Explicit Estimation of an Integer at a Domain in the Reciprocity Principle with the use of Inverse Operations

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Extended Abstract

The work presents application of the reciprocity principle to mesh-less function approximation at the Ω domain delimited with Γ boundary. As a starting point the Green formula in the following form is used

$$\int_{\Omega}(u \cdot \Delta v - v \cdot \Delta u)d\Omega = \oint_{\Gamma}\left(\frac{\partial u}{\partial n}v - \frac{\partial v}{\partial n}u\right)d\Gamma, \quad (x, y, z) \in \Omega \tag{1}$$

substituting $u = \Delta^k w$, $v = \Delta^{n-k} q$, $k = 0,1,2,\dots, n$ provides

$$\int_{\Omega}\left(\Delta^k w \cdot \Delta^{n+1-k} q \cdot \Delta^{n-k} q \cdot \Delta^{k+1} w\right)d\Omega = \oint_{\Gamma}\left(\frac{\partial}{\partial n}\Delta^k w \cdot \Delta^{n-k} q - \frac{\partial}{\partial n}\Delta^{n-k} q \cdot \Delta^k w\right)d\Gamma \tag{2}$$

or summing (2) for consecutive $k = 0,1,\dots, n$ gives

$$\int_{\Omega}\left(w \cdot \Delta^{n+1} q - q \cdot \Delta^{n+1} w\right)d\Omega = \sum_{k=0}^{n}\oint_{\Gamma}\left(\frac{\partial}{\partial n}\Delta^k w \cdot \Delta^{n-k} q - \frac{\partial}{\partial n}\Delta^{n-k} q \cdot \Delta^k w\right)d\Gamma \tag{3}$$

The assumption

$$\Delta^{n+1} q(x-\xi, y-\eta, z-\zeta) = \delta(x-\xi, y-\eta, z-\zeta), \quad \xi, \eta, \zeta \in \overline{\Omega} \tag{4}$$

gives the following particular integral

$$\Delta^{n-k} q = \Delta^{-(k+1)}\delta, \quad q = \Delta^{-(n+1)}\delta \tag{5}$$

while the formula (3) takes a form

$$\begin{aligned} c \cdot w(\xi, \eta, \zeta) = {} & \sum_{k=0}^{n}\oint\left[\frac{\partial}{\partial n}\Delta^k w \cdot \Delta^{-(k+1)}\delta - \frac{\partial}{\partial n}\Delta^{-(k+1)}\delta \cdot \Delta^k w\right]d\Gamma + \\ & + \int_{\Omega}\Delta^{n+1} w(x, y, z) \cdot \Delta^{-(n+1)}\delta(x-\xi, y-\eta, z-\zeta)d\Omega \end{aligned} \tag{6}$$

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Execution of consecutive inverse operations enables to show that

$$\Delta^{-j}\delta = \begin{cases} \dfrac{1}{2}\cdot\dfrac{(x-\xi)^{2j-1}}{(2j-1)!}\cdot \operatorname{sign}(x-\xi), & r = |x-\xi| \quad \textit{for the} \;\; \text{1D } \textit{case} \\ \dfrac{1}{2\pi}\cdot\dfrac{r^{2(j-1)}}{\left(2^{j-1}\cdot(j-1)!\right)^2}\left(\sum\limits_{k=1}^{j-1}\dfrac{1}{k} - \ln r\right), & r = \sqrt{(x-\xi)^2+(y-\eta)^2} \quad \textit{for the} \;\; \text{2D } \textit{case} \\ \dfrac{1}{3\pi}\cdot\dfrac{r^{2j-1}}{(2j)!}, & r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2} \quad \textit{for the} \;\; \text{3D } \textit{case} \end{cases}$$

The relationship (7) enables to estimate the integer error in the (6) formulation.
It should be noticed that the Δ^{-l}w operation expressed by the formula (6) is reduced to the determination of the $\delta^{-(l+k+1)}$ operation. In particular, for the Poisson's equation

$$\Delta\Phi = w \tag{8}$$

the solution may be written in the form

$$\Phi = \Delta^{-1}w + H \tag{9}$$

where H is a harmonic function, while the source function w may be presented in the (6) form, omitting the integral at the domain for sufficiently high values of n. Solutions of nonstationary heat conduction equation and of the hyperbolic equation may be reduced to inverse operation of the Δ^{-jw} type, with j=0,1,2,...
Numerical results obtained already for small values of n confirm high accuracy of the f function approximation carried out by means of (6) formulation.