

# The Method of Fundamental Solutions with Dual Reciprocity for Thermoelasticity

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**Abstract:** *Here the Method of Fundamental solutions is used to solve a thermoelastic problem, the inhomogeneous terms being modeled using the Dual Reciprocity Method, employing polyharmonic spline approximation functions with polynomial augmentation terms. It is found that results converge both for increasing numbers of internal points, and for increasing order of the polyharmonic splines. For the homogeneous solution, the circle of fictitious points first introduced by Bogomolny is employed, results are found to be practically constant over a range of values of the radius of this circle.*

## 1 Introduction

The thermoelastic problem is an example in which the Navier governing equations can be treated with the Method of Fundamental Solutions (MFS) and the inhomogeneous terms due to thermal forces treated using the Dual Reciprocity Method.

The Method of Fundamental Solutions is a meshless, indirect boundary technique which avoids singularities by defining a fictitious surface exterior to the problem geometry. The method was introduced by Kupradze and Aleksidze [1]. In its original form, the method is only applicable to homogeneous problems for which a complete fundamental solution is available, recent work has extended the method to more general problems, applications being reviewed by Fairweather and Karageorghis [2] and Golberg and Chen [3]. Here, the position of the fictitious points is fixed on a circle as first done by Bogomolny, [4], which has led to some very accurate solutions for potential problems. The position of the points can also be established using optimization. Recent applications include elasticity without body forces and fluids, see references cited in [2-3].

The MFS has been extended to more general cases including time-dependent and non-linear potential problems by including the non-homogeneous terms using particular solutions, [5]. The DRM, originally developed for taking domain integrals to the boundary in BEM, [6], can be applied for calculating approximate particular solutions in MFS in a similar way.

Here application is made to a thermoelastic problem, in which the inhomogeneous term due to the thermal forces is approximated using DRM considering Polyharmonic Spline approximation functions with polynomial augmentation terms, [7]. It is shown that for a given order of Polyharmonic Spline approximation function, convergence of the solution can be obtained by increasing the number of internal points. For a fixed number

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of internal points, further convergence can be obtained by increasing the order of the Polyharmonic spline function.

In relation to the MFS used to solve the homogeneous problem, it is found that results are practically constant over a range of values of the radius of the circle of fictitious points. Best results are obtained using the Singular Value Decomposition (SVD) algorithm [8], for obtaining the unknown coefficients.

## 2 Problem Formulation

### 2.1 Thermoelasticity

Consider the Navier equation with body force terms, which governs linear elastic problems,

$$Gu_{l,mm} + \frac{G}{(1-2\nu)}u_{m,ml} + b_l = 0 \quad \text{in } \Omega \quad (1)$$

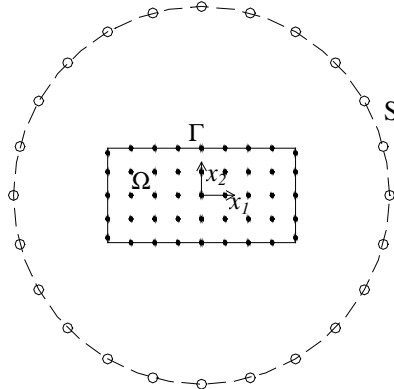
where  $G$  is the shear modulus and  $\nu$  is the Poisson Ratio,  $u_m$  are displacements and  $b_m$  are body forces. Here plane problems are considered, ( $m=1,2$ ). Equation (1) is subject to boundary conditions in displacements,  $u_m = \bar{u}_m$  on  $\Gamma_{u_m}$  and in surface tractions  $p_m = \bar{p}_m$  on  $\Gamma_{p_m}$  where  $\Gamma_{u_m} \cup \Gamma_{p_m} = \Gamma$ , the total boundary of the domain  $\Omega$ . The surface tractions are given by  $p_m = \sigma_{ml}n_l$  where  $n_l$  are the direction cosines of the outward normal on the boundary  $\Gamma$  and  $\sigma_{ml}$ , ( $m,l=1,2$ ) is the stress tensor. The overline signifies known values.

It has been shown, see for instance [9], that a thermal load of  $\theta^o$ , can be modeled using (1) if a pseudo body force of  $b_m = \alpha(3\lambda + 2\mu)\theta_{,m}$  is employed and if on the  $\Gamma_p$  boundary, a pseudo surface traction  $p''_m = \alpha(3\lambda + 2\mu)\theta n_m$  is added to the existing boundary condition in such a way that on this part of the boundary  $p_m = \bar{p}_m + p''_m$

In the above expressions,  $\alpha$  is the coefficient of thermal expansion and  $\lambda, \mu$  are Lamé constants. It should be noted that  $(3\lambda + 2\mu) = 2G(1+\nu)/(1-2\nu)$ .

### 2.2 Method of Fundamental Solutions

Equation (1) with the pseudo body force and pseudo surface traction defined as above can be solved using MFS by first defining a fictitious surface  $S$  exterior to  $\Omega$  as shown in Fig. 1.



**Figure 1:** Circle of Fictitious Points exterior to Problem Geometry

$N$  points are defined on the boundary  $\Gamma(i)$ , and an equal number of points evenly distributed on the fictitious surface  $S(j)$ .  $M$  points, where  $M \neq N$  are defined on the domain  $\Omega(k)$ .

Consider a point  $x^i \in \Gamma_{u_m}$ , on which a boundary condition  $u_m = \bar{u}_m$  is prescribed. One can express the known value of the variable as the sum of a series of fundamental solutions  $u_{1m}^{j*}$  and  $u_{2m}^{j*}$  with unknown coefficients,  $c(j)$ ,  $j=1, \dots, 2N$ , [2,3]. Body forces are included using particular solutions for displacements,  $\hat{u}_m^i$  as has been done for potential problems, [3,5]. At a point  $x^i \in \Gamma_{p_m}$ , having a boundary condition  $p_m = \bar{p}_m$ , a similar expression can be written, using fundamental solutions  $p_{1m}^{j*}$  and  $p_{2m}^{j*}$ , and particular solutions for surface tractions  $\hat{p}_m^i$

$$\sum_{j=1}^N \left[ u_{1m}^{j*}(x^i) \cdot c(j) + u_{2m}^{j*}(x^i) \cdot c(j+N) \right] = \bar{u}_m^i + \hat{u}_m^i \quad (2)$$

$$\sum_{j=1}^N \left[ p_{1m}^{j*}(x^i) \cdot c(j) + p_{2m}^{j*}(x^i) \cdot c(j+N) \right] = \bar{p}_m^i + \hat{p}_m^i \quad (3)$$

the coefficients  $c(j)$  are the same in (2) and (3). The fundamental solutions for displacements and surface tractions are given by

$$u_{lm}^{j*}(x) = \frac{1}{8\pi(1-\nu)G} \left\{ (3-4\nu) \ln(r^j(x)) \delta_{lm} - r^j(x)_{,m} r^j(x)_{,l} \right\}, \quad (4)$$

$$p_{lm}^{j*}(x) = \frac{C_1}{r^j(x)} \left\{ \left[ C_2 \delta_{lm} + 2r^j(x)_{,l} r^j(x)_{,m} \right] \frac{\partial r}{\partial n} - C_2 (r^j(x)_{,l} n_m(x) - r^j(x)_{,m} n_l(x)) \right\} \quad (5)$$

where  $C_1 = -1/4\pi(1-\nu)$  and  $C_2 = (1-2\nu)$  which are the usual expressions used in BEM, [10]. It should be noted that the expressions given in this section are written for the plane strain case. For plane stress,  $\nu$  should be substituted for  $\nu' = \nu/(1+\nu)$ , and  $\alpha$  for  $\alpha' = \alpha/(1+\nu)$  as is usually done in BEM, [10].

Equations (2) and (3) lead to a matrix equation

$$Ac = y \quad (6)$$

where  $A$  is of order  $2N$ . Once values of  $c$  are available, unknown values of  $p_m$  can be calculated, on the boundary  $\Gamma_{u_m}$  using

$$p_m(x^i) = \sum_{j=1}^N \left[ p_{1m}^{j*}(x^i) \cdot c(j) + p_{2m}^{j*}(x^i) \cdot c(j+N) \right] + \hat{p}_m^i \quad (7)$$

and unknown values of  $u_m$  can be obtained for points both on the boundary  $\Gamma_{p_m}$  and on the domain  $\Omega$  using

$$u_m(x^i) = \sum_{j=1}^N \left[ u_{1m}^{j*}(x^i) \cdot c(j) + u_{2m}^{j*}(x^i) \cdot c(j+N) \right] + \hat{u}_m^i \quad (8)$$

Stresses can be obtained for both boundary and internal points using

$$\begin{aligned}\sigma_{11}(x^i) &= \sum_{j=1}^N \left[ \sigma_{111}^{j*}(x^i) \cdot c(j) + \sigma_{112}^{j*}(x^i) \cdot c(j+N) \right] + \hat{\sigma}^i_{11} \\ \sigma_{12}(x^i) = \sigma_{21}(x^i) &= \sum_{j=1}^N \left[ \sigma_{121}^{j*}(x^i) \cdot c(j) + \sigma_{212}^{j*}(x^i) \cdot c(j+N) \right] + \hat{\sigma}^i_{12} \\ \sigma_{22}(x^i) &= \sum_{j=1}^N \left[ \sigma_{221}^{j*}(x^i) \cdot c(j) + \sigma_{222}^{j*}(x^i) \cdot c(j+N) \right] + \hat{\sigma}^i_{22}\end{aligned}\quad (9)$$

where, [10],

$$\sigma_{lmn}^{j*}(x) = \frac{-C_1}{r^j(x)} \left\{ C_2 (r^j(x)_{,m} \delta_{ln} + r^j(x)_{,n} \delta_{lm} - r^j(x)_{,l} \delta_{mn}) \right\} \\ + 2r^j(x)_{,l} r^j(x)_{,m} r^j(x)_{,n}$$

where  $C_1$  and  $C_2$  are as in (5). Due to the pseudo body force, pseudo surface traction approach adopted, the displacements given by (8) will be correct, however the stresses given by (10) need to be corrected using  $\sigma_{ij} = \sigma_{ij}' - (3\lambda + 2\mu)\alpha\theta\delta_{ij}$  where  $\sigma_{ij}'$  are the uncorrected values and  $\delta_{ij}$  is the Kronecker delta. Approximate particular solutions  $\hat{u}_m^i$ ,  $\hat{p}_m^i$  and  $\hat{\sigma}_{lm}^i$  are obtained using DRM as given below.

### 2.3 Dual Reciprocity Method

Approximate particular solutions for  $\hat{u}_m^i$ ,  $\hat{p}_m^i$  and  $\hat{\sigma}_{lm}^i$  are obtained in a similar way as is done in the BEM, [6]. For DRM, the fictitious surface is not considered. Approximate particular solutions are calculated at the  $N+M$  points located on the boundary  $\Gamma$ , and the domain  $\Omega$ . Let these points be called  $\bar{\Omega}(i) = (x_1^i, x_2^i)$ . The same set of points is used for DRM collocation,  $\bar{\Omega}(k) = (x_1^k, x_2^k)$ .

First consider the homogeneous Navier equation written using an operator  $L$

$$L(u_m) = Gu_{m,ll} + \frac{G}{(1-2\nu)} u_{l,lm} \quad (10)$$

The particular solutions for body forces required should be such that

$$L(\hat{u}_m) = b_m \quad (11)$$

In DRM, the body forces are approximated using known functions as follows

$$b_m(x) \approx \sum_{j=1}^Q f^j(x) \delta_{ml} \beta_l^j \quad (12)$$

where

$$f^j(x) = \begin{cases} g(r^j(x)) & , \quad 1 \leq j \leq N+M \\ a^{j-M}(x) & , \quad N+M+1 \leq j \leq N+M+A \end{cases} \quad (13)$$

In (13)  $r^k(x) = [(x_1^k - x_1)^2 + (x_2^k - x_2)^2]^{1/2}$ ,  $k=1, \dots, N+M$ ,  $g(r)$  are Radial Basis Functions, (RBF) and functions  $a^j(x)$  are augmentation functions, [7].  $Q=N+M+A$  where  $A$  is the number of augmentation functions to be considered. The functions  $f^j(x)$  will be considered below. The  $\beta_l^j$  are a set of initially unknown coefficients. The  $\delta_{lm}$  are introduced because there are no directions associated with the functions  $f^j(x)$ . The particular solutions for the DRM approximating functions  $\hat{U}_{lm}^j$  should be such that

$$L(\hat{U}_{lm}^j) = f^j \delta_{lm} \quad (14)$$

Combining equations (11-14) one obtains

$$L(\hat{u}_m) = b_m \approx \sum_{j=1}^Q f^j \delta_{lm} \beta_l^j = L \sum_{j=1}^Q (\hat{U}_{lm}^j \beta_l^j) \quad (15)$$

Considering the first and last terms in (15)

$$\hat{u}_m(x) = \sum_{j=1}^Q \hat{U}_{lm}^j(x) \beta_l^j \quad (16)$$

In the same way an approximation for  $\hat{p}_m(x)$  in (3) and (7) can be obtained

$$\hat{p}_m(x) = \sum_{j=1}^Q \hat{P}_{lm}^j(x) \beta_l^j \quad (17)$$

Where  $\hat{P}_{lm}^j$  is obtained from  $\hat{U}_{lm}^j$  as follows: .

$$\begin{aligned} \hat{E}_{lmn}^j &= \frac{1}{2} (\hat{U}_{lm,n}^j + \hat{U}_{ln,m}^j) \\ \hat{D}_{lmn}^j &= 2G\hat{E}_{lmn}^j + \frac{2G\nu}{(1-2\nu)} \hat{E}_{lpp}^j \delta_{mn} \end{aligned} \quad (18)$$

$$\hat{P}_{lm}^j = \hat{D}_{lmn}^j n_l$$

An approximate particular solution for stresses,  $\hat{\sigma}_{mn}^i$ , (9) is obtained using  $\hat{D}$  from the last equation (18)

$$\hat{\sigma}_{mn}^i = \hat{D}_{lmn}^i \beta_l^i. \quad (19)$$

#### 2.4 Approximation Functions

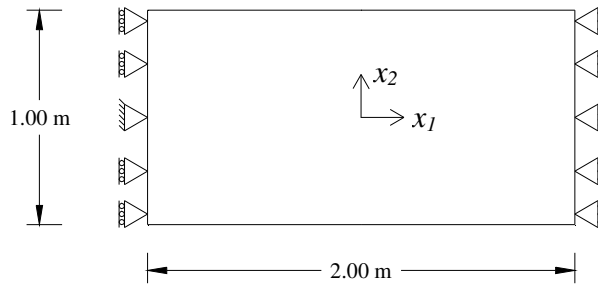
Here the DRM approximation functions considered are the Polyharmonic splines  $g(r) = r^{2n} \log r$  with augmentation functions  $a^j(x)$ . With  $n=1$ ,  $g(r)$  becomes the Thin Plate Spline (TPS) for which the particular solutions  $\hat{U}_{lm}^j$  and  $\hat{P}_{lm}^j$  are given in [11]. In the general case for 2D for  $g = r^{2n} \log r \delta_{lm}$  for  $n=1,2,3,\dots$  the relevant functions are given in [12].

The augmentation functions were first introduced by Golberg and Chen [7], who considered a TPS function augmented with the linear polynomial terms  $[1, x_1, x_2]$  calling the new function the Augmented Thin Plate Spline, ATPS. The implementation of the linear terms was first detailed for elasticity problems by Bridges and Wrobel, [11], where particular solutions for these terms were presented. Some other cases are considered in [12].

### 3 Numerical results

For the example considered the geometry shown in Fig. 2 is employed.

A thermal load of  $\theta = 50(x_2^3 + x_2^2 + x_2 + 1)$  is considered under plane stress. The exact solution is  $\sigma_{11} = -E\alpha\theta$ . Material properties were taken as  $E=10000\text{Mpa}$ ,  $\nu=0.3$  and  $\alpha=0.00001^\circ\text{C}^{-1}$ . 28 boundary points are used, and results given in Table 1 are calculated at 11 points along  $x_1=0$ , of which 9 points are on  $\Omega$  and two on  $\Gamma$ .



**Figure 2:** *Geometry for thermal load problem*

For linear augmentation, (Columns 2-5 in Table 1), for a fixed value of the order  $n$  of the polyharmonic spline approximation functions,  $g(r) = r^{2n} \log r$ , results improve increasing the number of internal points. Considering columns 2 and 3, results are given for  $n=1$  for 9 and 63 internal points (IP). Considering columns 3-5, for a fixed number of internal points, in this case 63, results converge as the order of the polyharmonic spline approximation functions is increased, in this case  $n=1, 2$  and 3 is considered. The values given in Table 1 are for the radius of the circle of fictitious points equal to 250. Consider column 5 for which IP=63 and  $n=3$ , with Linear Augmentation. The largest difference between calculated results and the expected solution is 0.0036 at  $x_2 = \pm 0.5$ . Over the range of values of the radius of the circle of fictitious points from 250 to 1250, this difference changes by only  $\pm 0.00019$ .

If the exact variation of the temperature field is known, an appropriate augmentation function may be used. In this case using cubic augmentation, considering column 6 in table 1, the expected values of the stresses are obtained, these results being independent of the order of the polyharmonic splines,  $n$ , and the number of internal points. In this case results are also independent of the value of the radius of the circle of fictitious points. The results given in column 2-5 of table 1 were obtained using Singular Value Decomposition, [8], which was found to produce best results. In the case of the cubic augmentation, either SVD or the usual LU decomposition may be employed.

**Table 1:** Values of  $\sigma_{11}$  (MPa) on  $x_1 = 0$  for thermal load case, for different augmentation, orders of polyharmonic spline,  $n$ , and numbers of internal points,  $IP$

$x_2$	Linear Augmentation				Cubic	Exact
	IP = 9	IP = 63			IP = 9	
	$n = 1$	$n = 1$	$n = 2$	$n = 3$	$n = 1$	
0.5	-9.5520	-9.3970	-9.3683	-9.3786	-9.3750	-9.3750
0.4	-8.2259	-8.1366	-8.1150	-8.1226	-8.1200	-8.1200
0.3	-7.1460	-7.0966	-7.0816	-7.0868	-7.0850	-7.0850
0.2	-6.2728	-6.2471	-6.2379	-6.2411	-6.2400	-6.2400
0.1	-5.5692	-5.5583	-5.5540	-5.5555	-5.5550	-5.5550
0.0	-5.0000	-5.0000	-5.0000	-5.0000	-5.0000	-5.0000
-0.1	-4.5308	-4.5416	-4.5460	-4.5445	-4.5450	-4.5450
-0.2	-4.1272	-4.1528	-4.1621	-4.1589	-4.1600	-4.1600
-0.3	-3.7540	-3.8033	-3.8184	-3.8132	-3.8150	-3.8150
-0.4	-3.3741	-3.4634	-3.4854	-3.4774	-3.4800	-3.4800
-0.5	-2.9480	-3.1030	-3.1317	-3.1214	-3.1250	-3.1250

## 4 Conclusions

Here the Method of Fundamental Solutions is applied to a thermoelastic problem, the inhomogeneous terms being modelled with Dual Reciprocity. Polyharmonic Spline approximation functions were employed with polynomial augmentation. It is found that results converge both for increasing numbers of internal points, and for increasing order of the polyharmonic splines. For the homogeneous solution, the circle of fictitious points first introduced by Bogomolny is employed, results are found to be practically constant over a range of values of the radius of this circle.

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