

Computation of Incompressible Navier-Stokes Equations by Local RBF-based Differential Quadrature Method

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Abstract: *Local radial basis function-based differential quadrature (RBF-DQ) method was recently proposed by us. The method is a natural mesh-free approach. Like the conventional differential quadrature (DQ) method, it discretizes any derivative at a knot by a weighted linear sum of functional values at its neighbouring knots, which may be distributed randomly. However, different from the conventional DQ method, the weighting coefficients in present method are determined by taking the radial basis functions (RBFs) instead of high order polynomials as the test functions. The method works in a similar fashion as conventional finite difference schemes but with “truly” mesh-free property. In this paper, we mainly concentrate on the multiquadric (MQ) radial basis functions since they have exponential convergence. The effects of shape parameter c on the accuracy of numerical solution of linear and nonlinear partial differential equations are studied, and how the value of optimal c varies with the number of local support knots is also numerically demonstrated. The proposed method is validated by its application to solve incompressible Navier-Stokes equations. Excellent numerical results are obtained on an irregular knot distribution.*

1 Introduction

The differential quadrature (DQ) method was introduced by Richard Bellman and his associates in the early of 1970's [1], following the idea of integral quadrature. The basic idea of the DQ method is that any derivative at a mesh point can be approximated by a weighted linear sum of all the functional values along a mesh line. The key procedure in the DQ method is the determination of weighting coefficients. As shown by Shu and Richards [2], when the solution of a partial differential equation (PDE) is approximated by a high order polynomial, the weighting coefficients can be computed by a simple algebraic formulation or by a recurrence relationship. The details of the DQ method can be found in the book of Shu [3].

On the other hand, it is noted that the polynomial approximation in the DQ method is along a straight line. This means that numerical discretization of derivatives by the DQ method is also along a straight line. Due to this feature, the DQ method cannot be directly applied to irregular domain problems. As will be shown in this paper, the radial

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basis functions (RBFs), which have “truly” meshless property, could be a good choice in the DQ approximation.

Initially, RBFs were developed for multivariate data and function interpolation. However, their “truly” mesh-free nature motivated researchers to use them to deal with partial differential equations. The first trial of such exploration was made by Kansa [4]. Subsequently, Fornberg [5], Hon and Wu [6], Chen et al [7], Fasshauer [8], Chen and Tanaka [9] also made a great contribution in this development. It should be noted that most of above works related to the application of RBFs for the numerical solution of PDEs are actually based on the function approximation instead of derivative approximation. In other words, these works directly substitute the expression of function approximation by RBFs into a PDE, and then change the dependent variables into the coefficients of function approximation. The process is very complicated, especially for non-linear problems. To remove this difficulty, Wu and Shu [10], Shu et al. [11] recently developed the RBF-based differential quadrature (RBF-DQ) method. The RBF-DQ method directly approximates derivatives of a PDE, which combines the mesh-free nature of RBFs with the derivative approximation of differential quadrature (DQ) method. In the local MQ-DQ approach, any spatial derivative at a knot is approximated by a linear weighted sum of all the functional values in the supporting region around the reference knot. The method is consistently well applied to linear and nonlinear problems. The weighting coefficients in derivative approximation are determined by MQ approximation of the function and linear vector space analysis. Some fundamental issues of this method are shown in [11]. This paper further explores the applicability of local MQ-DQ method for simulation of incompressible viscous flows.

2 Local MQ-DQ Method

The details and fundamental issues of local MQ-DQ method have been shown in [11]. Some basic formulations for derivative approximation will be shown below. Following the work of Shu et al. [11], the n th order derivative of a smooth function $f(x, y)$ with respect to x , $f_x^{(n)}$, and its m th order derivative with respect to y , $f_y^{(m)}$, at (x_i, y_i) can be approximated by local MQ-DQ method as

$$f_x^{(n)}(x_i, y_i) = \sum_{k=1}^N w_{i,k}^{(n)} f(x_k, y_k) \quad (1)$$

$$f_y^{(m)}(x_i, y_i) = \sum_{k=1}^N \bar{w}_{i,k}^{(m)} f(x_k, y_k) \quad (2)$$

where N is the number of knots used in the supporting region, $w_{i,k}^{(n)}$, $\bar{w}_{i,k}^{(m)}$ are the DQ weighting coefficients in the x and y directions. The determination of weighting coefficients is based on the analysis of function approximation and the analysis of linear vector space.

In the local MQ-DQ method, MQ approximation is only applied locally. At any knot, there is a supporting region represented by a circle, in which there are N knots randomly distributed. The function in this region can be locally approximated by MQ RBFs as

$$f(x, y) = \sum_{j=1, j \neq i}^N \lambda_j g_j(x, y) + \lambda_i \quad (3)$$

where λ_j is a constant, c_j is the shape parameter to be given by the user, and

$$g_j(x, y) = \sqrt{(x - x_j)^2 + (y - y_j)^2 + c_j^2} - \sqrt{(x - x_i)^2 + (y - y_i)^2 + c_i^2} \quad (4)$$

It is easy to see that $f(x, y)$ in equation (3) constitutes a N -dimensional linear vector space \mathbf{V}^N with respect to the operation of addition and multiplication. From the concept of linear independence, the bases of a vector space can be considered as linearly independent subset that spans the entire space. In the space \mathbf{V}^N , one set of base vectors is $g_i(x, y) = 1$, and $g_j(x, y)$, $j = 1, \dots, N$ but $j \neq i$ given by equation (4). From the property of a linear vector space, if all the base functions satisfy the linear equation (1) or (2), so does any function in the space \mathbf{V}^N represented by equation (3). There is an interesting feature. From equation (3), while all the base functions are given, the function $f(x, y)$ is still unknown since the coefficients λ_i are unknown. However, when all the base functions satisfy equation (1) or (2), we can guarantee that $f(x, y)$ also satisfies equation (1) or (2). In other words, we can guarantee that the solution of a partial differential equation approximated by local MQ satisfies equation (1) or (2). Thus, when the weighting coefficients of DQ approximation are determined by all the base functions, they can be used to discretize the derivatives in a partial differential equation. That is the essence of the MQ-DQ method.

Following the work of [11], the weighting coefficient matrix of the x -derivative can be determined by

$$[G][W^n]^T = \{G_x\} \quad (5)$$

where $[W^n]^T$ is the transpose of the weighting coefficient matrix $[W^n]$, and

$$[W^n] = \begin{bmatrix} w_{1,1}^{(n)} & w_{1,2}^{(n)} & \cdots & w_{1,N}^{(n)} \\ w_{2,1}^{(n)} & w_{2,2}^{(n)} & \cdots & w_{2,N}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{N,1}^{(n)} & w_{N,2}^{(n)} & \cdots & w_{N,N}^{(n)} \end{bmatrix}, \quad [G] = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ g_1(x_1, y_1) & g_1(x_2, y_2) & \cdots & g_1(x_N, y_N) \\ \vdots & \vdots & \ddots & \vdots \\ g_N(x_1, y_1) & g_N(x_2, y_2) & \cdots & g_N(x_N, y_N) \end{bmatrix}$$

$$[G_x] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ g_x^n(1,1) & g_x^n(1,2) & \cdots & g_x^n(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ g_x^n(N,1) & g_x^n(N,2) & \cdots & g_x^n(N,N) \end{bmatrix}$$

The elements of matrix $[G]$ are given by equation (4). For matrix $[G_x]$, we can successively differentiate equation (4) to get its elements. For example, the first order derivative of $g_j(x, y)$ with respect to x can be written as

$$\frac{\partial g_j(x, y)}{\partial x} = \frac{x - x_j}{\sqrt{(x - x_j)^2 + (y - y_j)^2 + c_j^2}} - \frac{x - x_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + c_i^2}} \quad (6)$$

With the known matrices $[G]$ and $[G_x]$, the weighting coefficient matrix $[W^n]$ can be obtained by using a direct method of LU decomposition. The weighting coefficient matrix of the y -derivative can be obtained in a similar manner.

In the local MQ-DQ method, the shape parameter c has a strong influence on the accuracy of numerical results. The optimal value of c is mainly affected by the number of supporting knots and the size of supporting region. Usually, the number of supporting

knots is fixed for an application. The size effect of supporting region can be removed by normalization of scale in the supporting domain. The idea is actually motivated from the finite element method, where each element is usually mapped into a regular shape in the computational space. The essence of this idea is to transform the local support region to a unit circle for the two dimensional case. The normalization can be made by the following transformation

$$\bar{x} = \frac{x}{D_i}, \quad \bar{y} = \frac{y}{D_i} \quad (7)$$

where (x, y) represents the coordinates of supporting region in the physical space, (\bar{x}, \bar{y}) denotes the coordinates in the unit circle, D_i is the diameter of the minimal circle enclosing all knots in the supporting region for the knot i . The corresponding MQ basis functions in the local support now become

$$\phi = \sqrt{\left(\bar{x} - \frac{x_i}{D_i}\right)^2 + \left(\bar{y} - \frac{y_i}{D_i}\right)^2 + \bar{c}^2}, \quad i = 1, \dots, N, \quad (8)$$

Compared with traditional MQ-RBF, we can find that the shape parameter c is actually equivalent to $\bar{c}D_i$. The coordinate transformation (7) also changes the formulation of the weighting coefficients in the local MQ-DQ approximation. For example, by using the differential chain rule, the first order partial derivative with respect to x can be written as

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \bar{x}} \frac{d\bar{x}}{dx} = \frac{1}{D_i} \frac{\partial f}{\partial \bar{x}} = \frac{1}{D_i} \sum_{j=1}^N w_{i,j}^{(1)} f_j = \sum_{j=1}^N \frac{w_{i,j}^{(1)}}{D_i} f_j \quad (9)$$

where $w_{i,j}^{(1)}$ are the weighting coefficients computed in the unit circle, $w_{i,j}^{(1)}/D_i$ are the actual weighting coefficients in the physical domain. Clearly, when D_i is changed, the equivalent c in the physical space is automatically changed. In our application, \bar{c} is chosen as a constant. Its optimal value depends on the number of supporting knots. In this study, the number of knots in the supporting region is fixed as 17. For this case, \bar{c}^2 is taken as 3.1, based on the previous work of Shu et al. [11].

3 Computation of Incompressible Navier-Stokes Equations by Local MQ-DQ Method

The two dimensional Navier-Stokes equations in the vorticity-stream function form are solved to evaluate the accuracy and reliability of local MQ-DQ method. The governing equations can be written as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \omega \quad (10)$$

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \frac{1}{\text{Re}} \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \quad (11)$$

where u, v denote the components of velocity in the x and y direction, which can be calculated from the stream function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (12)$$

Re is the Reynolds number. Similar to the conventional FD scheme, equations (10), (11) can be easily discretized by the local MQ-DQ method. In this work, the resultant algebraic equations are solved by SOR iteration method. The boundary condition for the stream function can be easily implemented. For this problem, the updating of vorticity on the wall involves the derivatives of stream function. Like the governing equations, these derivatives can also be discretized by the local MQ-DQ method. They can also be approximated by the conventional one-sided second order FD scheme.

Two test problems are considered in this work. The first case is the driven flow in a square cavity. We have conducted the computation for various Reynolds numbers, and compared the present results with those of Ghia et al. [12], which are considered as benchmark data in the literature. It was found that the agreement is very good. This can be observed in Fig. 1, which shows the velocity profiles through the center of cavity. Note that the present results are obtained by using 9573 knots.

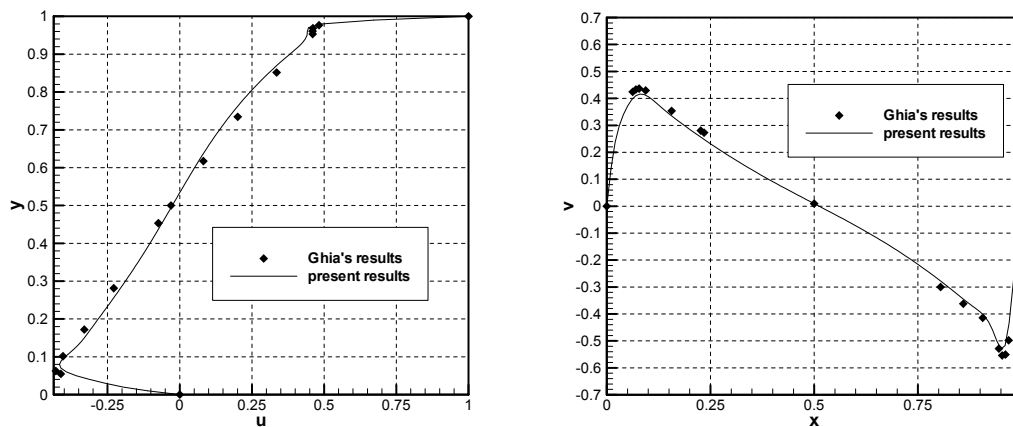


Figure 1 U (left) and V (right) velocity profiles along vertical and horizontal central lines at $Re=5000$, 9573 knots

The second case is the flow around a circular cylinder. We have conducted the computation for the steady and unsteady flows. Again, the present results agree very well with available data in the literature. Fig. 2 shows the streamlines for an unsteady case of $Re=200$. The comparison of drag and lift coefficients between the present work and Liu et al. [13] is shown in Table 1.

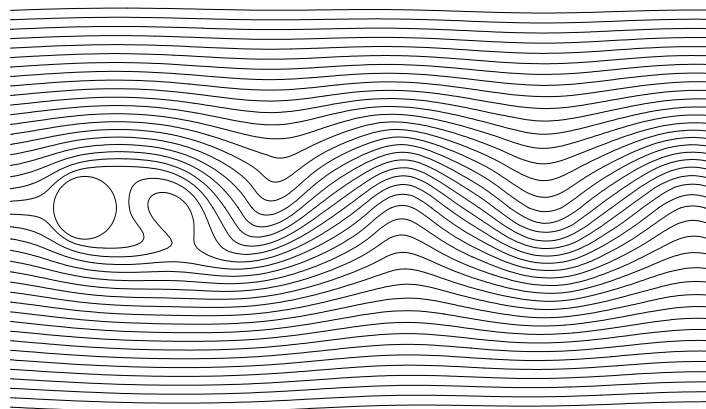


Figure 2 Streamlines of flow past a circular cylinder at $Re=200$

Table 1 Comparison of drag and lift coefficients for $Re=100$ and 200

Numerical results	Drag (C_d)		Lift (C_l)	
	$Re=100$	$Re=200$	$Re=100$	$Re=200$
Liu <i>et al</i> [13]	1.350 ± 0.012	1.31 ± 0.049	± 0.339	± 0.69
Present	1.362 ± 0.010	1.352 ± 0.049	± 0.32	± 0.62

4 Conclusions

The recently developed local MQ-DQ method is applied in this work to solve two-dimensional incompressible Navier-Stokes equations. Numerical results showed that the present method is an accurate and powerful approach for solution of partial differential equations.

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