

Special purpose Trefftz functions for the torsion of bars with regular polygonal cross-section

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Abstract: *A special purpose Trefftz functions for solutions of torsion problem of simply connected, two connected, and composite bars possessing regular polygon on cross section contour are proposed. Seven cases of bars are considered: 1) regular polygonal bars, 2) regular polygonal bars with circular centred holes, 3) cylindrical bars with regular polygonal centred holes, 4) regular polygonal bars with regular polygonal centred holes, 5) regular polygonal bars with circular centred reinforced rod, 6) cylindrical bars with regular polygonal reinforced rod, 7) regular polygonal bars with regular reinforced rod. Proposed Trefftz functions fulfil not only governing equation but also boundary conditions on part of boundary. The boundary collocation method in the least squares sense for solving appropriate boundary values problems for stress functions is used. By means of analytical integration of the stress functions, for the seven considered cases the analytical formulae for non-dimensional stiffness of bars are obtained.*

1 Introduction

The study of the torsion of rods is important and basic in design of structural elements and its one of classical problem in theory of elasticity [1-2]. Analytical solutions have been found for some simple cross-sectional shapes such as circle, annulus, ellipse, rectangle and triangle. Numerical methods is usually necessary for more complicated shapes. The most widely used methods are finite difference and the finite element methods. However, in recent years we have also witnessed the fast developed of the so-called meshfree methods. The Trefftz method can be summarised as consisting in using the exact solution to governing differential equation of the problem and satisfying the given boundary conditions approximately. Then this method can be treated as some version of meshfree method.

The present investigation is concerned with elastic torsion problems. Although the torsion of homogeneous isotropic prismatic bars has been studied very extensively and by numerous investigators, relatively little work has been done on the corresponding problems of cylinders consisting of two or more different materials

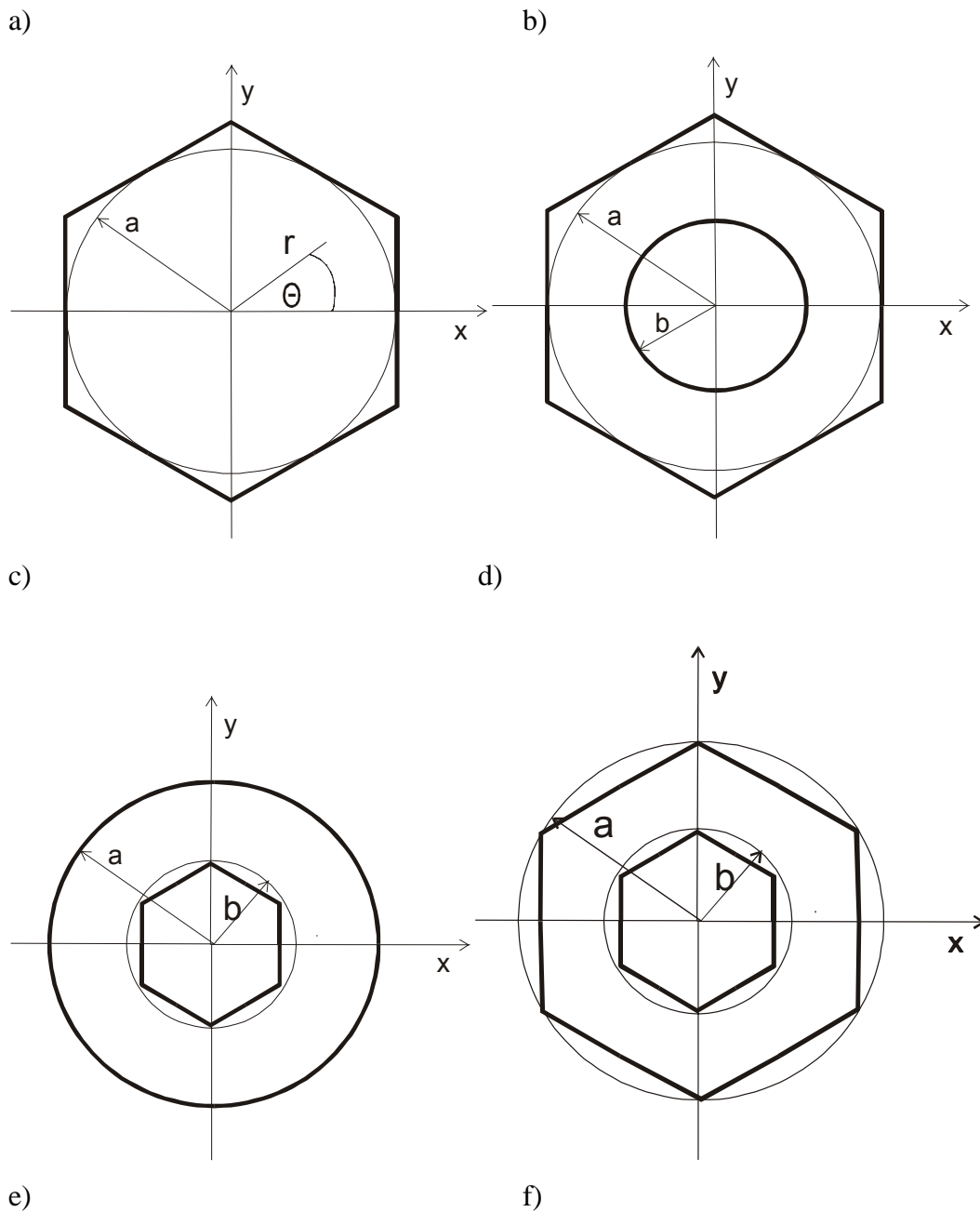
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bounded together. The purpose of the present paper is proposition of special purpose Trefftz functions for solutions of torsion problem of simply connected, two connected (rod with hole), and composite bars (two different materials) possessing regular polygon on cross section contour. These functions fulfil exactly not only governing equation of the torsion problem but also boundary condition on some parts of boundary of considered regions.

2 Formulation of boundary volume problems

Consider a family of prismatic bars of regular polygonal cross sections, which can be solid, with hole or with reinforced core made of different material. The outer, inner or both boundaries are regular polygon with L sides. The considered seven cases are presented on Fig. 1.



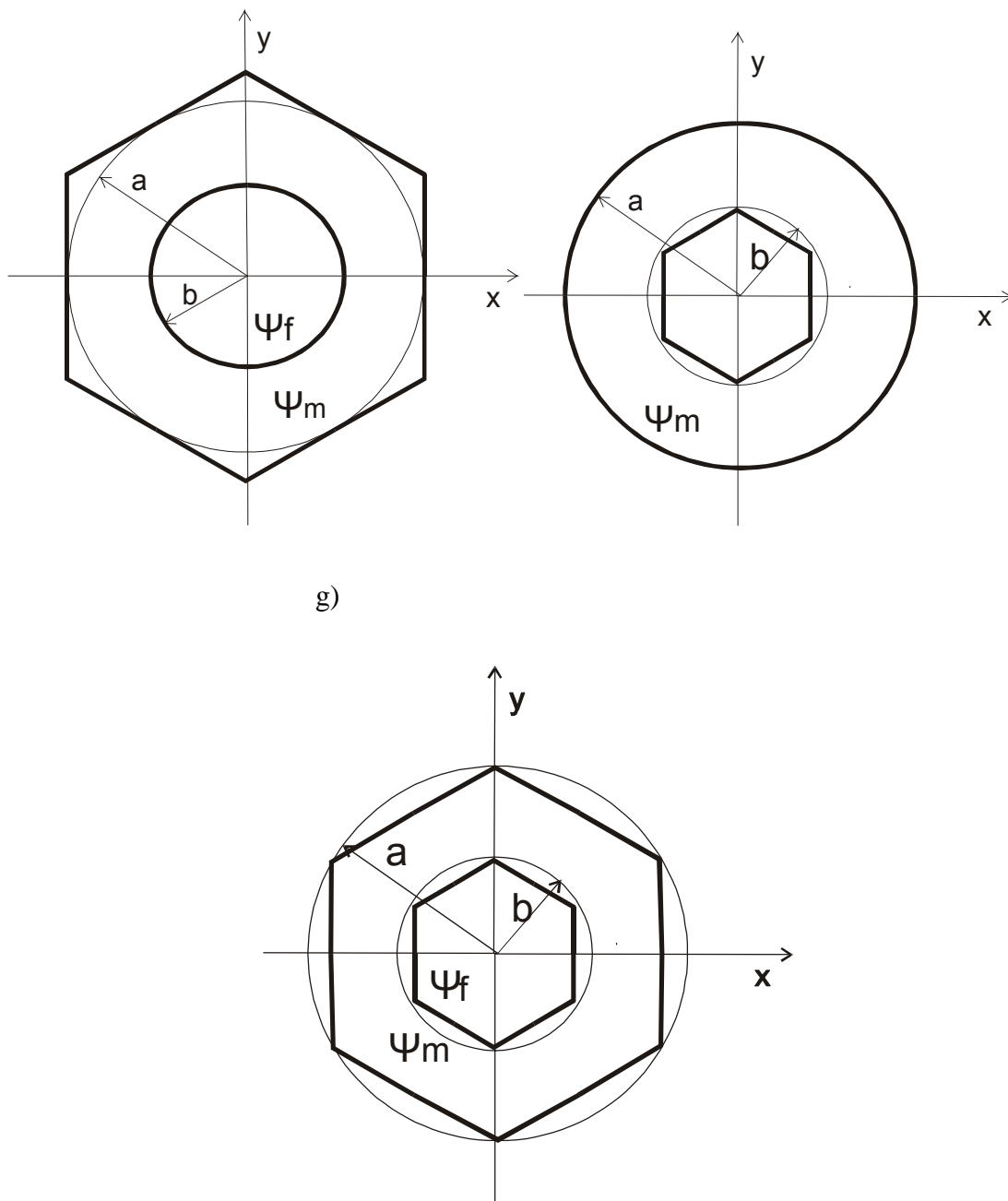


Figure 1: Considered cases: a) –Case 1. Full regular polygonal bars, b) - Case 2. Regular polygonal bars with circular centred holes, c) Case 3. Cylindrical bars with regular polygonal centred holes, d) Case 4. Regular polygonal bars with regular polygonal centred holes e) Case 5. Regular polygonal bars with circular centred reinforced rod, f) Case 6. Cylindrical bars with regular polygonal reinforced rod, g) Case 7. Regular polygonal bars with regular reinforced rod.

The torsion problem in prismatic bars are often formulated in terms of the stress functions, which satisfies Poisson's equation

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} = -2\mu\omega \quad (1)$$

where: Ψ is stress function, μ is shear modulus of rod material, ω is angle of twist of rod per unite length.

For one connected regions (Case 1) boundary condition is the following

$$\Psi(\Gamma) = 0 \quad (2)$$

where Γ is contour of cross-section.

For case 2-4 when cross-section is two connected the stress function on the boundaries must be constants:

$$\Psi(\Gamma_1) = 0 \quad , \quad \Psi(\Gamma_2) = \Psi_0 \quad (3)$$

where Γ_1 is outer contour of cross-section, Γ_2 is inner contour of cross-section, Ψ_0 is constant. This constant must be chosen in way that fulfil the condition

$$\int_0^{2\pi} \frac{\partial \Psi}{\partial n} \Big|_{\Gamma_i} d\Gamma = -2\mu\omega A_h. \quad (4)$$

where A_h is area of cross-section of hole.

For composite rods, case 5-7, we have two stress functions

$$\frac{\partial^2 \Psi_f}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi_f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi_f}{\partial \theta^2} = -2\mu_f \omega, \quad (5)$$

$$\frac{\partial^2 \Psi_m}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi_m}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi_m}{\partial \theta^2} = -2\mu_m \omega, \quad (6)$$

where Ψ_f is the stress function in the core, Ψ_m is the stress function in the cylinder bar, ω is angle of twist per unit length, μ_f and μ_m and are shear modules of the core and cylinder materials, respectively.

For composite cases the following boundary conditions have to be fulfilled:

$$\Psi_f = \Psi_m \quad \text{on} \quad \Gamma_i \quad (7)$$

$$\frac{1}{\mu_f} \frac{\partial \Psi_f}{\partial r} = \frac{1}{\mu_m} \frac{\partial \Psi_m}{\partial r} \quad \text{on} \quad \Gamma_i \quad (8)$$

$$\oint_{\Gamma_f} \frac{\partial \Psi_f}{\partial r} ds = -2\pi b^2 \mu_f \omega \quad (9)$$

3 Method of solution

Let introduce the nondimensional variables in the form:

$$R = \frac{r}{a}, \quad X = \frac{x}{a}, \quad E = \frac{b}{a}, \quad \delta = \frac{\mu_f}{\mu_m}, \quad \Psi_f = \frac{\Psi_f}{2\mu_f \omega a^2}, \quad \Psi_m = \frac{\Psi_m}{2\mu_m \omega a^2} \quad (10)$$

Since the problems are symmetric about the lines $\theta = 0$ and $\theta = \frac{\pi}{L}$, the solutions of above problems can be found only between these two lines in repeated elements. Taking into account general solution of Poisson equation with constant right hand in polar co-ordinate system and using consideration as in papers [3-4] one can obtained the following approximate solutions:

$$\text{Case 1} \quad \psi = -\frac{R^2}{4} + \sum_{k=1}^N Y_k R^{(k-1)L} \cos[(k-1)L\theta] \quad (11)$$

$$\text{Case 2} \quad \psi = -\frac{1}{4}(R^2 - E^2) + Y_1 + \sum_{k=2}^N Y_k \left(R^{(k-1)L} - \frac{E^{2(k-1)L}}{R^{(k-1)L}} \right) \cos[(k-1)L\theta] \quad (12)$$

$$\text{Case 3} \quad \psi = -\frac{R^2}{4} + \frac{1}{4} + \sum_{k=1}^N Y_k \left(R^{(k-1)L} - \frac{1}{R^{(k-1)L}} \right) \cos[(k-1)L\theta] \quad (13)$$

$$\text{Case 4} \quad \psi = -\frac{R^2}{4} + Y_1 + \sum_{k=2}^N (Y_k R^{(k-1)L} + Y_{N+k} R^{-(k-1)L}) \cos[(k-1)L\theta] \quad (14)$$

$$\text{Case 5} \quad \psi_f = -\frac{R^2}{4} + Y_1 + \sum_{k=2}^N Y_k R^{(k-1)L} \cos[(k-1)L\theta] \quad (15)$$

$$\begin{aligned} \psi_m = & -\frac{R^2}{4} + \frac{E^2}{4}(1-\delta) + \delta Y_1 + \frac{1+\delta}{2} \sum_{k=2}^N Y_k R^{(k-1)L} \cos[(k-1)L\theta] \\ & - \frac{1-\delta}{2} \sum_{k=2}^N Y_k E^{2(k-1)L} R^{-(k-1)L} \cos[(k-1)L\theta] \end{aligned} \quad (16)$$

$$\text{Case 6} \quad \psi_f = -\frac{R^2}{4} + Y_1 + \sum_{k=2}^N Y_k R^{(k-1)L} \cos[(k-1)L\theta] \quad (17)$$

$$\psi_m = -\frac{R^2}{4} + \frac{1}{4} + \sum_{k=1}^N Y_{N+k} (R^{kL} + R^{-kL}) \cos(kL\theta) \quad (18)$$

$$\text{Case 7} \quad \psi_f = -\frac{R^2}{4} + Y_1 + \sum_{k=2}^N Y_k R^{(k-1)L} \cos[(k-1)L\theta] \quad (19)$$

$$\Psi_m = -\frac{R^2}{4} + Y_{N+1} + \sum_{k=2}^N Y_{N+k} R^{(k-1)L} \cos[(k-1)L\theta] + \sum_{k=1}^N Y_{2N+k} R^{-kL} \cos(kL\theta) \quad (20)$$

These solutions fulfil exactly not only governing equation but also boundary conditions on lines of symmetry and circles sector (Trefftz function of special purpose). The boundary collocation method in the least squares sense for solving appropriate boundary values problems for stress functions is used. It means that one choose M points on this part of boundary where the boundary condition are not fulfil exactly and satisfy boundary condition in these points in the least square sense one has system of linear equation for unknown constants Y_k

By means of analytical integration of the stress functions, for the seven considered cases the analytical formulae for non-dimensional stiffness of bars are obtained. For example stiffness for case one has a form:

$$M = 8L\mu\omega a^4 \left\{ -\frac{1}{48} \frac{\sin \frac{\pi}{L}}{\cos \frac{\pi}{L}} \left(2 + \frac{1}{\cos^2 \frac{\pi}{L}} \right) + \sum_{k=1}^N \frac{Y_k \sin \left\{ [(k-1)L+1] \frac{\pi}{L} \right\}}{[(k-1)L+2][(k-1)L+1] \left(\cos \frac{\pi}{L} \right)^{(k-1)L+1}} \right\} \quad (21)$$

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