

Trefftz Solution for Steady-State Heat Conduction Problem in Functionally Gradient Materials

E. Kita ⁽¹⁾ Y. Ikeda ⁽²⁾ and N. Kamiya⁽³⁾

Abstract: *This paper describes the application of Trefftz method to the steady-state heat conduction problem on the functionally gradient materials. An inhomogeneous term of the governing equation is approximated by the polynomial in the global coordinates to derive the related particular solution. An unknown function is approximated with the superposition of the particular solution and the T-complete functions of the Laplace equation. Unknown parameters are determined so that the function satisfies the governing equation and the boundary conditions by means of the collocation method. Finally, the scheme is applied to a numerical example.*

Keywords: Trefftz Method, Computing Point Analysis Scheme, Steady-State Heat Conduction, Functionally Gradient Materials.

1 Introduction

The steady-state heat conduction problem in the functionally gradient material can be modeled as the boundary value problem of the Poisson equation if the heat conductivity changes gradually. When Trefftz methods are applied to the boundary value problem of the nonlinear Poisson equation, it is very difficult to derive the T-complete function due to the inhomogeneous term of the Poisson equation. For overcoming the difficulty, the combinational method of the Trefftz method and the computing point analysis scheme is presented in this paper.

According to the computing point analysis scheme[1, 2], the inhomogeneous term of the governing equation is approximated with the 5-order polynomial in the global coordinates to derive the particular solution of the governing equation. The use of the particular solution transforms the boundary value problem of the Poisson equation into that of the Laplace equation. Once the Laplace problem is solved for the homogeneous solution by the boundary data alone, an unknown function of the problem is estimated from the homogeneous and the particular solutions. Finally,

¹Nagoya University, Nagoya 464-8601, Japan (kita@is.nagoya-u.ac.jp).

²Daido Institute of Technology, Nagoya 457-8530, Japan (ikeda@daido-it.ac.jp).

³Nagoya University, Nagoya 464-8601, Japan.

we shall consider as the numerical examples the functionally gradient materials of which heat conductivities are a linear and quadratic functions in order to examine the property.

2 Formulation[3]

2.1 Boundary Value Problem and Transformation of Governing Equation

When the heat conductivity λ is given as a continuous function, the steady-state heat conduction problem of the functionally gradient material can be modeled by the governing equation:

$$\nabla\{\lambda\nabla u\} = \nabla^2 u + \frac{1}{\lambda}\nabla\lambda\nabla u \equiv \nabla^2 u + b = 0 \quad (\text{in } \Omega) \quad (1)$$

and the boundary conditions:

$$u = \bar{u} \quad (\text{on } \Gamma_u) \quad , \quad q = \bar{q} \quad (\text{on } \Gamma_q) \quad (2)$$

where $q \equiv \partial u / \partial \mathbf{n}$ and Ω, Γ_u and Γ_q denote the object domain under consideration, its potential u and the flux q specified boundaries, respectively. \mathbf{n} denotes the unit normal vector on the boundary and $(\bar{\cdot})$ the specified value.

The inhomogeneous term b is approximated by 5-order complete function in the global coordinates;

$$b = c_1 + c_2 x + c_3 y + \cdots + c_{20} x y^4 + c_{21} y^5 = \mathbf{c}^T \mathbf{r} \quad (3)$$

where \mathbf{c} and \mathbf{r} mean the unknown parameter and the coefficient vectors, respectively. Applying Eq.(3) to (1), we have

$$\nabla^2 u + \mathbf{c}^T \mathbf{r} = 0 \quad (4)$$

Assuming the homogenous solution of Eq.(4) as u^h and the particular solution related to r_i as u_i^p , the solution of the problem u is given as

$$u = u^h + c_1 u_1^p + c_2 u_2^p + \cdots + c_{21} u_{21}^p = u^h + \mathbf{c}^T \mathbf{u}^p \quad (5)$$

where $\mathbf{u}^p = \{u_1^p, u_2^p, \cdots, u_{21}^p\}^T$ and u^h and u_i^p satisfy the following equations.

$$\nabla^2 u^h = 0 \quad , \quad \nabla^2 u_i^p + r_i = 0 \quad (6)$$

Since r_i is polynomial, the particular solution u_i^p can be estimated easily.

Substituting Eq.(5) to Eqs.(1) and (2), we have

$$\nabla^2 u^h = 0 \quad (\text{in } \Omega) \quad (7)$$

$$u^h = \bar{u} - \mathbf{c}^T \mathbf{u}^p \equiv \bar{u}^h \quad (\text{on } \Gamma_1) \quad , \quad q^h = \bar{q} - \mathbf{c}^T \mathbf{q}^p \equiv \bar{q}^h \quad (\text{on } \Gamma_2) \quad (8)$$

where $q^h \equiv \partial u_i^h / \partial \mathbf{n}$ and $q_i^p \equiv \partial u_i^p / \partial \mathbf{n}$.

2.2 Trefftz Formulation

T-complete functions are determined so as to satisfy the governing equation[4];

$$\mathbf{u}^* = \{u_1^*, \dots, u_{2\mu-1}^*, u_{2\mu}^*, \dots\}^T = \{1, \dots, \Re[z^\mu], \Im[z^\mu], \dots\}^T \quad (9)$$

where $z = x + jy$ and $j = \sqrt{-1}$.

The potential u^h is approximated by the superposition of the T-complete functions u_j^* ;

$$u^h \simeq \tilde{u}^h = a_1 u_1^* + a_2 u_2^* + \dots + a_N u_N^* = \mathbf{a}^T \mathbf{u}^* \quad (10)$$

where N is the total number of the T-complete functions and $\mathbf{a} = \{a_1, \dots, a_N\}^T$ denotes the unknown parameter vector. Since Equation (10) dose not satisfy the boundary condition (8), the residual expressions yield. In this study, the parameter \mathbf{a} is determined so that the residuals are minimized by means of the collocation method. So, we have

$$\mathbf{K}\mathbf{a} = \mathbf{f} \quad (11)$$

The row and the column of the coefficient matrix \mathbf{K} are equal to the total numbers of the collocation points and the T-complete functions, respectively. Equation (11) is solved by the singular value decomposition[5].

2.3 Update Rule of Unknown Parameter \mathbf{c}

The boundary value problem defined by Eqs.(7) and (8) includes the unknown parameter \mathbf{c} . Since \mathbf{c} depends on the unknown function u , the iterative process is necessary for solving the problem. We shall describe here the iterative process.

Holding Eq.(3) at the iteration steps (k) and $(k+1)$, we have

$$b^{(k+1)} = \mathbf{r}^T \mathbf{c}^{(k+1)} \quad , \quad b^{(k)} = \mathbf{r}^T \mathbf{c}^{(k)}$$

Subtracting both sides of the equations, we have

$$b^{(k+1)} - b^{(k)} = \mathbf{r}^T (\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}) \equiv \mathbf{r}^T \Delta \mathbf{c} \quad (12)$$

$\Delta \mathbf{c}$ is determined so that Equation (12) is satisfied by means of the collocation method at the collocation points placed in the domain and on the boundary, which are referred as ‘‘computing point’’. Equation (12) is held at the computing point Q_i ;

$$\mathbf{r}^T(Q_i) \Delta \mathbf{c} = b^{(k+1)} - b^{(k)} = b(u^{(k)}) - \mathbf{r}^T \mathbf{c}^{(k)} \equiv \Delta b(Q_i)$$

The above equation is held at all computing points;

$$\mathbf{D} \Delta \mathbf{c} = \mathbf{f} \quad (13)$$

where \mathbf{D} and \mathbf{f} mean the coefficient matrix and vector. Equation (13) is solved for $\Delta \mathbf{c}$ by the singular value decomposition[5].

The parameter \mathbf{c} is updated by

$$\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} + \Delta \mathbf{c} \quad (14)$$

The convergence criterion is defined as

$$\eta \equiv \frac{1}{M_c} \sum_{i=1}^{M_c} |\Delta b(Q_i)| < \eta_c \quad (15)$$

where η_c is the positive number specified by a user.

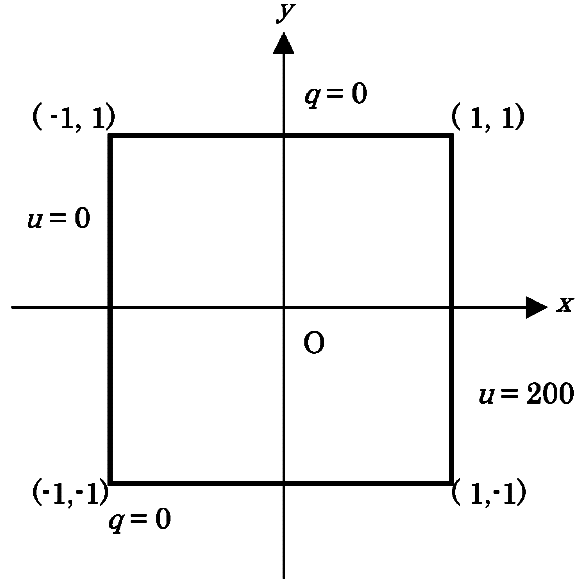


Figure 1: Numerical example

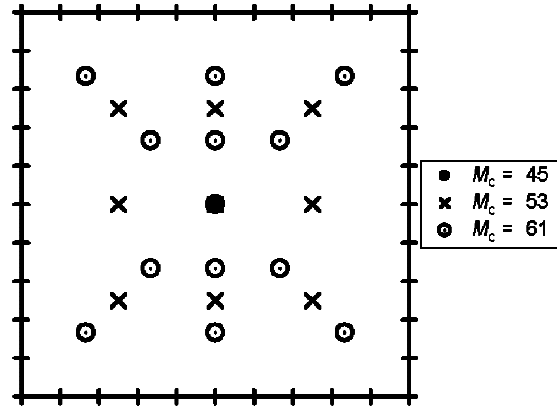


Figure 2: Placement of collocation and computing points

3 Numerical Examples

We shall consider as the first example that the heat conductivity λ is a linear function in x -coordinate;

$$\lambda = d_1 + d_2x$$

where d_1 is specified to be 200 and d_2 is taken as 20, ± 100 , ± 140 . The boundary condition is specified as shown in Fig.1. The analysis is carried out with 15 T-complete functions, 44 boundary collocation and 0, 1, 9 or 17 inner points. The placement of the points is shown in Fig.2. All boundary collocation and inner points are taken as the computing points. The initial values of the parameter c_i is specified as zero.

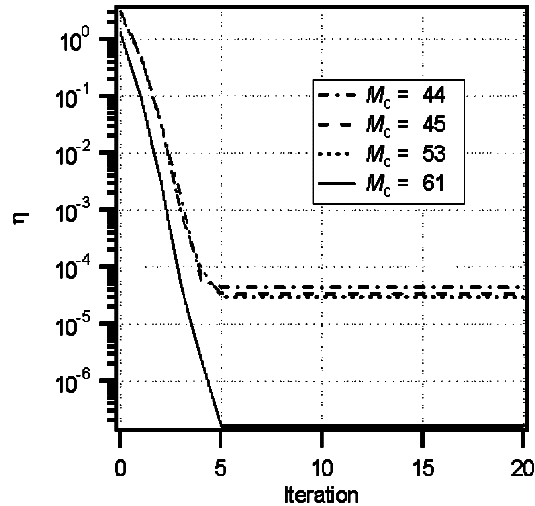
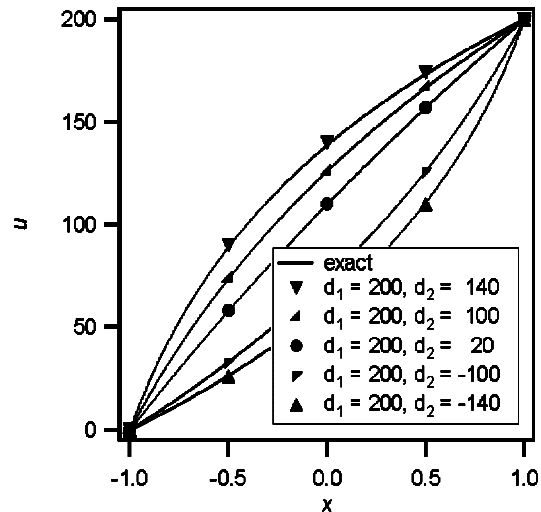
Figure 3: Convergence property of η ($d_2 = 20$)

Figure 4: Distribution of potential value

The computational error is estimated at the 25 estimation points which are distributed uniformly in the object domain. One can take as the error estimator

$$E_u^1 = \frac{1}{25} \sum |u - u^{ex}|$$

Figure 3 shows the convergence history of η in the case of $d_2 = 20$. The abscissa and the ordinate indicate the number of the iteration and η , respectively. In all cases, η converges at 5 iteration step. The converged value in the case of $M_c = 61$ inner points is much smaller than the other cases. Figure 4 shows the distribution of u . The abscissa and the ordinate indicate the x -coordinate of the estimation points and u , respectively. We notice that the numerical solutions well agree with the theoretical ones.

4 Conclusions

This paper describes the application of the Trefftz method to the steady-state heat transfer problem of the functionally gradient material. Since this problem is governed with Poisson equation, this paper presented the following scheme. An inhomogeneous term of the governing equation is approximated by 5-order complete polynomial in the global coordinates to determine the particular solution related to the inhomogeneous term. The boundary value problem of the Poisson equation is transformed into that of the Laplace equation by introducing the particular solution. Since the T-complete functions of the Laplace equation are known, the boundary value problem of the Laplace equation can be solved easily for the homogeneous solution. We considered as the numerical examples the functionally gradient materials of which heat conductivities are a linear and quadratic functions. The computational accuracy is better as the number of the inner points increases. The numerically-predicted expressions of inhomogeneous terms well agree with the theoretical ones. We can conclude that the validity of the present formulation can be confirmed.

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