On Knot Colorings: *The Turk's Head Knot*

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September 12, 2009

1. Introduction

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   - Reidemeister Moves
   - Colorings

Knots...

Definition (Knot)

A *knot* is a closed curve in $\mathbb{R}^3$ which does not intersect itself.

- Two knots are **equivalent** if they can be obtained one of each other through a continuous deformation, during which self-intersection does not occur.

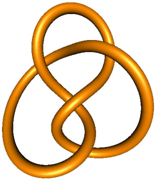
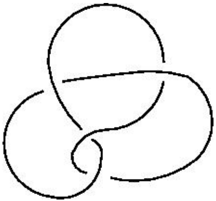


Figure: Figure-Eight-Knot

Knot Diagrams



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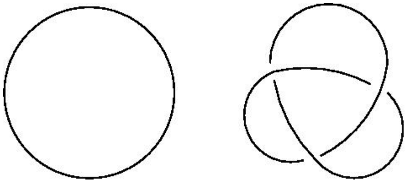
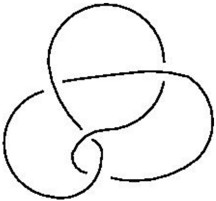


Figure: Trivial Knot and Trefoil



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Local transformations in the diagram of a knot, turning it in a diagram of an equivalent knot. There are three types:

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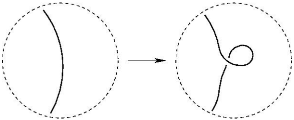

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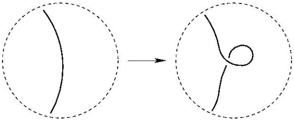

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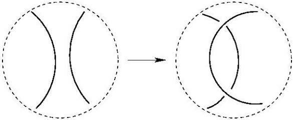

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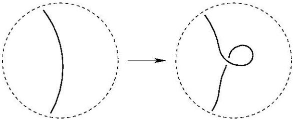

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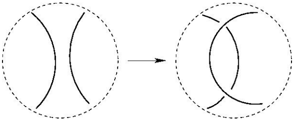

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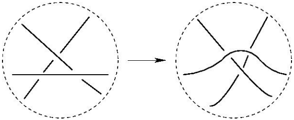

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We use *invariants*!

Colorings

Definition (Coloring)

Given a positive integer r, and a knot diagram D, a $r-$coloring of D is an assignement of integers in $\mathbb{Z}_r$, called colors, to the arcs of D, such that, in each crossing the double of the color of the upper arc equals $(mod\ r)$ the sum of the colors of the other two arcs.

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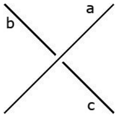
- $2a \equiv_r b + c \Leftrightarrow 2a - b \equiv_r c$



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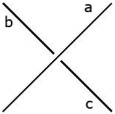

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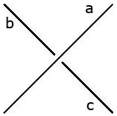


- Colorings are solutions of a linear homogeneous system, which consists of all crossings equation.
- Assigning the same color to every arc always sets a coloring. These are called trivial colorings.

Colorings (Examples)

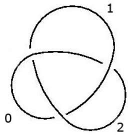
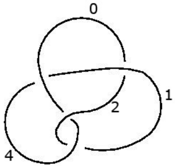


Figure: 3-Coloring



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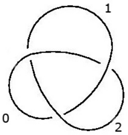

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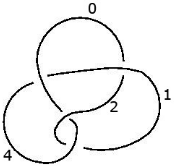

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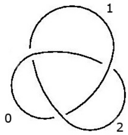

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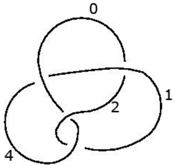

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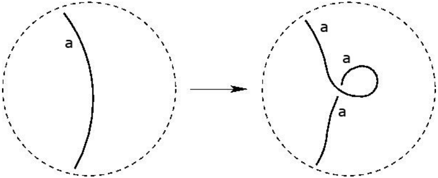

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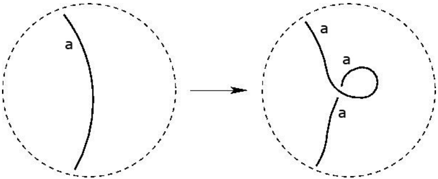

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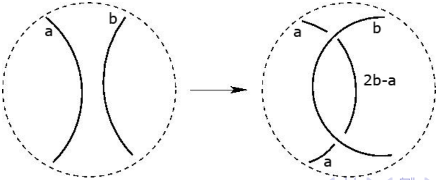

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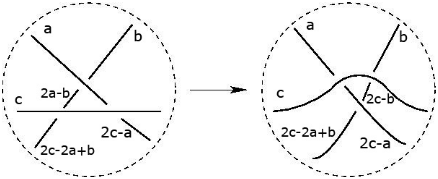

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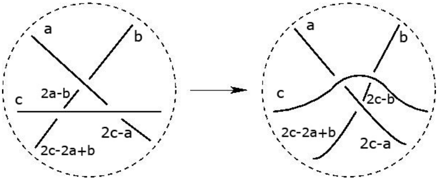

By Reidemeister's Theorem it exists a bijection between the colorings of two equivalent diagrams. Therefore, the *number of colorings* of a diagram is a knot invariant.

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Definition (Minimum Number of Colors)

Given a knot K, its *minimum number of colors*, $mincol_r K$, is given by:

$$\min\{n(D_K) \mid D_K \text{ is diagram of } K\}$$

Turk's Head Knot

- As we have seen before, the number of $r-$colorings is a knot invariant.
- The minimum number of colors is another one.

Turk's Head Knot

- Next, we will work with these invariants for the Turk's Head Knot with 3 strands.
- We wil start by seeing how is the standard diagram of the $THK(m,3)$.

The Turk's Head Knot

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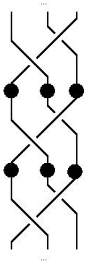
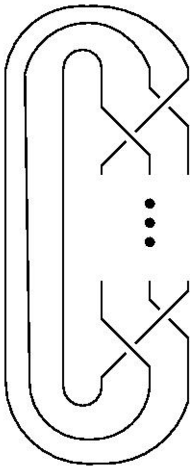


2 Juxtaposition m copies of the basic piece.

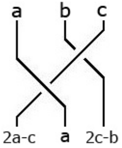
Turk's Head Knot



3 Close the braid connecting the correspondent ends of the strands.

Colorings of the $THK(m,3)$

The colors assigned at the top of a basic piece induce colors at its bottom, as presented bellow.



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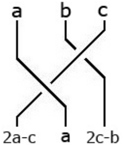

Also, the colors assigned at the top of a braid (like in step 2, previous slide) induce colors for the rest of its arcs. Furthermore, these colors form a coloring of the $THK(m, 3)$, if the colors induced at the bottom equal the colors at the top of the braid.

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- For m odd:

$$u_{m-1}\begin{bmatrix}1 & 0 & -1\\ 0 & -1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} \equiv_r \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

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With,

$$u_n=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2}-\left(\frac{-1+\sqrt{5}}{2}\right)^{n+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}+\left(\frac{-1-\sqrt{5}}{2}\right)^{n}\right).$$

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Theorem

The number of $r-$colorings of $THK(m,3)$ is given by:

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Corollary

The $THK(m,3)$ has non-trivial $r-$colorings if and only if:

- $(u_{m-1}, r) > 1$;
  or
- m is even and $5 \mid r$.

Minimum Number of Colors of the $THK(m,3)$

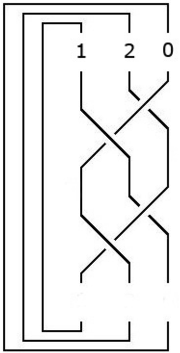

Figure: 5-Coloring of the $THK(2,3)$

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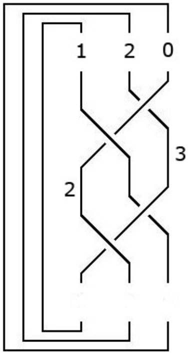

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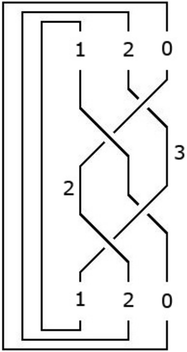

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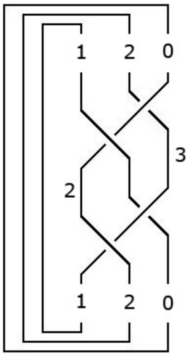

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- $THK(2,3)$ is non-trivially 5-colorable with 4 colors;

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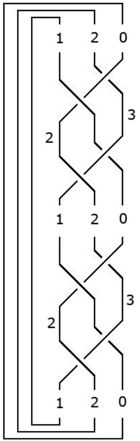

Figure: Stacking of the $THK(2, 3)$

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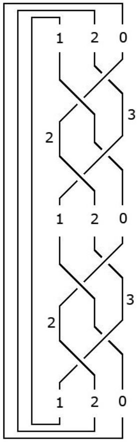

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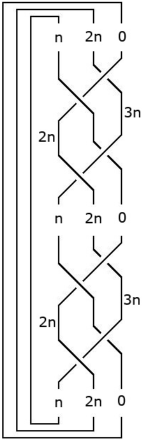

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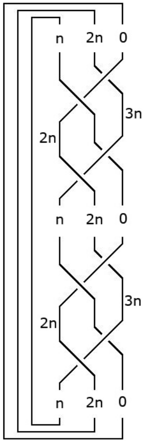

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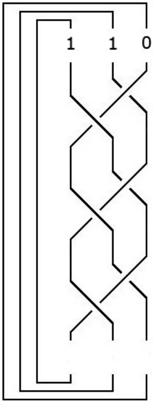

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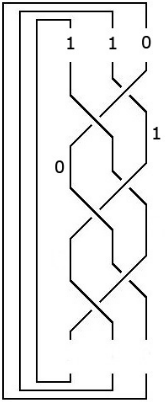

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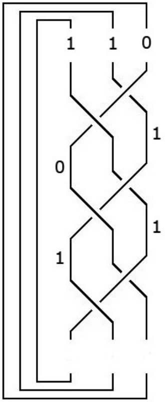

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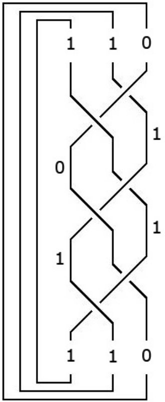

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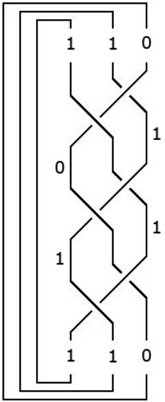

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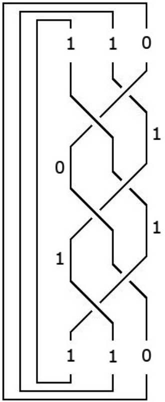

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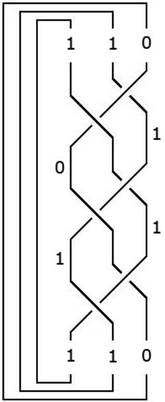
Minimum Number of Colors of the $THK(m,3)$



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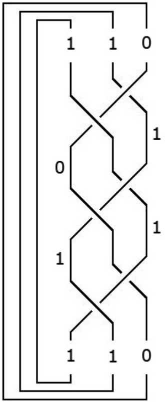

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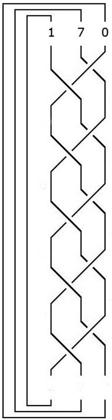
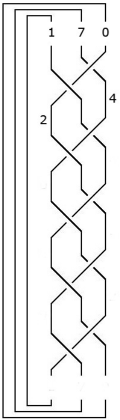


Figure: 11-Coloring of the $THK(5, 3)$

Minimum Number of Colors of the $THK(m,3)$



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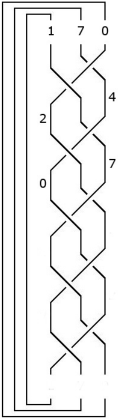

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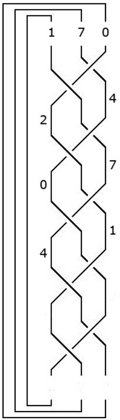

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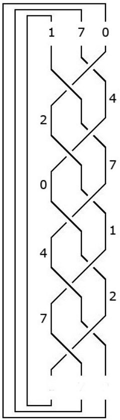

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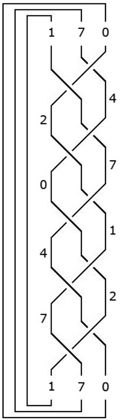

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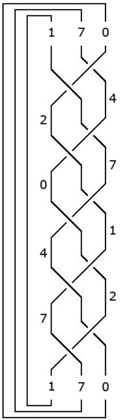

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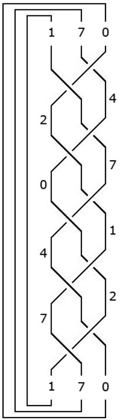

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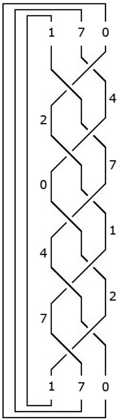

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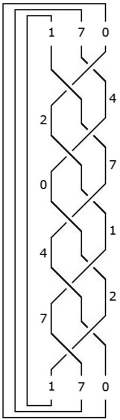

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- $(u_4 = 11)$

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Theorem

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*neither of the previous cases stand

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- If $2 \mid m$ and $5 \mid r$, or $8 \mid n$ and $7 \mid r$ (*), then $mincol_r THK(m, 3) = 4$

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- If $2 \mid m$ and $5 \mid r$, or $8 \mid n$ and $7 \mid r$ (*), then $mincol_r THK(m,3) = 4$
- If $5 \mid m$, and $11 \mid r$ (*), then $mincol_r THK(m,3) = 5$

*neither of the previous cases stand

Minimum Number of Colors of the $THK(m, 3)$

Definition ($\psi(.)$)

Let $\psi : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function defined by:

$$\psi(r) := \min\{q \in \mathbb{Z}^+ \mid r \mid u_{q-1}\}, r \in \mathbb{Z}^+$$

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  - As $(u_{\psi(p)-1}, r) \geq p$.

Minimum Number of Colors of the $THK(m,3)$

Proposition

Let $p \neq 5$ be an odd prime, then we have:

$$\begin{cases} p \mid u_p & \text{if and only if } 5^{\frac{p-1}{2}} \equiv_p -1 \\ p \mid u_{p-2} & \text{if and only if } 5^{\frac{p-1}{2}} \equiv_p 1 \end{cases}$$

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Corollary

Let $p \neq 5$ be an odd prime, then:

$$\psi(p) \leq p+1$$

Theorem

Given $p \neq 5$ with $\psi(p)$ odd, we have:

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$$mincol_p THK(\psi(p),3) \leq \psi(p) - 1$$

Definition ($\langle ., . \rangle_\psi$)

Given positive integers a, b, we define $\langle a, b \rangle_\psi$ as the least common prime factor that minimizes ψ.

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For n odd and $r \in \mathbb{Z}^+$, such that, $(u_{n-1}, r) > 1$, we have:

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