

Radial Basis Functions on Grids and Beyond

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Abstract: *Radial basis functions are useful approximation tools especially for multi-dimensional interpolation. There is a variety of ways to get away from the standard approach with data placed on equally spaced lattices. We investigate some of those.*

1 Overview

Radial basis function interpolation is a well-known and popular tool for multivariate approximation. Its principal purpose is the interpolation to data and functions in multiple variables, and its success has been noted both in applications and analysis (Buhmann, 2003). The main reason why these methods are popular in applications is their availability in any dimension $n = 1, 2, \dots$. The basic form of these is

$$S(x) = \sum_k \lambda_k \phi(\|x - x_k\|), \quad x \in \mathbb{R}^n,$$

where $x_k \in \mathbb{R}^n$ are finitely or infinitely many regularly or arbitrarily distributed “centres”, λ_k are suitable real coefficients usually fixed by interpolation conditions at the centres, the norm is Euclidean and ϕ is the radial basis functions, e.g. the ubiquitous multiquadric $\phi(r) = \sqrt{r^2 + c^2}$ for a positive parameter c and reciprocal multiquadric, which are extremely popular in applications (Hardy, 1990). If the data are given through $F : \mathbb{R}^n \rightarrow \mathbb{R}$ at the centres, the interpolation conditions are usually $S(x_k) = F(x_k)$ for all x_k . In many cases these interpolation conditions fix the coefficients λ_k uniquely.

Much interest has been paid to the study of interpolation on infinite, square grids because this approach is particular amenable to theoretical analysis of the approximation power of radial basis functions (cf., e.g., Buhmann, 2003). The next step is the study of interpolation on half-spaces, i.e. integer grids where for example the first component of the integer-vectors is restricted to the nonnegative integers, thereby introducing a boundary to the set of data. This is what we wish to do in generalisation of the work on (inverse) multiquadrics in (Buhmann, 1993), using mainly methods from (Buhmann, 1990). A similar approach using Wiener-Hopf factorisations specifically adapted to “polyharmonic” radial basis functions, for example,

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$\phi(r) = r^2 \log r$ in even dimensions, is due to (Bejancu, 2000), also in a multidimensional set-up and with results that are related to ours, but we do not address the class of polyharmonic radial functions.

We study interpolation by translates of such functions $\phi(\|\cdot - k\|)$, where the norm is Euclidean and k is now from the “half-space” $\mathbb{Z}_+ \times \mathbb{Z}^{n-1}$, i.e. a vector in n dimensions whose components are all integers except that the first one is restricted to the non-negative integers \mathbb{Z}_+ . Thus, our [semi-]cardinal interpolants, where F and S are required to agree on $\mathbb{Z}_+ \times \mathbb{Z}^{n-1}$, to data $F(j)$, $j \in \mathbb{Z}_+ \times \mathbb{Z}^{n-1}$, which must satisfy suitable growth conditions, have the form

$$S(x) = \sum_{j \in \mathbb{Z}_+ \times \mathbb{Z}^{n-1}} F(j) L_j(x), \quad x \in \mathbb{R}_+ \times \mathbb{R}^{n-1}.$$

In the above, the L_j are so-called Lagrange functions from the space spanned by the radial basis function’s translates. Incidentally, the procedure of “Lagrange interpolation” was in fact first described by Edward Waring, Fellow of Magdalene College, Cambridge, in 1779, some 16 years before Lagrange, and therefore they might be called Waring’s functions. It is, of course, too late to change that. Most importantly, the Lagrange functions have to satisfy the Lagrange conditions $L_j(k) = \delta_{jk}$ for $j, k \in \mathbb{Z}_+ \times \mathbb{Z}^{n-1}$.

Such a Lagrange representation is achieved and well-defined when we interpolate on the whole grid \mathbb{Z}^n (Buhmann, 1990, for instance) rather than on $\mathbb{Z}_+ \times \mathbb{Z}^{n-1}$ with Lagrange functions L^* with includes multiquadrics and inverse multiquadrics. They satisfy $L^*(k) = \delta_{0k}$ for all $k \in \mathbb{Z}^n$. We define $L_j^* := L^*(\cdot - j)$ and may then interpolate by

$$S(x) = \sum_{j \in \mathbb{Z}^n} F(j) L_j^*(x), \quad x \in \mathbb{R}^n.$$

However, in our case here, the infinite summation is not well-defined because the c_k^j may not decay sufficiently fast and we have to define L_j differently. We study L_j and their coefficients’ properties. Whenever the radial function is square-integrable and if we are in *one dimension* ($n = 1$), the existence of Lagrange functions with square-integrable coefficients – albeit without a study of their asymptotic behaviour – follows from the seminal work of (Calderón, Spitzer and Widom, 1959) which was generalised in (Buhmann, 1993).

References

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