A Meshfree Weak-Strong-form (MWS) method

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Abstract: *A novel meshfree weak-strong (MWS) form method is proposed based on a combined formulation of both the strong-form and the local Petrov-Galerkin weak-form. In the MWS method, the problem domain and its boundary is represented by a set of distributed field nodes. The strong-form or collocation method is used for all nodes whose local quadrature domains do not intersect with Neumann boundaries. Therefore, no numerical integration is required for these nodes. The local Petrov-Galerkin weak-form, which needs the local numerical integration, is only used for nodes on or near the Neumann boundaries. The Neumann boundary conditions can then be easily imposed to produce stable and accurate solutions. The RPIM is used to construct the meshfree shape functions. Numerical examples of incompressible flow problems are presented to demonstrate the efficiency, stability and accuracy of the proposed MWS method.*

1 Introduction

In recent years, meshfree or meshless methods have been developed and used to solve partial differential equations (PDE) of mechanics problems[1]. Current meshfree methods can be largely categorized into two major categories: meshfree methods based on strong-forms (or short for meshfree strong-form methods) and meshfree methods based on weak-forms (or short for meshfree weak-form methods). There are also meshfree methods based on the integral representation method for functional approximations, such as the particle methods, many of which are introduced in the book of GR Liu and Liu [2].

The mesh free strong-form method kown as collocation method is simple to implement and computationally efficient. They are truly meshless methods without using any mesh for both field variable approximation and integration. However, it is often found unstable and less accurate, especially for problems governed by partial differential equations with Neumann (derivative) boundary conditions.

On the other hand, the mesh free weak-form methods exhibits very good stability and excellent accuracy. However, the numerical integration makes them computational

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expensive, and the background mesh (global or local) for integration is responsible for not being “truly” mesh free.

The meshfree weak-strong (MWS) form method, is originated by GR Liu and Gu[3] based on a combined formulation of both the strong-form and the local Petrov-Galerkin weak-form that was used by Atluri in developing the MLPG method[4]. The MWS method has been successfully developed and used for the static and dynamic anaylysse of structures[5][6]. In the MWS method, the problem domain and its boundary is represented by a set of points or nodes. The strong-form or collocation method is used for all nodes whose local quadrature domains do not intersect with Neumann boundaries (the internal nodes and the nodes on the essential (Dirichlet) boundaries). There is no need for numerical integrations for these nodes. The local numerical integration is performed only for the nodes on the Neumann boundaries. The Neumann boundary conditions can then be easily imposed to produce stable and accurate solutions.

This paper details the MWS method for incompressible flow problems. The MWS formulatiomns are obtained from PDEs of incompressible flow problems. The radial point interpolation method (RPIM) is used to construct the shape functions. The final system matrix will be sparse and banded for computational efficiency. Numerical examples of incompressible flow problems are presented to demonstrate the efficiency, stability and accuracy of the proposed MWS method.

2 Meshfree weak-strong (MWS) method for incompressible flow problems

As shown in Figure 1, the problem domain and boundaries are represented by properly scattered nodes. The key idea of the MWS method is that in establishing the discrete system equations, both the strong-form and the local Petrov-Galerkin weak-form are used for the same problem, but for different nodes. In Figure 1, Ω_q is the local quadrature domain for a field node. If Ω_q does not intersect with the Neumann boundaries, the strong-form is used for this node. Otherwise, the local Petrov-Galerkin weak-form is used.

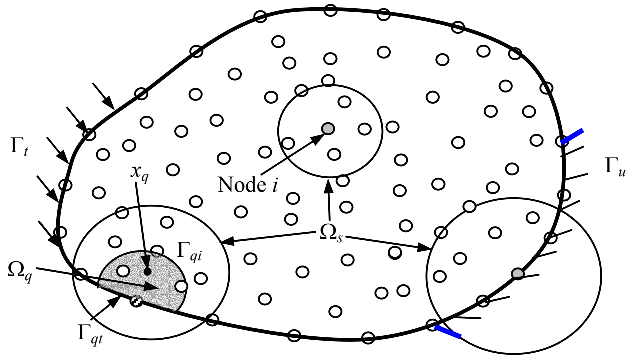


Figure: 1 *The local support domain Ω_s and local quadrature domain Ω_q used in the MWS method*

2.1 Simulation of natural convection in enclosed domain

The problem domain is given in Figure 2 (a). For a field node whose local quadrature domain does not intersect with the Neumann boundaries the strong-form is used. The standard strong-form for problems of natural convection in an enclosed domain can be written in the Cartesian coordinate system:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\omega \tag{1}$$

$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\mathrm{Pr}(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2})-\mathrm{Pr}\cdot\mathrm{Ra}\cdot\frac{\partial T}{\partial x} \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where

$$u=\frac{\partial\psi}{\partial y}, v=-\frac{\partial\psi}{\partial x} \tag{4}$$

Using radail point interpolation (RPIM) and MQ RBF, the discretized strong-form for the *i*th field node can be written as:

$$\sum_{k=1}^{n}(\phi_k)_{xx}\psi_k+\sum_{k=1}^{n}(\phi_k)_{yy}\psi_k=\omega_i \tag{5}$$

$$u_i\sum_{k=1}^{n}(\phi_k)_x\omega_k+v_i\sum_{k=1}^{n}(\phi_k)_y\omega_k=\mathrm{Pr}\left(\sum_{k=1}^{n}(\phi_k)_{xx}\omega_k+\sum_{k=1}^{n}(\phi_k)_{yy}\omega_k\right)-Ra\,\mathrm{Pr}\sum_{k=1}^{n}(\phi_k)_x T_k \tag{6}$$

$$u_i\sum_{k=1}^{n}(\phi_k)_x T_k+v_i\sum_{k=1}^{n}(\phi_k)_y T_k=\sum_{k=1}^{n}(\phi_k)_{xx}T_k+\sum_{k=1}^{n}(\phi_k)_{yy}T_k \tag{7}$$

$$u_i=\sum_{k=1}^{n}(\phi_k)_x\psi_k \quad , \quad v_i=\sum_{k=1}^{n}(\phi_k)_y\psi_k \tag{8}$$

Note here $\phi_k \equiv \phi_i(\mathbf{x}_k)$.

For a field node whose local quadrature domain intersects with the Neumann boundaries the following local Petrov-Galerkin weak-form that formed from the weighted residual method is used.

$$C_{ik}\psi_k-E_{ik}\psi_k=-A_{ik}\omega_k \tag{9}$$

$$B_{ik}\omega_k+\mathrm{Pr}\cdot C_{ik}\omega_k-\mathrm{Pr}\cdot E_{ik}\omega_k=-\mathrm{Pr}\cdot\mathrm{Ra}\cdot D_{ik}T_k \tag{10}$$

$$B_{ik}T_k+C_{ik}T_k-E_{ik}T_k=0 \tag{11}$$

where

$$A_{ik}=\iint_{\Omega_Q}\phi_k w_i d\Omega, \tag{12}$$

$$B_{ik}=\iint_{\Omega_Q}[u\cdot\frac{\partial\phi_k}{\partial x}+v\cdot\frac{\partial\phi_k}{\partial y}]\cdot w_i d\Omega \tag{13}$$

$$C_{ik}=\iint_{\Omega_Q}(\frac{\partial\phi_k}{\partial x}\frac{\partial w_i}{\partial x}+\frac{\partial\phi_k}{\partial y}\frac{\partial w_i}{\partial y})d\Omega \tag{14}$$

$$D_{ik} = \iint_{\Omega_Q} \frac{\partial \phi_k}{\partial x} w_i d\Omega \tag{15}$$

$$E_{ik} = \int_{\Gamma_{Qu}} \frac{\partial \phi_k}{\partial \vec{n}} w_i d\Gamma \tag{16}$$

The following quantities are calculated in the computation:

$\lvert\psi_{\max}\rvert$	maximum absolute value of the stream function
$u_{\max}$	maximum horizontal velocity on the vertical mid-plane of the cavity
$v_{\max}$	maximum vertical velocity on the horizontal mid-plane of the cavity
$Nu_{\max}$	maximum value of the local Nusselt number on the boundary at x=0
$Nu_{\min}$	minimum value of the local Nusselt number on the boundary at x=0

First, we compared the rates of convergence for the present MWS method, LRPIM method and FD method in the case of Ra=10^3, using the same uniform nodal distribution in Figure 3 (a). From Figure 4(a), it is very clear that MWS method is much more accurate than FD when h is decreased and less accurate than the LRPIM in which the local Petrov-Galerkin weak-form is used for all field nodes. However, it is found that the point iterative scheme, such as SOR scheme can be adopted in the MWS method to solve the algebraic equations systems. Therefore, the computational complexity for the MWS is about $O(N)$, while in LRPIM method, the more expensive direct solver has to be used to solve the algebraic equations systems, whose computational complexity is about $O(N^3)$. Figure 4 (b) confirms this conclusion. Hence, the MWS method is computationally much more efficient than the LRPIM method.

2.2 Simulation of the flow around a cylinder

The problem addressed here is simulation of an incompressible, viscous fluid flow at a constant velocity U_∞ past a stationary cylinder of radius a, as shown Figure 3(b).

The boundary conditions of the problem are:

i) free stream velocity U at the in-flow boundary:
ii) no slip on the surface of the cylinder;
iii) uniform flow at infinity except downstream boundary
iv) Zero-gradient condition at infinity downstream

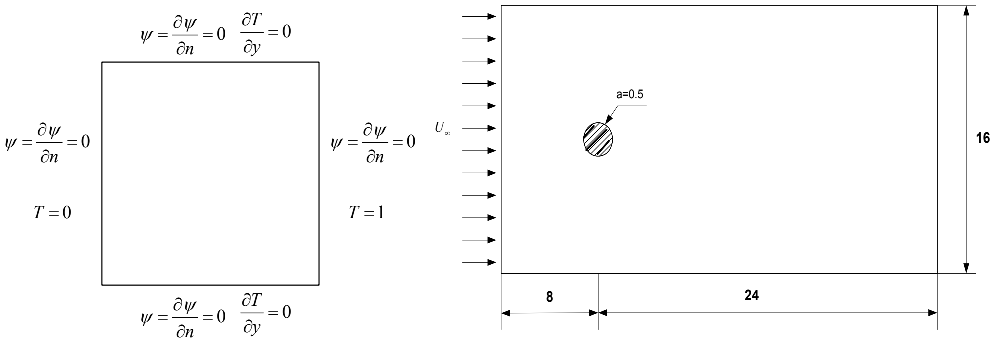
The initial condition for this problem is taken as an unsymmetrical initial flow field, i.e.

$$\psi\big|_{t=0} = \sqrt{x^2 + y^2} \tag{17}$$

which serves as an artificial initiator for the numerical simulation.

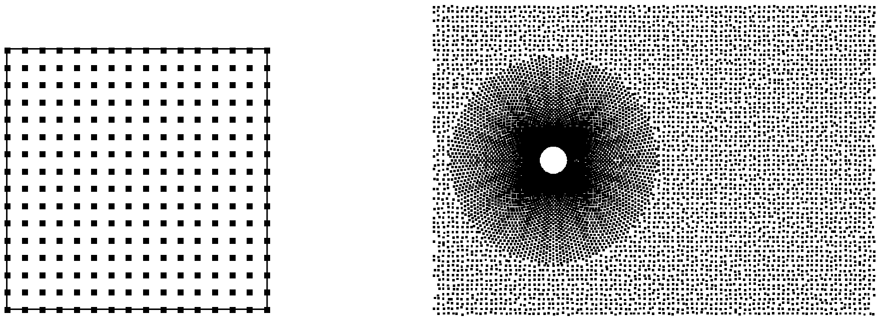
There is a time derivative for the unsteady problem. In the present model, the time derivative is approximated using an explicit three-step formulation based on a Taylor series expansion in time.

It is found that for Re=20, the unsymmetrical initial flow field evolves to be symmetrical and flow appear to be laminar steady flow (Figure 5a); for Re=100, the flow field eventually settle into a periodic oscillatory pattern (Figure 5b). This confirms the other experimental and numerical results.



a) Natural convection in square cavity b) Flow past a circular cylinder

Figure 2: *Problem domain for the simulation*



a) 256 nodes for cavity flow b) 23177 nodes for flow past cylinder

Figure 3: *Nodal distributions*

3 Conclutions

A novel meshfree method, the meshfree weak-strong (MWS) form method, is proposed for incompressible flow problems. In the MWS method, both the strong-form and the local Petrov-Galerkin weak-form are used. Numerical examples of problems for natural convection and the flow around a cylinder are presented to demonstratethe effectiveness of the present MWS method. Compared with other meshfree methods, the present MWS method is an improvement for the following reasons:

(a) The strong-form and the local Petrov-Galerkin weak-form are novelly combined together.

(b) The MWS method is a stable meshless method using least local "meshes" only for Newmann boundaries.

(c) The MWS method takes fully the advantages of meshfree strong-form methods and meshfree weak-form methods to achieve both accuracy and efficiency.

References

[1] G.R. Liu. *Mesh Free Methods: Moving Beyond the Finite Element Method*. CRC press, Boca Raton, USA (2002).

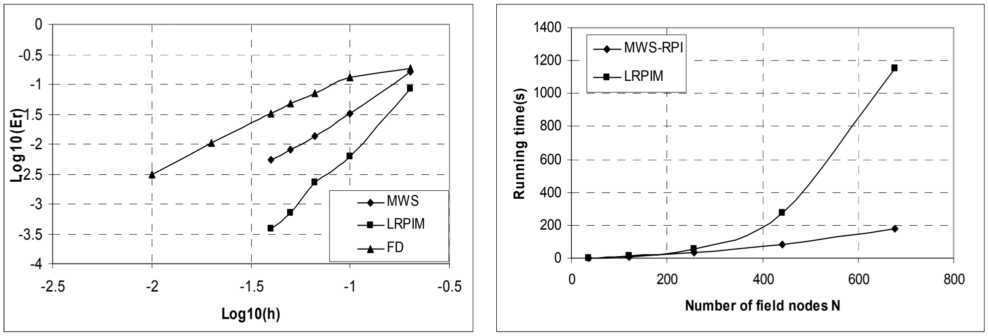
[2] G.R. Liu and M.B. Liu. *Smoothed particle hydrodynamics: a Meshfree Particle Method*, *World Scientific, Singapore* (2003).

[3] G.R. Liu and Y.T. Gu. A truly meshless method based on the strong-weak form, *in Advances in Meshfree and X-FEM Methods, Proceeding of the 1st Asian Workshop in Meshfree Methods* (Ed. GR Liu), *World Scientific, Singapore*, pp259-261(2002).

[4] S.N. Atluri and T. Zhu. A new meshless local Petrov-Galerkin (MLPG) approach in computational mechanics. *Computational Mechanics*, 22, pp117-127 (1998)..

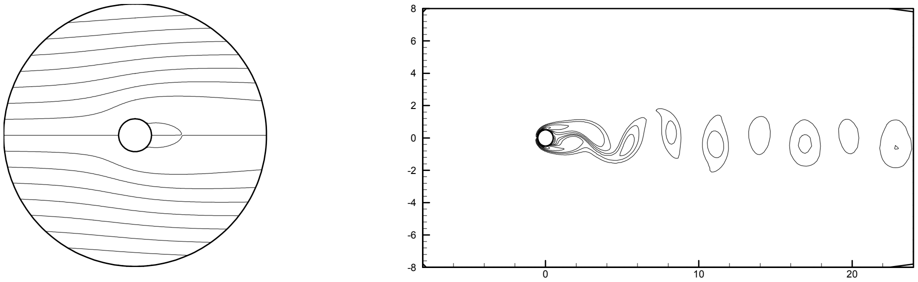
[5] G.R. Liu and Y.T. Gu. A meshfree method: Meshfree Weak-Strong (MWS) form method, for 2-D solids. *Computational Mechanics* (revised) (2003).

[6] Y.T. Gu and G.R. Liu. A Meshfree Weak-Strong (MWS) form method for static and dynamic analyses of structures. *Journal of Sound and Vibration* (submitted)(2003).



(a) convergence rates (b) Running time

Figure 4: *Comparison of convergence rates and running time between the MWS and their full weak-form counterparts for different field nodes*



(a) (b)

Figure 5: *a) Streamlines at the final steady state for Re=20， b)Vorticity distribution for flow across a cylinder, Re=100*