Pseudo-divergence-free element free Galerkin method for incompressible fluid flow

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Abstract: *Incompressible modelling in finite elements has been a major concern since its early developments and has been extensively studied. However, incompressibility in mesh-free methods is still an open topic. Thus, instabilities or locking can preclude the use of mesh-free approximations in such problems. Here, a novel mesh-free formulation is proposed for incompressible flow. It is based on defining a pseudo-divergence-free interpolation space. That is, the finite dimensional interpolation space approaches a divergence-free space when the discretization is refined. Note that such an interpolation does not include any overhead in the computations. The numerical evaluations are performed using the inf-sup numerical test and two well-known benchmark examples for Stokes flow.*

Keywords: Locking, Element Free Galerkin, Diffuse derivatives, Moving Least Squares, Incompressible flow, LBB condition.

1 Introduction

Accurate and efficient modelling of incompressible flows is an important issue in finite elements. Various formulations have been proposed in the literature to deal with incompressible flow problems (1; 2). Mixed finite elements present numerical difficulties caused by the saddle-point nature of the resulting variational problem. Solvability of the problem depends on a proper choice of finite element spaces for velocity and pressure. They must satisfy the so-called LBB (or inf-sup) condition which is not trivial to verify analytically for a given interpolation. This has spurred the use of numerical inf-sup testing (3; 4; 5)

Incompressibility in mesh-free methods is still an open topic. Until now the remedies proposed in the literature are extensions of the methods developed for finite elements.

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Here a novel approach is explored: the pseudo-divergence-free (PDF) EFG method. It consists in using interpolation functions that verify approximately the divergence-free constraint for a given discretization, and asymptotically become divergence-free as the discretization is refined. This method is based on diffuse derivatives (6). One of the key advantages of this approach is that the PDF interpolation functions are computed *a priori* so there is no extra computational cost.

Preliminary results in incompressible elasticity (8; 9) were encouraging. In this paper convergence of the approximation in incompressible flows is studied.

2 Preliminaries of the EFG method

There are well known references (10) of mesh-free methods. Here only some notions will be recalled in order to introduce the notation employed in following sections.

The moving least squares approach is based on the local approximation (i.e., at any point z in the neighborhood of x) of the unknown scalar function $u(z)$ by u^ρ as

$$u(z) \simeq u^\rho(x,z) = \mathbf{P}^\mathsf{T}(z)\,\mathbf{a}(x) \qquad \text{for } z \text{ near } x, \tag{1}$$

where the coefficients $\mathbf{a}(x) = \{\mathrm{a}_0(x), \mathrm{a}_1(x), \dots, \mathrm{a}_l(x)\}^\mathsf{T}$ are not constant, they depend on point x, and $\mathbf{P}(z) = \{p_0(z), p_1(z), \dots, p_l(z)\}^\mathsf{T}$ includes a complete basis of the subspace of polynomials of degree m. The coefficients $\mathbf{a}$ are obtained by minimization of the functional $J_x(\mathbf{a})$ centered in x and defined by

$$J_x(\mathbf{a}) = \sum_{i \in I_x} \phi(x,x_i)\left[u(x_i) - \mathbf{P}(x_i)\,\mathbf{a}(x)\right]^2, \tag{2}$$

where $\phi(x,x_i)$ is a weighting function that characterizes the mesh-free method. The particles cover the computational domain Ω, $\Omega \subset \mathbb{R}^{\mathrm{n_{sd}}}$, and, in particular, a number of particles $\{x_i\}_{i \in I_x}$ belong to the support of $\phi(x,x_i)$. The minimization of $J_x(\mathbf{a})$ induces the standard normal equations in a weighted least-squares problem

$$\mathbf{M}(x)\,\mathbf{a}(x) = \sum_{i \in I_x} \phi(x,x_i)\,u(x_i)\,\mathbf{P}(x_i) \tag{3}$$

where the Gram matrix $\mathbf{M}(x)$ is the scalar product of the interpolation polynomials.

Since the weighting function ϕ usually favors the central point x, it seems reasonable to assume that such an approximation is more accurat at $z = x$:

$$u(x) \simeq u^\rho(x) = \mathbf{P}^\mathsf{T}(x)\,\mathbf{a}(x) = \mathbf{P}^\mathsf{T}(x)\,\mathbf{M}^{-1}(x) \sum_{i \in I_x} \phi(x,x_i)\,u(x_i)\,\mathbf{P}(x_i). \tag{4}$$

3 The diffuse derivative

The approximation of the derivative of u in each spatial direction is the corresponding derivative of u^ρ. This requires to derive (4), that is

$$\frac{\partial u}{\partial x} \simeq \frac{\partial u^\rho}{\partial x} = \frac{\partial \mathbf{P}^\mathsf{T}}{\partial x}\,\mathbf{a} + \mathbf{P}^\mathsf{T}\,\frac{\partial \mathbf{a}}{\partial x}. \tag{5}$$

On one hand, derivatives of the coefficients **a** require the knowledge of the cloud of particles surrounding each point x. On the other hand, the first term of the r.h.s of (5) is easily evaluated. The derivative of the polynomials in **P** is trivial and can be evaluated *a priori*. The concept of diffuse derivative proposed by (7) consist in approximating the derivative by this first term. (7) shows that the diffuse derivative converges at optimal rate to the derivative of u. In (8) the proof is developed in several spatial dimensions.

4 Diffuse divergence

Incompressible computations require a divergence-free approximating field. That is, the solution $u(x)$, now a vector $u : \mathbb{R}^{\mathtt{n_{sd}}} \to \mathbb{R}^{\mathtt{n_{sd}}}$, verifies $\nabla \cdot u = 0$, and the approximation $u^{\rho}(x)$ should also be divergence-free. Here, instead of imposing a divergence-free interpolation, the diffuse divergence of the approximation, u^{ρ}, is imposed equal to zero. In (8; 9) details can be found.

5 Stationary Stokes problem

The model problem, steady Stokes flow, is used to analyze the performance of the PDF EFG formulation. It is well-known that continuous and discrete spaces for Stokes equations are subject to an inf-sup condition (1). This stability requirement is evidenced in practical computations by the existence of spurious pressure modes. The pseudo-divergence-free velocity field and the pressure field employed should comply asymptotically with the LBB condition.

5.1 Statement of the problem

Let Ω denote an open bounded region of $\mathbb{R}^2$ with boundary $\partial\Omega$. The 2D Stokes problem in Ω seeks a velocity field $u = (u_1, u_2)$ and a pressure field p such that:

$$\begin{cases} -\nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where ν is the viscosity of the fluid and f is the body force. Only Dirichlet boundary conditions are considered here but imposed tractions could also be considered.

5.2 Weak form

Given the problem defined in (6) and taking $g = \mathbf{0}$ the weak form of the Stokes problem is: find $u \in \mathcal{V}$ and $p \in \mathcal{Q}$, where $\mathcal{V} := [\mathcal{H}^1(\Omega)]^2$ and $\mathcal{Q} := \mathcal{L}_2(\Omega)$, such that

$$a(u,v) + b(v,p) + b(u,q) = (f,v) \qquad \forall (v,q) \in \mathcal{V} \times \mathcal{Q}$$

and $u = g$ on $\partial\Omega$, where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined as

$$a(u,v) := \int_\Omega \nabla v : \nu \nabla u \, d\Omega = \nu(\nabla u, \nabla v), \quad \text{and}$$

$$b(v,p) := -\int_\Omega p \nabla \cdot v \, d\Omega \; = -(p, \nabla \cdot v).$$

In order to impose the Dirichlet boundary conditions Nitsche's method (11; 12) is used because it produces reasonable results in mesh-free methods and is more stable than Lagrange multipliers or penalty methods without introducing another discretization on the boundary (13). Under this circumstances, the weak form becomes

$$\begin{aligned}
&a(u,v)+b(v,p)+b(u,q)\\
&-(\nu\partial_n u-pn,v)_{\partial\Omega}-(u,\nu\partial_n v-qn)_{\partial\Omega}+\nu\hat{\gamma}(u,v)_{\partial\Omega}\\
&\qquad\qquad\qquad\qquad=(f,v)-(g,\nu\partial_n v-qn)_{\partial\Omega}+\nu\hat{\gamma}(g,v)_{\partial\Omega},
\end{aligned}\tag{7}$$

where $\hat{\gamma}$ is not a penalty parameter, but must be large enough to ensure stability.

Let $\mathcal{V}_\rho$ and $\mathcal{Q}_\rho$ denote finite dimensional subspaces of $\mathcal{V}$ and $\mathcal{Q}$ respectively. The index ρ refers to a characteristic measure of the support of the interpolation functions. It is related to the characteristic measure between particles, h (recall, ρ/h is assumed constant). The discrete version of the problem reads: find $u^\rho\in\mathcal{V}_\rho$ and $p^\rho\in\mathcal{Q}_\rho$ such that, $\forall(v^\rho,q^\rho)\in\mathcal{V}_\rho\times\mathcal{Q}_\rho$,

$$\begin{aligned}
&a(u^\rho,v^\rho)+b(v^\rho,p^\rho)+b(u^\rho,q^\rho)\\
&-(\nu\partial_n u^\rho-p^\rho n,v^\rho)_{\partial\Omega}-(u^\rho,\nu\partial_n v^\rho-q^\rho n)_{\partial\Omega}+\nu\frac{\gamma}{\rho}(u^\rho,v^\rho)_{\partial\Omega}\\
&\qquad\qquad\qquad\qquad=(f,v^\rho)-(g,\nu\partial_n v^\rho-q^\rho n)_{\partial\Omega}+\nu\frac{\gamma}{\rho}(g,v^\rho)_{\partial\Omega}.
\end{aligned}$$

Now, $(\cdot,\cdot)_{\partial\Omega}$ denotes the $\mathcal{L}_2(\partial\Omega)$-scalar product. Note that the scalar $\hat{\gamma}$ is replaced by $\hat{\gamma}=\gamma/\rho$ because the threshold of this parameter is inversely proportional to ρ. To guarantee that γ is large enough an eigenvalue problem is solved.

5.3 The inf-sup condition

The previous problem, see (7), can be written in the following form: find $(u,p)\in\mathcal{V}\times\mathcal{Q}$ such that

$$\begin{cases} A(u,v)+B(v,p)=F(v) & \forall v\in\mathcal{V},\\ \qquad\qquad\; B(u,q)=G(q) & \forall q\in\mathcal{Q},\end{cases}\tag{8}$$

where the new forms are defined as

$$\begin{aligned}
A(u,v)&:=a(u,v)-(\nu\partial_n u,v)_{\partial\Omega}-(u,\nu\partial_n v)_{\partial\Omega}+\nu\hat{\gamma}(u,v)_{\partial\Omega},\\
B(v,p)&:=b(v,p)+(pn,v)_{\partial\Omega},\\
F(v)&:=(f,v)-(g,\nu\partial_n v)_{\partial\Omega}+\nu\hat{\gamma}(g,v)_{\partial\Omega},\\
G(q)&:=(g,qn)_{\partial\Omega}.
\end{aligned}$$

(3) show that the variational problem is well-posed if:

i) $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are continuous, i.e.,

$$\begin{aligned}
&\exists M \text{ such that } \forall u\in\mathcal{V},\ \forall v\in\mathcal{V} &&A(u,v)\le M\|u\|\|v\|,\\
&\exists M \text{ such that } \forall u\in\mathcal{V},\ \forall q\in\mathcal{Q} &&B(v,q)\le M\|v\|\|q\|.
\end{aligned}$$

ii) $A(\cdot,\cdot)$ is coercive, i.e.,

$$\exists\alpha \text{ such that } \forall v \in \mathcal{V}\ A(v,v) \geq \alpha \|v\|^2.$$

iii) $B(\cdot,\cdot)$ satisfies:

$$\inf_{q\in\mathcal{Q}} \sup_{v\in\mathcal{V}} \frac{B(v,q)}{\|v\|\,\|q\|} \geq \beta > 0.$$

Here consideration is given to the discrete problem arising when (8) is discretized using $\mathcal{V}_\rho$ and $\mathcal{Q}_\rho$ finite dimensional subspaces of $\mathcal{V}$ and $\mathcal{Q}$, respectively. Conditions *i*) and *ii*) carry over to the discrete model and condition *iii*) becomes:

$$\inf_{q^\rho\in\mathcal{Q}_\rho} \sup_{v^\rho\in\mathcal{V}_\rho} \frac{B(v^\rho,q^\rho)}{\|v^\rho\|\,\|q^\rho\|} =: k_\rho > 0. \tag{9}$$

The important stability condition to the convergence of the discrete model is:

$$\lim_{\rho\to 0} k_\rho \geq \beta > 0.$$

To verify analytically (9) for a given pair $(\mathcal{Q}_\rho, \mathcal{V}_\rho)$ is not trivial. For this reason the numerical inf-sup test is used.

5.4 The numerical inf-sup test

The matrix form of the discretized problem reads:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}$$

where **A** and **B** are the matrices associated to the bilinear forms $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ respectively. The numerical inf-sup test is based in the next theorem: *Let* $\mathbf{M}_v$ *and* $\mathbf{M}_q$ *be the mass matrices associated to the scalar products of* $\mathcal{V}_\rho$ *and* $\mathcal{Q}_\rho$ *respectively and let* μ_{min} *be the smallest non zero eigenvalue defined by the eigenproblem:*

$$\mathbf{B}^\top \mathbf{M}_q^{-1} \mathbf{B} v = \mu^2 \mathbf{M}_v v$$

then the value of k_ρ *is simply* $\mu_{\min}$.

Proof can be found in (3). The numerical test proposed by (4) consists in testing a particular formulation by calculating k_ρ using meshes of increasing refinement.If a steady decrease in $\log(k_\rho)$ is observed when h goes to zero, the element is predicted to violate the inf-sup condition but if the $\log(k_\rho)$ stabilizes the test is passed.

5.5 Numerical test of the PDF EFG method

In order to perform the numerical inf-sup test a sequence of successive refined meshes is considered. Figure 1 compares FEM, EFG and the PDF EFG. Some curves present rate of decrease close 1 in the log/log graph, clearly indicating that the numerical inf-sup condition fails. As expected, the finite element interpolations indicate that $Q1P0$, $Q1Q1$ and $Q2Q2$ elements do not verify the inf-sup condition and the $Q2Q1$ element is LBB compliant. Standard EFG method does not improve the finite element results. For the PDF EFG method $\log(k_\rho)$ appears to be bounded in every case.

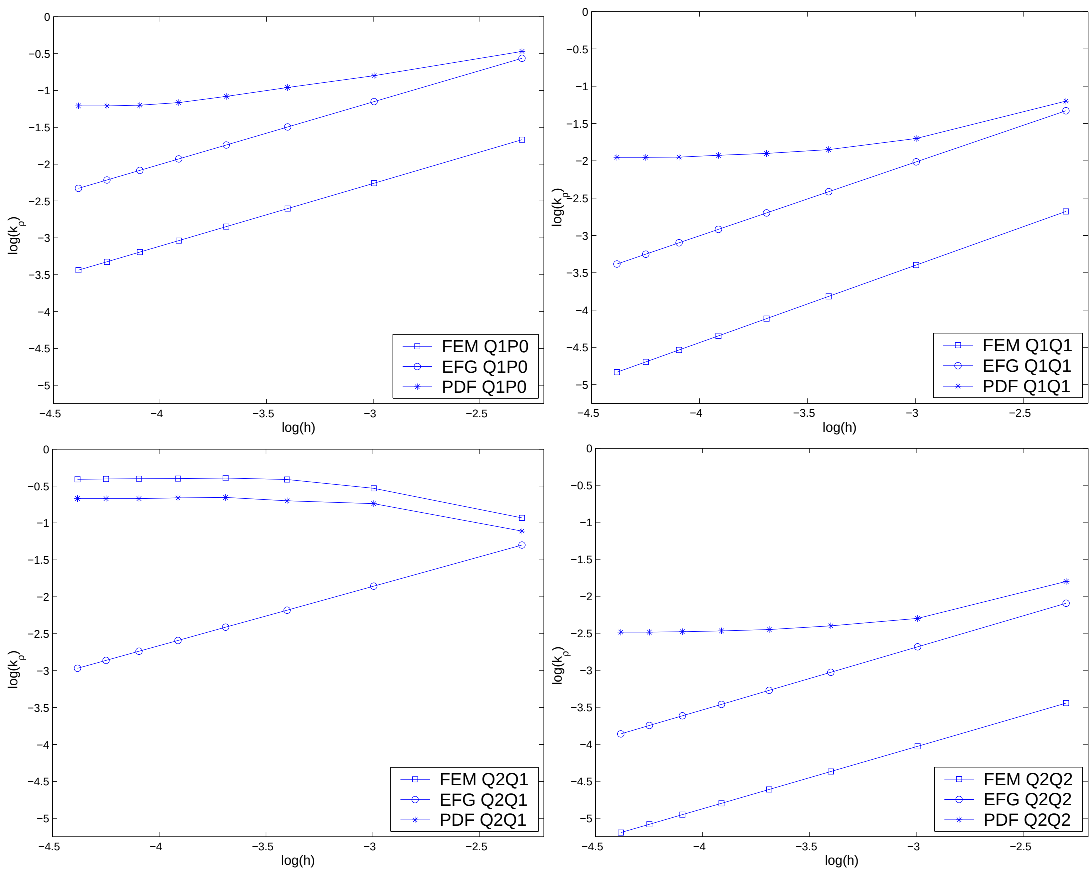

Figure 1: Inf-sup results: comparing FEM, EFG and the PDF EFG method.

6 Numerical example:Driven cavity flow problem

This problem models a plane flow of an isothermal fluid in a square lid-driven cavity. The upper side of the cavity moves in its own plane at unit speed, while the other sides are fixed. The domain is $\Omega =]0,1[\times]0,1[$, the body forces are neglected, $f = (0,0)^{\mathsf{T}}$, and the boundary conditions are formally defined as: $g = (0,0)^{\mathsf{T}}$ on $\partial\Omega \setminus \{y = 1\}$ and $g = (1,0)^{\mathsf{T}}$ on $\partial\Omega \cap \{y = 1\}$.

We solve this problem using $\rho/h = 2.1$ and a biquadratic interpolation to approximate velocity and pressure. Streamlines, pressure distribution and divergence of u are depicted in Figures 2 and 3 for standard EFG and the PDF EFG method, respectively. EFG presents oscillations on the pressure field. Nevertheless, for the PDF approach reasonable results are obtained. No spurious pressure modes are observed.

7 Conclusions

The PDF EFG method does not require extra computations given a particle distribution, thus the computational overhead is negligible. In fact, only the interpolating polynomials are modified compared with standard EFG.

The comparison with standard EFG is clear. While EFG fails the numerical inf-sup test the PDF EFG allows to use any combination of interpolating spaces in a mixed formulation. Moreover, the cavity example for Stokes flow corroborates these conclusions.

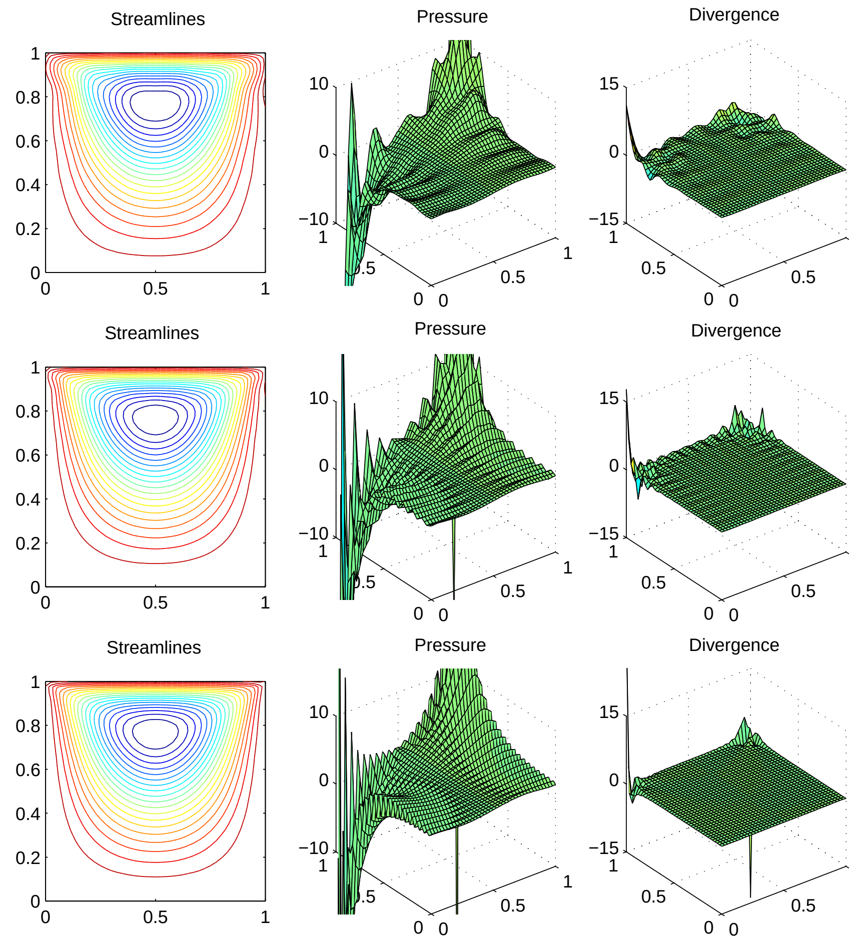

Figure 2: Standard EFG solution for a uniform distribution of 11×11 (top), 21×21 (middle) and 41×41 (bottom) particles.

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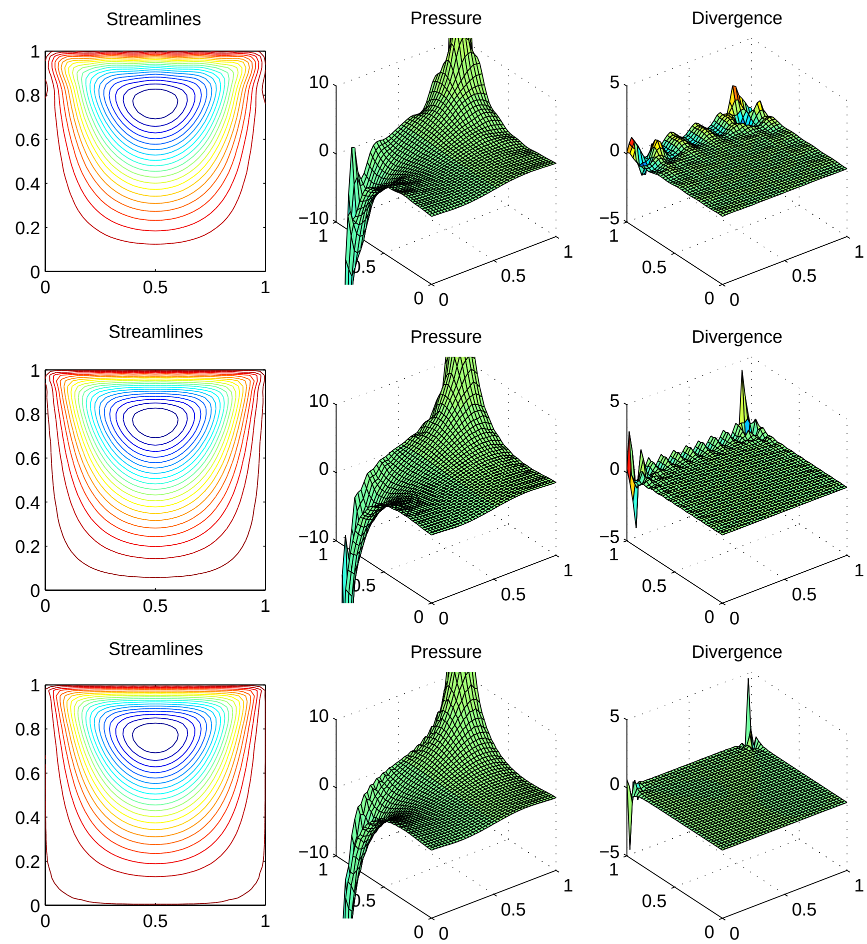

Figure 3: PDF EFG solution for a uniform distribution of 11x11 (top), 21x21 (middle) and 41x41 (bottom) particles.

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