24th eigenvalues) now considering a domain Ω_2 with boundary given by

$$t \mapsto \left(\cos(t), \sin(t) + \frac{\sin(2t)}{3} \right)$$

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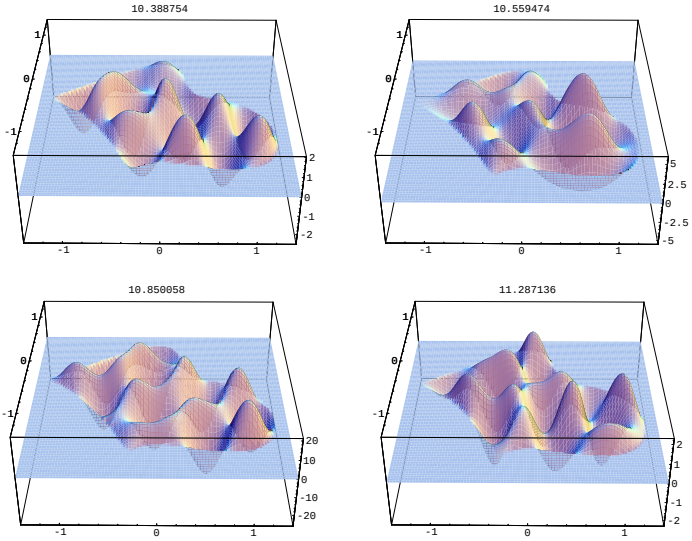

Figure 1: *3D plots of the 21th, ... , 24th eigenfunctions associated to* Ω_1

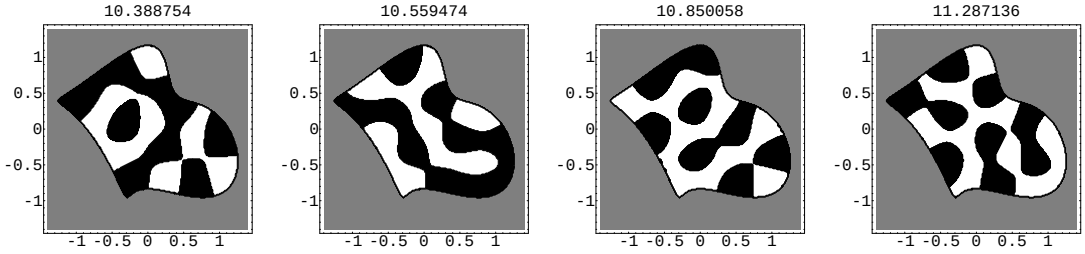

Figure 2: *Plots of the 21th, ... , 24th nodal domains associated to* Ω_1

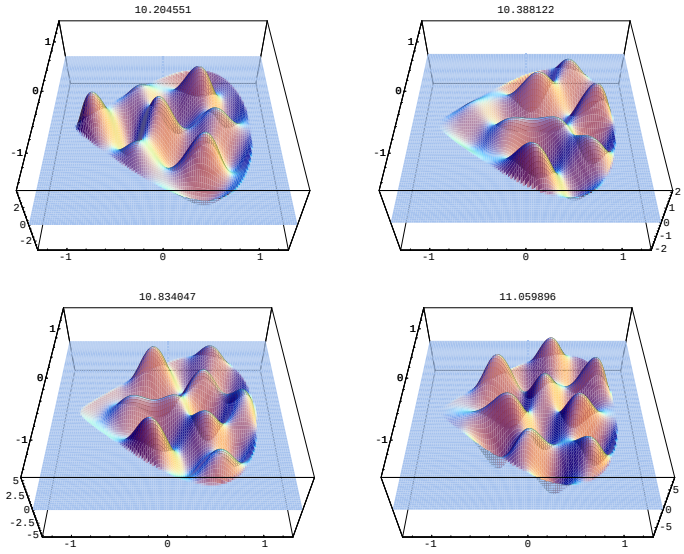

Figure 3: *3D plots of the 21th, ... , 24th eigenfunctions associated to* Ω_2

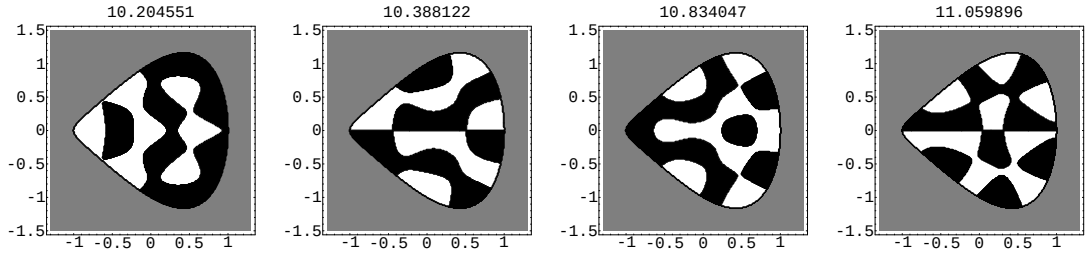

Figure 4: *Plots of the 21th, ... , 24th nodal domains associated to* Ω_1

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