

## Density results using Stokeslets and the method of fundamental solutions applied to fluid flow

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**Abstract:** *The method of fundamental solutions has been used to solve fluid flow problems by reduction to Laplace problems. In this work we propose to use the fundamental solution of the Stokes system, using the so-called Stokeslets, to solve problems with conservative and non-conservative forces (homogeneous and nonhomogeneous Stokes system). In the nonhomogeneous case, we will also consider fundamental solutions of eigenvalue equations associated to the Stokes operator. We establish new density results in terms of fundamental solutions for the functional spaces used in the Stokes equations, by extending some density results recently obtained by Alves and Chen. Such density results are used to choose suitable basis functions for the method of fundamental solutions to approach the solution of the boundary value problem for the Stokes equations. We also show the convergence of this MFS based on the density results. Numerical simulations will be presented.*

**Keywords:** Stokes equations, meshless method, fundamental solutions.

### 1 Introduction

The numerical solution of the Stokes equations

$$\begin{cases} \nu \Delta \mathbf{u} - \nabla p = \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma. \end{cases} \quad (1)$$

has been addressed with classical numerical methods such as finite differences, finite elements or boundary elements. More recently, with the development of meshless methods some other simpler and fast methods have appeared in the literature. In particular, the Method of Fundamental Solutions (MFS) has been applied to the two-dimensional Stokes equations by reformulating the system as a scalar equation involving the biharmonic operator (e.g. [6], [7]). The resulting equation is then solved by the MFS for the biharmonic

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equation or by the MFS for a pair of Laplace equations.

We recall that in the two-dimensional case, the Stokes system can be reduced to a scalar biharmonic equation, by using a stream function  $\psi$  such that  $\mathbf{u} = \mathbf{curl}(\psi) = (\partial_2\psi, -\partial_1\psi)$ , and

$$\begin{cases} \nu\Delta^2\psi = \mathbf{curl}(\mathbf{f}), & \text{in } \Omega \\ \psi = \xi, & \text{on } \Gamma \\ \partial_n\psi = -\mathbf{g} \cdot \boldsymbol{\tau}, & \text{on } \Gamma \end{cases} \quad (2)$$

where  $\mathbf{curl}(\mathbf{f}) = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$  and the auxiliary function  $\xi$  is calculated from  $\partial_\tau\xi = \mathbf{g} \cdot \mathbf{n}$  on  $\Gamma$ , and  $\xi(x_0) = 0$  for some  $x_0 \in \Gamma$ . Although it is possible to apply the classical methods of fundamental solutions to (2), the calculation of the auxiliary function  $\xi$  requires an integration on the boundary. In the three-dimensional case, it is still possible to use a similar formulation, now in a vectorial form, but a divergence free condition has to be included

$$\begin{cases} \nu\Delta^2\Psi = \mathbf{curl}(\mathbf{f}), & \text{in } \Omega \\ \nabla \cdot \Psi = 0, & \text{in } \Omega \end{cases}$$

The corresponding boundary conditions are not trivial, consisting on  $\mathbf{curl}(\Psi) \times \mathbf{n} = \mathbf{g} \times \mathbf{n}$  together with conditions involving the surface gradient and the surface curl (for further details, see e.g. [5]).

Here, we present an alternative approach for the numerical treatment of the Stokes equations, keeping the meshless feature of the MFS and using the fundamental solution of the Stokes equations, for both two and three-dimensional cases.

## 2 Density results and the method of fundamental solutions

We recall that the fundamental solution of the Stokes equations is given by

$$U_{ij}(x) = \begin{cases} \frac{1}{4\pi\nu} \left( \delta_{ij} \log \frac{1}{|x|} + \frac{x_i x_j}{|x|^2} \right), & \text{if } d = 2 \\ \frac{1}{8\pi\nu} \left( \delta_{ij} \frac{1}{|x|} + \frac{x_i x_j}{|x|^3} \right), & \text{if } d = 3 \end{cases}, \quad (i, j = 1, \dots, d).$$

$$P_i(x) = \begin{cases} \frac{1}{2\pi} \frac{x_i}{|x|^2}, & \text{if } n = 2 \\ \frac{1}{4\pi} \frac{x_i}{|x|^3}, & \text{if } n = 3 \end{cases}, \quad (i = 1, \dots, d).$$

and verifies

$$\begin{cases} \nu\Delta\mathbf{U}_{(j)} - \nabla P_j = -\delta\mathbf{e}_j & \text{in } \mathbb{R}^d \\ \nabla \cdot \mathbf{U}_{(j)} = 0 & \text{in } \mathbb{R}^d \end{cases}$$

where  $\delta\mathbf{e}_j$  stands for the Dirac delta distribution in the  $\mathbf{e}_j$  direction, and  $\mathbf{U}_{(j)} = \mathbf{U} \cdot \mathbf{e}_j$ .

We will consider the MFS using directly the fundamental solution of the Stokes equations and we justify this approach by presenting some new density results. These density results, are similar to those in [1], and may be extended to domain functions as in [2]. In a recent work in collaboration with C.J.S. Alves, concerning the numerical solution of the Stokes problem for conservative forces (cf.[3])

$$\begin{cases} \nu \Delta \mathbf{u} - \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma, \end{cases}$$

we have shown the following result

**Theorem 1** Let  $\Omega, \hat{\Omega}$  be connected bounded domains with regular boundaries  $\Gamma = \partial\Omega$  and  $\hat{\Gamma} = \partial\hat{\Omega}$ , such that  $\Omega \subset \hat{\Omega}$ . Then, for  $d = 3$ ,

$$\mathbf{S}_I(\Gamma, \hat{\Gamma}) = \text{span} \{ \mathbf{U}_{(k)}(x-y)|_{x \in \Gamma} : y \in \hat{\Gamma}, k = 1, \dots, d \}$$

is dense in

$$H_{\mathbf{n}}^{1/2}(\Gamma) = \left\{ \mathbf{v} \in (H^{1/2}(\Gamma))^d : \int_{\Gamma} \mathbf{v} \cdot \mathbf{n} = 0 \right\}.$$

In the 2D case, the density also holds considering  $\mathbf{S}_I(\Gamma, \hat{\Gamma}) + \mathbb{R}^2$ .

The previous density results will be used to choose basis functions in the Method of Fundamental Solutions for solving nonhomogeneous boundary value problems for the Stokes equations. Essentially, the MFS for the boundary value problem (1) in a domain  $\Omega$  consists in (i) taking a domain  $\hat{\Omega}$  with boundary  $\hat{\Gamma}$  located outside the domain  $\Omega$ , and then (ii) a finite number of point is choosen in  $\hat{\Gamma}$  for approaching the boundary data by a finite linear combination of fundamental solutions associated with these points. Since the singularities of the fundamental solutions are located outside  $\Omega$ , the previous linear combination of fundamental solutions will solve exactly the partial differential equation in  $\Omega$ . It is important to emphasize that this approach is applied directly to the Stokes equations, without reducing them to another type of problem.

The convergence of the MFS for the Stokes equations can be partially justified by density results as it has been shown for the Laplace equation in [4] using a standard  $\hat{\Gamma}$  set and functions defined by single layer potentials, or in [1], for the Helmholtz equation, using only the fundamental solutions and more general type of source sets  $\hat{\Gamma}$ .

We show examples in  $\mathbb{R}^2$  that validate this MFS method in a simple (but non trivial) bounded domain.

The case of non-conservative forces can be solve in two steps: first, we solve the problem

$$\begin{cases} \nu \Delta \mathbf{u}_1 - \nabla p_1 = f & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_1 = 0 & \text{in } \Omega \end{cases} \quad (3)$$

without imposing boundary conditions, and then, we solve the nonhomogeneous boundary value problem

$$\begin{cases} \nu \Delta \mathbf{u}_2 - \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u}_2 = 0 & \text{in } \Omega \\ \mathbf{u}_2 = \mathbf{g} - \mathbf{u}_1 & \text{in } \Gamma \end{cases}$$

using the MFS described above.

Problem (3) will also be solved by a meshless method, based on the following density result

**Theorem 2** Let  $\Omega, \hat{\Omega}$  be connected bounded domains with regular boundaries  $\Gamma = \partial\Omega$  and  $\hat{\Gamma} = \partial\hat{\Omega}$ , such that  $\Omega \subset \hat{\Omega}$ . Then, for  $d = 2, 3$ ,

$$\mathbf{S}(\Omega, \hat{\Gamma}) = \text{span} \left\{ \mathbf{U}_{(k)}^\lambda (x - y)|_{x \in \Omega} : y \in \hat{\Gamma}, k = 1, \dots, d, \lambda \in [a, b] \right\}$$

with  $(\mathbf{U}_{(k)}^\lambda, \mathbf{P}_k^\lambda)$  solution of the problem

$$\begin{cases} \nu \Delta \mathbf{U}_{(k)}^\lambda - \nabla P_k^\lambda = \lambda \mathbf{U}_{(k)}^\lambda - \delta \mathbf{e}_k & \text{in } \mathbb{R}^d \\ \nabla \cdot \mathbf{U}_{(k)}^\lambda = 0 & \text{in } \mathbb{R}^d, \end{cases}$$

is dense in  $L^2(\Omega)$ .

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