On the use of a wave based prediction technique for steady-state structural-acoustic radiation analysis

B. Pluymers[(1)], W. Desmet, D. Vandepitte and P. Sas

Abstract: *Conventional element based methods for modelling structural-acoustic radiation problems are limited to low-frequency applications. Recently a novel prediction technique has been developed based on the Trefftz approach. This new wave based method is computationally more efficient than the element based methods and, as a consequence, can tackle problems also at higher frequencies.*
This paper discusses the basic principals of the new method and illustrates its performance for the two-dimensional radiation analysis of a bass-reflex loudspeaker.

Keywords: Trefftz, structural-acoustics, radiation, wave based method, mid-frequency analysis

1 Introduction

The use of element based prediction techniques such as the finite element (FE) method, the infinite element (IE) method and the boundary element (BE) method, is generally accepted for the steady-state dynamic analysis of coupled structural-acoustic radiation problems. The FE based methods [1] truncate the unbounded radiation domain by introducing an artificial boundary surface. At this boundary surface appropriate impedance boundary conditions are applied to avoid spurious acoustic reflections [2]. The IE method [3] models explicitly the domain, exterior to the truncation surface, by coupling infinite elements to the bounded FE domain. Since model sizes increase with frequency, the use of these element based methods is restricted to low-frequency applications.
BE methods [4] discretize only the boundaries of the considered problem and obtain their solutions from a boundary integral formulation that inherently satisfies the Sommerfeld radiation condition. In this way no truncation surfaces must be introduced. Drawbacks of these methods are the fully populated, frequency dependent and not always symmetric system matrices which lead to computationally demanding calculations and restrict the use of the BE methods also to low-frequency applications.
Recently a new wave based method (WBM) [5], which is based on the Trefftz approach [6], has proven to be successful for low- and mid-frequency applications in bounded domains [7], [8]. Instead of using simple, approximating shape functions to describe the dynamic

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variables, exact solutions of the governing differential equations are used. No fine discretization of the domains is necessary so the model size and subsequent computational efforts are much smaller than with the element based methods. This allows to handle also mid-frequency applications.
This paper discusses how the WBM can be extended for radiation problems in unbounded domains. The technique is illustrated for the two-dimensional analysis of the sound radiation of a bass-reflex loudspeaker and its performance is compared with the conventional element based techniques.

2 Problem definition

Figure 1 shows a two-dimensional (2D) bass-reflex loudspeaker. The thin flexible membrane Γ_s is excited by a normal line force F at circular frequency ω and radiates sound into the loudspeaker and its unbounded surroundings. The loudspeaker back-cavity panels Γ_0 are rigid.
The steady-state *normal displacement* $w(\underline{\mathbf{r}}_s)$ of the loudspeaker membrane is governed by the dynamic plate equation

$$(\nabla^4 - k_b^4)w(\underline{\mathbf{r}}_s) = \frac{F}{D}\delta(\underline{\mathbf{r}}_s, \underline{\mathbf{r}}_F) + \frac{\Delta p(\underline{\mathbf{r}}_s)}{D} \tag{1}$$

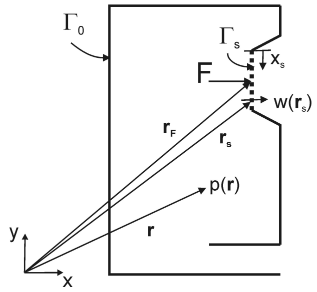

Figure 1: A 2D bass-reflex loudspeaker

$k_b = \sqrt[4]{\frac{\rho_s t \omega^2}{D}}$ is the bending wavenumber and $D = \frac{Et^3(1+j\eta)}{12(1-\nu^2)}$ is the bending stiffness, with t the plate thickness, ρ_s the density, E the elasticity modulus, j the imaginary unit $\sqrt{-1}$, ν the Poisson coefficient and η the loss factor. $\Delta p(\underline{\mathbf{r}}_s)$ is the pressure difference over the plate, acting as an external load.
To uniquely define the normal displacement, four structural boundary conditions must be specified, i.e. two conditions at both plate edges. For clamped plates, for instance, the boundary conditions are ($\underline{\mathbf{r}}_s \rightarrow x_s \in [0, L]$)

$$w(0) = w(L) = \frac{dw(0)}{dx_s} = \frac{dw(L)}{dx_s} = 0 \tag{2}$$

Assuming that the system is linear, non-viscous, and adiabatic, the steady-state *acoustic pressure* $p(\underline{\mathbf{r}})$ is governed by the homogeneous Helmholtz equation

$$\nabla^2 p(\underline{\mathbf{r}}) + k^2 p(\underline{\mathbf{r}}) = 0, \tag{3}$$

with $k = \omega/c$ the acoustic wavenumber and c the speed of sound.
The acoustic boundary conditions for the rigid back-cavity panels are:

$$\underline{\mathbf{r}} \in \Gamma_0 : \frac{j}{\rho_0 \omega}\frac{\partial p(\underline{\mathbf{r}})}{\partial n} = 0, \tag{4}$$

with ρ_0 the ambient fluid density and $\partial/\partial n$ the derivative in the normal direction.
To ensure the normal velocity continuity along the fluid-plate coupling interface Γ_s, the following relationships must apply at the interface:

$$\underline{\mathbf{r}}_s \in \Omega_s : \frac{j}{\rho_0 \omega}\frac{\partial p(\underline{\mathbf{r}}_s)}{\partial n} = j\omega w(\underline{\mathbf{r}}_s) \tag{5}$$

The Sommerfeld radiation condition ensures that no reflections occur at infinity,

$$\lim_{|\underline{\mathbf{r}}| \rightarrow \infty} (|\underline{\mathbf{r}}|\frac{\partial p(\underline{\mathbf{r}})}{\partial |\underline{\mathbf{r}}|} + jkp(\underline{\mathbf{r}})) = 0 \tag{6}$$

3 The Wave Based Method

3.1 Acoustic pressure approximation

As is done in the IE method, the acoustic domain is divided into two regions (see figure 2): an unbounded region outside a circular truncation boundary surface (subdomain 12) and a bounded region inside the boundary surface (subdomains 1 - 11). A sufficient condition for the WBM to converge towards the exact solution is convexity of the considered problem domains [5]. Non-convex domains have to be partitioned into a number of convex subdomains, applying continuity conditions at the induced subdomain interfaces [10]. The non-convex region inside the circular boundary surface of the considered loudspeaker problem is therefore partitioned into 11 convex subdomains (see figure 2).

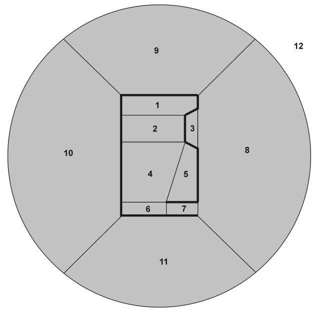

Figure 2: Domain decomposition

The steady-state pressure fields $p_i(\mathbf{r})$ $(i = 1...11)$ in the 11 *bounded* acoustic subdomains are approximated as solution expansions $\widehat{p}_i(\underline{\mathbf{r}})$,

$$p_i(\underline{\mathbf{r}}) \simeq \widehat{p}_i(\underline{\mathbf{r}}) = \sum_{a=1}^{n_{a_i}} p_{a_i} \Phi_{a_i}(\underline{\mathbf{r}}) \tag{7}$$

Each function $\Phi_{a_i}(\underline{\mathbf{r}})$ is an acoustic wave function, which satisfies the homogeneous Helmholtz equation:

$$\Phi_{a_i}(\underline{\mathbf{r}}) = \begin{cases} \Phi_{a_{i_r}}(x,y) = cos(k_{xa_{i_r}} x) e^{-jk_{ya_{i_r}} y} \\ \Phi_{a_{i_s}}(x,y) = e^{-jk_{xa_{i_s}} x} cos(k_{ya_{i_s}} y) \end{cases} \tag{8}$$

Since the only requirement for k_{xa_i} and k_{ya_i} is that $k_{xa_i}^2 + k_{ya_i}^2 = k_i^2$ with $k_i = \omega / c_i$, an infinite number of wave functions (8) can be found for expansion (7). It is proposed to select the following wavenumber components,

$$(k_{xa_i}, k_{ya_i}) = \left[\left(\frac{n_{a1_i}\pi}{L_{x_i}}, \pm\sqrt{k_i^2 - \left(\frac{n_{a1_i}\pi}{L_{x_i}}\right)^2} \right), \left(\pm\sqrt{k_i^2 - \left(\frac{n_{a2_i}\pi}{L_{y_i}}\right)^2}, \frac{n_{a2_i}\pi}{L_{y_i}} \right) \right], \tag{9}$$

with n_{a1_i}, $n_{a2_i} = 0, 1, 2, ...$ The dimensions L_{x_i} and L_{y_i} in (9) represent the dimensions of the (smallest) rectangular domain, enclosing the considered subdomain. The wave function contributions p_{a_i} in (7) are the unknowns.

In the *unbounded* domain outside the circular boundary surface Γ_R, the steady-state pressure $p(\underline{\mathbf{r}})$ is approximated as a solution expansion of wave functions that are solutions of the Helmholtz equation (3) and that satisfy the Sommerfeld radiation condition (6). It has been proven in [9] that, for a 2D acoustic domain, exterior to a circular boundary surface with radius R, the expansion

$$p(r,\theta) \simeq \widehat{p}(r,\theta) = p_{c0} H_0^{(2)}(kr) + \sum_{n=1}^{N} p_{cn} H_n^{(2)}(kr) cos(n\theta) + p_{sn} H_n^{(2)}(kr) sin(n\theta) \tag{10}$$

converges for $N \rightarrow \infty$. $H_n^{(2)}(*)$ is the n-th order Hankel function of the second kind. The contributions p_{c0}, p_{cn} and p_{sn} are the unknowns.

3.2 Structural displacement approximation

Based on the above pressure approximations, the steady-state normal displacement w is approximated as a solution expansion $\widehat{w}$

$$w(x_s) \simeq \widehat{w}(x_s) = \sum_{s=1}^{4} w_s \Psi_s(x_s) + \widehat{w}_F(x_s) - \sum_{a=1}^{n_{a_3}} p_{a_3} \frac{\Phi_{a_3}(\underline{\mathbf{r}}_s)}{D(k_{xa_3}^4 - k_b^4)} + \sum_{a=1}^{n_{a_2}} p_{a_2} \frac{\Phi_{a_2}(\underline{\mathbf{r}}_s)}{D(k_{xa_2}^4 - k_b^4)} \tag{11}$$

The four structural wave functions Ψ_s are four linearly independent solutions of the homogeneous part of the fourth-order dynamic plate equation (1),

$$\Psi_s(x_s) = e^{-j^s k_b x_s} \ , \ (s = 1...4) \tag{12}$$

and $\widehat{w}_F$ is the normal displacement of an infinite plate, excited by a normal line force F,

$$\widehat{w}_F(x_s) = \tfrac{-jF}{4Dk_b^3}(e^{-jk_b|x_s - x_F|} - je^{-k_b|x_s - x_F|}) \tag{13}$$

The structural wave function contributions w_s are the structural unknowns.

3.3 Coupled vibro-acoustic wave model

By using the proposed pressure and displacement expansions (7), (10) and (11), the dynamic plate equation (1), the Helmholtz equation (3) and the Sommerfeld radiation condition (6) are always exactly satisfied, irrespective of the values of the unknown wave function contributions w_s $(s = 1...4)$, p_{a_i} $(i = 1...11)$, p_{c0}, p_{cn} and p_{sn}. These contributions are merely determined by the structural and acoustic boundary conditions.
For a structural domain in a 2D problem, structural boundary conditions are specified at discrete edge locations. The boundary conditions (2) can be imposed exactly using expansion (11). This leads to 4 algebraic equations in $(4 + \sum_i^{11} n_{a_i} + 2N + 1)$ unknowns. Due to the introduction of 11 subdomains, continuity conditions along the subdomain interfaces must be taken into account, in addition to the problem boundary conditions (4) and (5). Since both the acoustic boundary conditions and the continuity conditions are defined at an infinite number of boundary positions, while only finite sized prediction models are amenable to numerical implementation, the acoustic boundary conditions are transformed into a weighted residual formulation yielding a set of $(\sum_i^{11} n_{a_i} + 2N + 1)$ algebraic equations in the $(4 + \sum_i^{11} n_{a_i} + 2N + 1)$ unknown wave function contributions. The combination of the structural boundary conditions and the transformed acoustic boundary conditions for each of the acoustic subdomains yields a square matrix equation in the unknown wave function contributions w_s $(s = 1..4)$, p_{a_i} $(i = 1..10)$, p_{c0}, p_{cn} and p_{sn}. As for the BE method and in contrast with the FE method, the proposed technique yields a fully populated matrix, whose elements are complex and which cannot be decomposed into frequency-independent submatrices. The big advantage of the WBM is, however, that the system matrices are substantially smaller in comparison with the element based techniques. This property, combined with the fast convergence of the WBM, make it a less computationally demanding method for dynamic response calculations, and make it possible to tackle problems also in the mid-frequency range. The beneficial convergence characteristics of the WBM for radiation problems, in comparison with the IE and BE methods, are illustrated in the next section.

4 Numerical results

4.1 Validation example

In order to illustrate the high accuracy that can be obtained with the WBM, a 2D bass-reflex as shown in figure 3 is considered (dimensions in mm). A unit normal line force F is applied at the center of the loudspeaker membrane ($E = 70.10^9 \frac{N}{m^2}$, $\rho_s = 700 \frac{kg}{m^3}$, $\nu = 0.3$, $t = 3mm$). The membrane edges are clamped. The loudspeaker is surrounded with air ($c = 340 \frac{m}{s}$, $\rho_0 = 1.225 \frac{kg}{m^3}$).

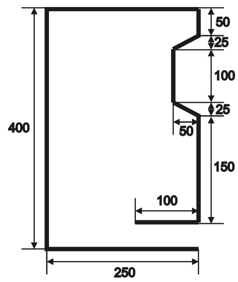

Figure 3: A 2D bass-reflex loudspeaker

Figures 4 and 5 show the calculated pressure field and the active intensity field at $120Hz$. These results are obtained with a wave model consisting of 495 wave functions with a truncation boundary surface with a radius of $0.5m$. The pressure contour plots show that the rigid boundary conditions are correctly taken into account by the WBM since the pressure contour lines are perpendicular to the rigid walls. Also, no pressure field discontinuities are observed, which indicates that the continuity conditions at the subdomain interfaces are correctly taken into account. Figure 5 shows that active intensity is flowing from both the membrane and the reflex-channel towards infinity which clearly illustrates the working of the bass-reflex channel. For the given loudspeaker dimensions, the considered frequency of $120Hz$ indeed corresponds to the reflex-frequency of the loudspeaker.

4.2 Comparison with element based techniques

To compare the performances of the WBM and the existing element based techniques, several coupled FE/indirect BE and FE/(8^{th} order conjugated) IE models of the considered problem have been solved using LMS/SYSNOISE Rev.5.5. The structural FE meshes consist of 2-noded plate elements, the acoustic BE meshes of 2-noded linear fluid elements and the acoustic FE meshes of 3- and 4-noded linear fluid elements.
Figure 6 plots the relative prediction errors for the radiated sound power W at $5000Hz$ against the CPU time needed for a direct response calculation at one frequency on a

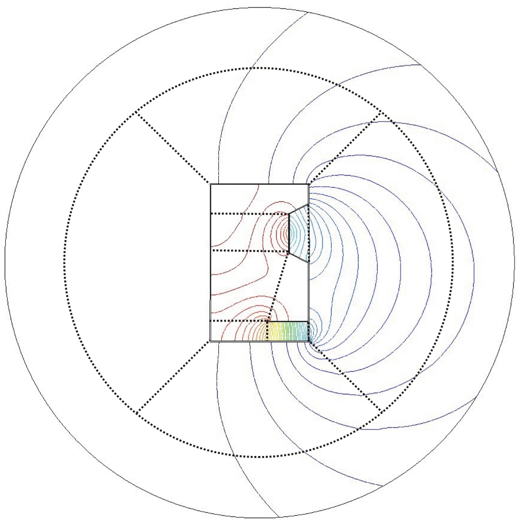

Figure 4: Contour plot of the pressure amplitudes at $120Hz$

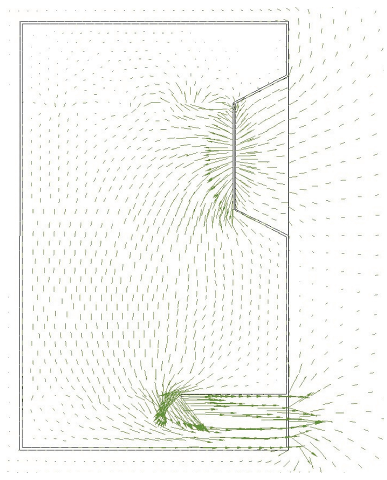

Figure 5: Vector plot of the active intensity at $120Hz$

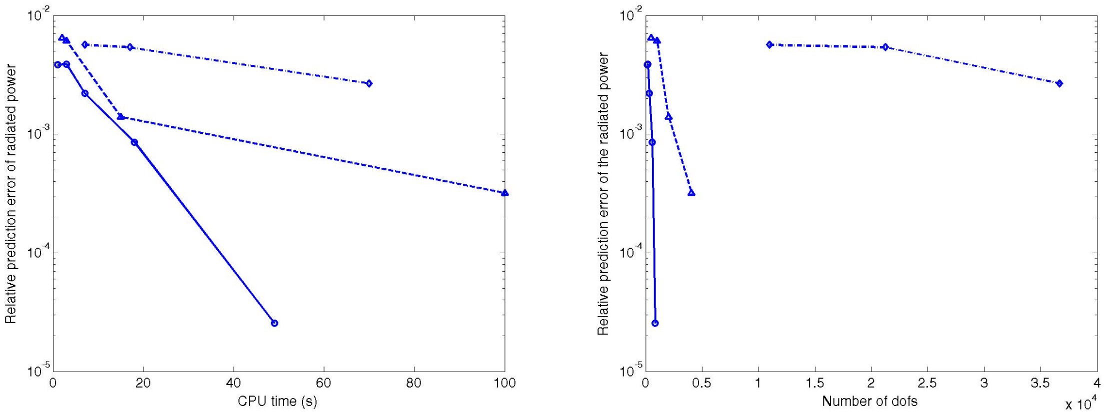

Figure 6: Convergence curves at $5000Hz$ in terms of CPU time and in terms of number of degrees of freedom,(solid (o): WBM, dashed ($\triangle$): FE/BE, dot-dashed ($\diamondsuit$): FE/IE)

Windows XP system (INTEL $1.8GHz, 1Gb$ RAM) and against the number of degrees of freedom to model the problem with the different methods. The indicated CPU times for the WBM and the FE/BE models include both the times for constructing the model as well as for solving the resulting matrix equation since the matrices are frequency dependent. This is in contrast with the frequency independent FE/conjugated IE models where only the solution time is taken into account. Both in terms of the CPU time and in terms of the number of degrees of freedom, the WBM has a beneficial convergence rate, compared with the element based techniques, which confirms the findings of [5],[7] and [8].

5 Conclusions

This paper applies a novel wave based method for the steady-state dynamic analysis of structural-acoustic problems with unbounded fluid domains. It is illustrated through a bass-reflex loudspeaker example that the method converges towards the exact solution. A comparison with corresponding FE/indirect BE and FE/conjugated IE models indicates the enhanced convergence rate of the new method. In this way, the proposed technique offers an adequate way to comply with the current challenge in structural-acoustic modelling. Due to its beneficial convergence rate, the practical frequency threshold may become substantially higher than for the element based methods, resulting in a significant narrowing of the currently existing mid-frequency twilight zone.

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