DIRECTIONAL DEPENDENCE AND RADIAL BASIS FUNCTIONS

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Abstract: *The Radial Basis Function (RBF) interpolator is a linear combination of translates of basis functions, the basis functions being invariant with respect to rotations on the underlying space. This invariance is not necessary either in the derivation of the estimator nor for the existence of a unique solution for the coefficients in the linear combination.*

One of the easiest ways to see why directional dependence might be appropriate is to transform the interpolator into a weighted linear combination of the data values, in that case the interpolating function is only defined implicitly. A positive definite radial basis function is also a covariance function and in the data value form for the RBF, the weights are determined by the spatial correlation between the values at the data locations and the data locations vs the location to be interpolated. In that context then directional dependence is very plausible. One of the ways in which a covariance can be directionally dependent is in the range, i.e., the distance at which the covariance is zero or nearly so. That is, the range depends on direction. T

A second more complicated form of directional dependence for a basis function is when the dependence changes with respect to dimension. For example, the dependence in the horizontal direction might be different than in the vertical direction. This suggests combining basis functions each of which are defined on lower dimensional subspaces. Sums and products are the two simplest constructions. By combining the two a much more general class of basis functions can be constructed. This construction also extends models valid on lower dimensional spaces to non-radial form on higher dimensional space. The key tool is the "marginal" which can be estimated and fitted from the data

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1 Introduction

This work is motivated by the following observations about radial basis functions and the radial basis functions interpolator:

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1. The derivation of the form of the interpolator and the equations to determine the coefficients is the same whatever the choice of the basis function.

2. The "data", i.e., the values of the function to be interpolated/approximated are not directly used in the derivation.

3. The sufficient condition for the existence and the uniqueness of the solution of the equations is the positive definiteness (of appropriate order)of the basis function.

4. Continuous positive definite functions may be interpreted as (auto)covariance functions of second order stationary random functions, likewise continuous conditionally positive definite functions may be interpreted as generalized covariances of intrinsic random functions, [4].

5. Neither covariances nor generalized covariances have to be "radial", i.e. they can depend on both distance and direction.

Moreover the usual form of the interpolator and the system of equations can be re-written in a form which is more natural for the stochastic interpretation, [5], [7], [9]. Let $u_1,\ldots,u_n$ be points in R^k and $f(u_1),\ldots,f(u_n)$ be the values of an unknown function. Let ϕ be the basis function and $P_j; j=0,\ldots,p$ the usual polynomials . Using simple linear algebra the radial basis interpolator can be re-written as

$$f*(u) = \Sigma_{i=1}^{n}\lambda_i(u)f(u_i) \tag{1}$$

where the system of equations is re-written in the form

$$\Sigma_{i=1}^{n}\lambda_i(u)\phi(u_i-u_m)+\Sigma_{j=0}^{p}\mu_j P_j(u_m) = \phi(u-u_m); \quad m=1,\ldots,n \tag{2}$$

and

$$\Sigma_{i=1}^{n} b_i P_j(u_i) = P_j(u); \quad j=0,\ldots,p \tag{3}$$

The basis function, i.e., covariance, can be dependent on direction in several different ways. Two problems associated with this will be discussed: (1) constructing valid basis functions incorporating this directional dependence and (2) ways to use the data to determine an appropriate model for the basis function.

2 Covariances, Variograms and Estimation

Let $C(h)$ be a real valued covariance function, i.e., $C(h) = Cov[Z(u+h),Z(u)]$ where $Z(u)$ is a second order stationary random function. Then $|C(h)| \le C(0)$. Set $\gamma(h) = C(0)-C(h)$, $-\gamma(h)$ is conditionally positive definite. Functions of this type can also be obtained as follows, $\gamma(h) = 0.5Var[Z(u+h)-Z(u)]$ for an intrinsic random function. $\gamma(h)$ is called a variogram. Unlike covariances which must be bounded, generalized covariances (e.g., $\gamma(h)$) may be unbounded and may exist for a wider class of random functions. For covariances and generalized covariances it is natural to estimate them from data. Moreover $\gamma(h)$ can be estimated without separately estimating the mean of $Z(u)$.

$$\gamma*(r,\theta) = [1/2N(r,\theta)]\Sigma[Z(u+h)-Z(u]^2 \tag{4}$$

The sum is taken over all pairs (of data locations) where $|h|$ is "close" to r and where the argument of h is close to θ. $N(h)$ is the number of such pairs

3 Anisotropic Models

3.1 Model Parameters

Let

$$\gamma(r) = S[(3r)/(2a) - r^3/2a^3]; \quad 0 \leq r \leq a \quad and = S \quad \text{otherwise} \tag{5}$$

Then $\gamma(r)$ is a a radial conditionally negative definite function on R^k. Equivalently $C(r) = S - \gamma(r)$ is a covariance function with compact support. The parameter C_1 is usually called the *sill* and a the *range* (of dependence). This model is sometimes called the Spherical model. Let

$$\gamma(r) = Sr; \quad 0 \leq r \leq 1 \quad \text{and} = S \quad \text{otherwise} \tag{6}$$

. In variogram form this is called the Truncated Linear model and is conditionally negative definite only in R^1, the corresponding covariance is sometimes called the "hat" function. The range and sill are the same as for the Spherical. A crucial difference is that only the Spherical has a "radial" extension to higher dimensions. Some covariances will have only an "effective" range, e.g., the Exponential and Gaussian models, usually taken as the distance where the covariance is $.95 \times sill$

3.2 Geometric Anisotropies

The simplest form of an anisotropy is when the *range* depends on the direction. Then the "level curves" of the covariance are ellipses (ellipsoidal in higher dimensions). A covariance with a geometric anisotropy can be obtained from a "radial" by applying an affine transformation. The anisotropy is characterized by the angles and the distortion ratios. The sill does not change with direction.

3.3 Zonal Anisotropies

An extreme form of a geometric anisotropy is obtained when the dependencies in different orthogonal directions are separated. The simplest way to do this is to write the covariance as a product. For example, consider the gaussian model on R^2, $C(x,y) = \exp -(x^2 + y^2)$ which is the product of two covariances each on defined an valid on R^1. In contrast the Exponential model on R^2 is not the product of the Exponential models each on R^1 although that product is a valid model. The disadvantage to the product model is that each factor must have the same sill. More generally if R^k is written as $R^{k_1} \times R^{k_2}$ where $k = k_1 + k_2$, $C_1(h_1)$ a covariance defined on R^{k_1} and $C_2(h_2)$ a covariance defined on R^{k_2} then for $a > 0, b \geq 0, c \geq 0$

$$C(h_1, h_2) = aC_1(h_1) \times C_2(h_2) + bC_1(h_1) + cC_2(h_2) \tag{7}$$

is a covariance on $R^{k_1} \times R^{k_2}$ In terms of the associated variograms this becomes

$$\gamma(h_1, h_2) = \gamma_1(h_1, 0) + \gamma_2(0, h_2) - K\gamma_1(h_1, 0) \times \gamma_2(0, h_2) \tag{8}$$

where $0 < K \leq 1/(max(Sill_1, Sill_2))$. This can be generalized to multiple factors. $\gamma_1(h_1, 0)$ and $\gamma_2(0, h_2)$ are called the marginals and they are valid models on the lower dimensional spaces. Note that either or both of the marginals may have a geometric anisotropy.

3.4 Extensions

There are at least two ways to extend the product sum. The first way is to allow higher powers for two factors,

$$C(h_1,h_2) = \Sigma_i^\infty = 0\Sigma_j^\infty = 0[C_1(h_1)]^i[C_2(h_2)]^j p_{ij} \tag{9}$$

where p_{ij} is a discrete bivariate probability distribution. Ma [3] proposed this model for space-time modeling. This could be converted into the variogram form. The second way is to allow a further decomposition of R^k, e.g., $R^k = R^{k_1} \times R^{k_2} \times \ldots \times R^{k_t}$ where $k_1,\ldots,k_t$ are non-negative integers whose sum is k. Both of these extensions might be applied to the multivariate case analogous to the space-time application in [2]

4 Some Non-geometric Anistropic Models

4.1 The Truncated Linear Model

Let $k_1 = k_2 = 1, i.e., k = 2$. Let both marginals be the Truncated Linear model. then the product-sum model extension to $R^2$is

$$\gamma(h_1,h_2) = S|h_1| + S|h_2| - KS^2|h_1||h_2| \quad \text{if} \quad 0 \le |h_1| \le 1 \quad \text{and} \quad 0 \le |h_2| \le 1$$

$$= S|h_1| + S - KS^2|h_1| \quad \text{if} \quad 0 \le |h_1| \le 1 \quad \text{and} \quad |h_2| > 1$$

$$= S + S|h_2| - KS^2|h_2| \quad \text{if} \quad 0 \le |h_1| \le 1 \quad \text{and} \quad |h_2| > 1$$

$$= S|h_1| + S - KS^2|h_1| \quad \text{if} \quad 0 \le |h_2| \le 1 \quad \text{and} \quad |h_1| > 1$$

$$= 2S - KS^2 \quad \text{if} \quad |h_1| > 1 \quad \text{and} \quad |h_2| > 2$$

4.2 Gaussian

Let

$$\gamma_1(h_1,0) = S[1 - \exp - |h_1|^2]$$

and

$$\gamma_1(0,h_2) = S[1 - \exp - |h_2|^2]$$

then

$$\gamma(h_1,h_2) = S[1 - \exp - |h_1|^2] + S[1 - \exp - |h_2|^2] - KS^2[1 - \exp - |h_1|^2] \times [1 - \exp - |h_2|^2]$$

In the special case where $K = 1/S$

$$\gamma(h_1,h_2) = S[1 - \exp - |h_1|^2 - |h_2|^2] = S[1 - \exp - |(h_1,h_2)|^2]$$

Let $K = 1/S$. Then the usual ”radial” extension to the higher dimensional space is a special case of the product sum.

4.3 Exponential

Let

$$\gamma_1(h_1,0) = S[1-\exp -|h_1|]$$

and

$$\gamma_1(0,h_2) = S[1-\exp -|h_2|]$$

then

$$\gamma(h_1,h_2) = S[1-\exp -|h_1|] + S[1-\exp -|h_2|] - KS^2[1-\exp -|h_1|] \times [1-\exp -|h_2|]$$

. The special case $K = 1/S$ does not lead to the usual exponential model on the higher dimensional space.

5 Estimation and Fitting of Product-Sum Models

Each of the marginals can separately be estimated and fitted by the sample variograms

$$\gamma_1 * (r_1,\theta_1,0) = (1/2N(h_1))\Sigma[Z(u+h_1,v) - Z(u,v)]^2$$

$$\gamma_2 * (0,r_2,\theta_2) = (1/2N(h_2)\Sigma[Z(u,v+h_2) - Z(u,v)]^2$$

6 Evaluating the Fit

In the stochastic form, the interpolator is unbiased and has minimal error variance. By using a form of jackknifing multiple statistics can be computed to compare alternative models, [6]

7 Summary

Neither the form of the radial basis function interpolator nor the system of equations requires the basis function to be ”radial”. By using the relationship between positive definite functions and covariances a more general class of basis functions can be constructed This construction is a generalization of the space-time models found in [1]. Because in the stochastic form the interpolator is unbiased and has minimal error variance, multiple statistics can be used to compare alternative choices,

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