

# Numerical simulation of acoustic wave scattering using a meshfree plane waves method

Carlos J. S. Alves<sup>(1)</sup> and Svilen S. Valtchev<sup>(2)</sup>

**Abstract:** *Density results using an infinite number of plane acoustic waves allow to derive meshless methods for solving the homogeneous or the nonhomogeneous Helmholtz equation. In this work we consider the numerical simulation of acoustic source problems in a bounded domain using this method. We present several tests comparing with the method of fundamental solutions and a recent extension to nonhomogeneous problems.*

*Keywords:* method of fundamental solutions, plane waves, acoustic scattering

## 1 Introduction

In this work we present a meshfree method of the Trefftz type, based on density results for plane wave functions, to approximate the solution of a Helmholtz equation in a bounded connected domain  $\Omega \subset \mathbb{R}^N$ . For simplicity, we will consider Dirichlet conditions on the regular boundary  $\partial\Omega$ ,

$$\begin{cases} (\Delta - \mu)u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (1)$$

It is well known that this problem is well posed with solution in  $H^1(\Omega)$  for  $g \in H^{1/2}(\partial\Omega)$  if  $\mu$  is not a Dirichlet eigenfrequency for the Laplace operator in the domain  $\Omega$  (e.g. [6]). In the following we will assume that  $\mu = -\kappa^2$  is not an eigenfrequency, where  $\kappa$  is known to be the wavenumber or frequency (considering unitary constant speed of the wave propagation).

We will consider both homogeneous ( $f = 0$ ) and non homogeneous ( $f \neq 0$ ) problems. In the case of the homogeneous problems, the application of the Method of Fundamental Solutions is straightforward and has been used widely (e.g. [7]). More recently, in [2], the Method of Fundamental Solutions has been extended for non homogeneous problems,

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<sup>1</sup>Departamento de Matemática e CEMAT, Instituto Superior Técnico, 1049-001 Lisboa, Portugal (calves@math.ist.utl.pt).

<sup>2</sup>CEMAT, Instituto Superior Técnico, 1049-001 Lisboa, Portugal (ssv@math.ist.utl.pt).

presenting good results compared to other meshfree methods (cf. [3]). The mathematical justification for these approaches can be made using density results (cf. [2, 4]).

Similar density results are established for plane waves, and allow us to present the *plane wave method*, a meshfree method of the Trefftz type, which shares features that are quite similar to a Fourier transform, due to the expression of the plane waves.

– In the homogeneous case, for a fixed  $\kappa$ , we search for solutions of the form

$$u_m(x) = \sum_{j=1}^m \alpha_j e^{i\kappa x \cdot d_j} \quad (2)$$

with unitary directions  $d_j \in S^{N-1}$ . Since each plane wave  $e^{i\kappa x \cdot d_j}$  is a solution of the Helmholtz equation in  $\Omega$ , we just have to fit the unknown coefficients  $\alpha_j$  to get  $u_m(x_i) = g(x_i)$  on boundary collocation points  $x_i \in \partial\Omega$ . This is partially justified by the following density result (e.g. [6])

**Theorem 1.**

$$L^2(\partial\Omega) = \overline{\text{span} \{e^{i\kappa x \cdot d} : d \in S^{N-1}\}},$$

if  $\kappa$  is not a resonance frequency for the bounded domain  $\Omega$ .  $\square$

– In the non homogeneous case, we will search for a particular solution  $u_P$  of the Helmholtz equation  $(\Delta + \kappa^2)u_P = f$  adapting an idea in [2]. From an approximation

$$\tilde{f} = \sum_{k=1}^p \sum_{j=1}^m a_{k,j} e^{i\omega_k x \cdot d_j} \quad (3)$$

the function

$$\tilde{u}_P = \sum_{k=1}^p \sum_{j=1}^m b_{k,j} e^{i\omega_k x \cdot d_j} \quad (4)$$

with the coefficients  $b_{k,j} = \frac{a_{k,j}}{\kappa^2 - \omega_k^2}$  verifies  $(\Delta + \kappa^2)\tilde{u}_P = \tilde{f}$ .

Therefore  $\tilde{u}_P$  is an approximation of the particular solution, noticing that  $\kappa$  must be different from the chosen frequencies  $\omega_k$ . In particular, this approach also works for the Poisson equation, taking  $\kappa = 0$ . The approximation can be made either for complex or real valued functions. In the later case, we may choose the real or the imaginary part of the plane wave. A similar observation, concerning the use of the nonsingular part of the Helmholtz fundamental solution, has been made in [5].

We recall the fundamental solution of the Helmholtz equation,

$$\Phi_\lambda(x) = \begin{cases} \frac{i}{4} H_0^{(1)}(\sqrt{-\lambda}|x|) & \text{in 2D} \\ \frac{e^{i\sqrt{-\lambda}|x|}}{4\pi|x|} & \text{in 3D} \end{cases}$$

where  $H_0^{(1)} := J_0 + iY_0$  is the first Hänkel function, defined with the Bessel functions of first and second kind,  $J_0$  and  $Y_0$ . The  $J_0$  function and  $\text{sinc}(r) = \sin(r)/r$  are non singular at the origin. Although this allows to avoid the singularity (of  $Y_0$  or  $\cos(r)/r$ ), the use of this approach produces similar results.

## 2 Density results

To justify the approximation of  $f$  by (3), we need the following density theorem that holds for bounded simply connected domains (cf. [1]):

**Theorem 2.** *Let  $I$  be an interval in  $] -\infty, 0]$ . Then*

$$L^2(\Omega) = \overline{\text{span}\{e^{\sqrt{\lambda}x \cdot d}|_{x \in \Omega} : d \in S^{N-1}, \lambda \in I\}}. \quad \square \quad (5)$$

To ensure that the matrices generated by the *plane waves method* have independent columns (for an appropriate choice of collocation points), we prove that the plane waves are linear independent functions on  $\partial\Omega$ .

**Theorem 3.** *Let  $d_1, \dots, d_m \in S^2$ . The plane wave functions  $e^{i\kappa x \cdot d_1}, \dots, e^{i\kappa x \cdot d_m}$ , restricted to  $x \in \partial\Omega$ , are linear independent functions.*

**Proof.** Suppose  $v(x) := \sum_{j=1}^m \alpha_j e^{i\kappa x \cdot d_j} = 0$ , ( $\forall x \in \partial\Omega$ ), then by the well-posedness of problem (1), for non resonance frequencies, we have  $v = 0$  in  $\Omega$ . By the unique analytic extension outside the domain,  $v \equiv 0$ , in  $\mathbb{R}^N$ . Considering the Fourier transform  $\mathcal{F}$ , we get

$$0 = v(x) = \int_{\mathbb{R}^N} e^{-ix \cdot y} \sum_{j=1}^m \alpha_j \delta(y + \kappa d_j) dy = \mathcal{F} \left( \sum_{j=1}^m \alpha_j \delta_{-\kappa d_j} \right) (x) \quad (6)$$

and the result follows from the linear independence of the Dirac delta distributions centered on the distinct points  $-\kappa d_1, \dots, -\kappa d_m$ .  $\square$

**Remark.** Obviously, by the well posedness of the problem in  $\Omega$  (for non resonance frequencies), the plane waves are also linear independent in  $\Omega$ . Using a recursive argument it is also possible to conclude the linear independence in  $\Omega$  considering plane waves with different frequencies.

## 3 Numerical remarks

Take  $n$  points  $x_i$  in an domain  $W \supset \Omega$ ,  $m$  unitary directions  $d_j$  and  $p$  values  $\lambda_k$ . The coefficients  $a_{i,j}$  in (3) will be obtained by solving a least-squares linear system of the form:

$$\mathbb{F}^\top \mathbb{F} \mathbf{a} = \mathbb{F}^\top \mathbf{f}$$

where  $\mathbf{a} = [a_{i,j}]_{mp \times 1}$ ,  $\mathbf{f} = [f(x_i)]_{n \times 1}$  and the matrix  $\mathbb{F}$  is given by:

$$\mathbb{F} = \left[ [e^{\sqrt{\lambda_k x_1} \cdot d_j}]_{1 \times (mp)}, \dots, [e^{\sqrt{\lambda_k x_m} \cdot d_j}]_{1 \times (mp)} \right]_{n \times (mp)}$$

The particular solution  $u_P$  of the initial problem is then easily obtained using equation (4).

Having obtained an approximation of  $u_P$  we approximate the solution  $u_H$  of the homogeneous Helmholtz problem:

$$\begin{cases} (\Delta - \mu)u_H = 0 & \text{in } \Omega \\ u_H = g - \tilde{u}_P & \text{on } \partial\Omega. \end{cases} \quad (7)$$

This can be done using standard methods or choosing between the method of fundamental solutions and the plane wave method. In this work we will compare both possibilities. In particular, for  $\mu < 0$ , we may use the plane wave method to obtain an approximation of the homogeneous solution of problem (7),

$$\tilde{u}_H(x) = \sum_{j=1}^m c_j e^{\sqrt{\mu}x \cdot d_j} \quad (8)$$

Take  $n_{\text{hom}}$  collocation points  $x_i$  on the boundary  $\partial\Omega$  and  $m_{\text{hom}}$  unitary directions  $d_j \in S^{N-1}$ . The coefficients  $c_j$  in (8) will be obtained by interpolation (if  $m = n$ ) or by least-squares (if  $n > m$ ) solving the linear system,  $\mathbb{U}^\top \mathbb{U} \mathbf{c} = \mathbb{U}^\top \mathbf{h}$  where  $\mathbf{c} = [c_j]_{m \times 1}$ ,  $\mathbf{h} = [g(x_i) - \tilde{u}_P(x_i)]_{n \times 1}$  and the matrix  $\mathbb{U}$  is given by:  $\mathbb{U} = [e^{\sqrt{\mu}x_i \cdot d_j}]_{n \times m}$ .

## 4 Numerical Simulations

In this section we consider two illustrative numerical examples. The first example considers the *plane waves method* applied to a homogeneous 3D Helmholtz equation. In the second example we consider the resolution of a nonhomogeneous 2D Helmholtz equation.

### 4.1 Helmholtz homogeneous equation in 3D

Consider a domain  $\Omega_1 \subset \mathbb{R}^3$  with boundary parametrization  $(t, s) \rightarrow 0.03(\sin(4s) + 6)\{(3 + \cos(t))\cos(s); (3 + \cos(t))\sin(s); \sin(t)\}$  (see Fig.1-left). We take as exact solution:

$$u_1(x) = (2 + i)\Phi_\mu(x - y_1) + 2\Phi_\mu(x - y_2) + (2 - i)\Phi_\mu(x - y_3)$$

with point sources  $y_i \in \{(2, 0, 3); (0, 0, -3); (-2, 0, 3)\}$  and  $\kappa = 5$  (thus  $\mu = -\kappa^2 = -25$ ). In Fig.1(center) we show the absolute errors obtained with the *plane waves method* on  $\partial\Omega_1$ . In the same figure, on the right, we also show the plot of the absolute error on an interior surface  $0.5\partial\Omega_1$ . It is clearly seen that the errors are small and get smaller inside the domain. We considered  $40 \times 20$  points ( $n_t \times n_s$ ) on the boundary of the domain and  $30 \times 15$  ( $m_t \times m_s$ ) directions on the unitary sphere.

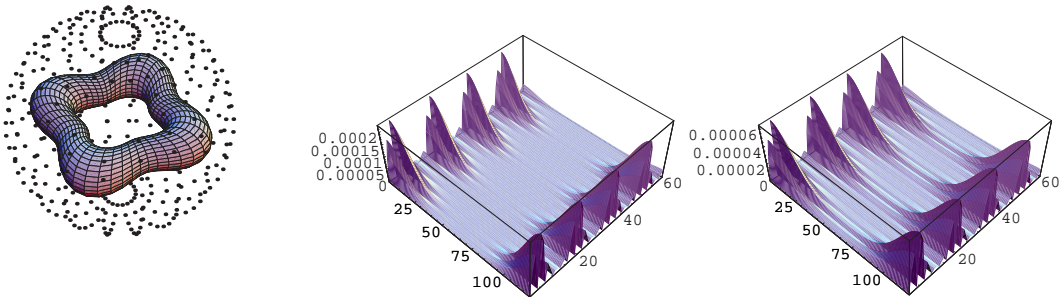


Figure 1: The domain  $\Omega_1$  and points representing the directions of the plane waves (left). Absolute error on the boundary (center) and in the interior, on the surface  $0.5\partial\Omega_1$  (right)

$n_t$	$n_s$	$m_t$	$m_s$	$\mu = -1$	$\mu = -25$	$\mu = -100$
40	20	30	15	$9.305E-7$	$2.220E-4$	$5.766E-1$
50	30	30	15	$7.136E-7$	$1.040E-6$	$1.552E-5$
50	30	40	20	$6.529E-7$	$1.377E-6$	$8.551E-4$
60	40	40	20	$6.703E-7$	$7.896E-8$	$5.931E-5$

Table 1: The absolute error on  $\partial\Omega_1$  for several tests.

In Table 1 we show some results testing several possibilities by changing the number of collocation points or the number of directions. It is rather difficult to analyze properly the behavior of the method. There exist particular combinations of collocation points and unitary directions for which the method shows better performance and more depth analysis must be carried.

## 4.2 Helmholtz nonhomogeneous equation

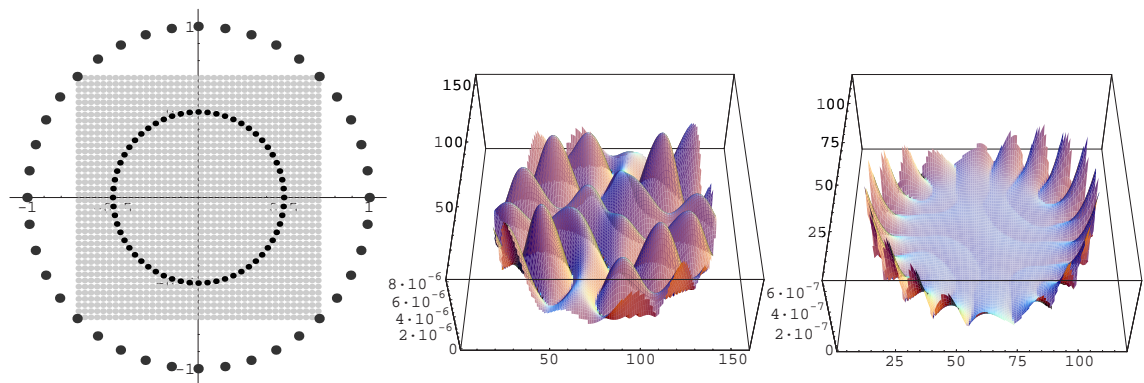
We now consider a 2D example, considering a disk  $\Omega_2 = B(0, 0.5)$ . The solution of (1) is given by  $u_2(x, y) = \cos(x^2 - y^2)$  therefore the boundary data is given by the function  $g_2 = u_2|_{\partial\Omega_2}$  and  $f_2(x, y) = -4(x^2 + y^2) \cos(x^2 - y^2) - \mu u_2(x, y)$ .

Since we have a nonhomogeneous Helmholtz equation, we must proceed in two steps.

(i) First, we will approximate  $f_2$  considering  $n$  collocation points on  $W \supset \Omega_2$ ,  $p$  frequencies  $\lambda_k = -\omega_k^2$  and  $m$  unitary directions on  $S^1$ .

(ii) Second, we will approximate  $h = g_2 - u_P$  using  $n_{\text{hom}}$  points on the boundary  $\partial\Omega_2$ , and  $m_{\text{hom}}$  unitary directions on  $S^1$ .

In Fig.2 we show some results for  $\mu = -11$ . We considered  $n = 40^2$  points on the domain  $W = [-0.7, 0.7]^2 \supset \Omega_2$ ,  $m = 20$  unitary directions,  $n_{\text{hom}} = 60$  and  $m_{\text{hom}} = 40$  and  $p = 12$  frequencies. The absolute error is of magnitude  $10^{-6}$ .

Figure 2: Domain  $\Omega_2$  and collocation points (left). Absolute error for  $f_2$  (center) and absolute error for  $u_2$  (left).

Finally we present some numerical results in order to compare the behavior of the *plane wave method* (PWM) and the *method of fundamental solutions* (MFS) for the current example. On Table 2 we present the maximum absolute errors of the approximated  $f_2$  and  $u_2$  by each of the methods for some different values of  $\mu$ . From the Table 2 we conclude that there is no significant difference between using the MFS or the PWM, with small advantage for the PWM in the cases shown.

$\mu$	PWM $\rightarrow f$	PWM $\rightarrow u$	MFS $\rightarrow f$	MFS $\rightarrow u$
$\mu = -5$	$1.631E-6$	$1.555E-6$	$1.297E-6$	$5.446E-5$
$\mu = -10.5$	$3.277E-6$	$1.967E-7$	$4.505E-6$	$1.761E-5$
$\mu = -60$	$5.616E-5$	$1.080E-5$	$4.164E-5$	$1.398E-6$
$\mu = -110$	$4.887E-5$	$1.481E-6$	$6.939E-5$	$1.215E-6$

Table 2: Absolute error analysis on the domain  $\Omega_2$  for the MFS and PWM

Although the accuracy of the *plane waves method* is in general similar to the method of fundamental solutions, the application of this method is restricted to bounded simply connected domains, and it does not present the versatility on the choice of source points. The *plane waves method* presents an advantage in 2D, since the plane waves are much faster to calculate than the Bessel functions. In 3D a better choice of the directions (for instance, using Gauss points in the sphere) may lead to even better results.

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