Probability and Hydrodynamics

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I. The Brownian motion

Observed by Robert Brown:

an extremely irregular, apparently endless motion of particles suspended in a fluid;

" A Brief Account of Microscopical Observations Made in the Months of June, July and August, 1827, on the Particles Contained in the Pollen of Plants; and on the General Existence of Active Molecules in Organic and Inorganic Bodies" , Edinb. J. of Sc. (1828)

and many others before him (the merit of Brown: showed that the motion was independent of the fluid and not of organic origin); many polemics around the origin of the motion, but Brown had "given the subject" to physicists.

Louis Bachelier (1900) mathematical model for the fluctuations in the stock market

Albert Einstein (1905) (independent work):

Gives a description of the motion of a particle subject to the forces coming from the molecules of the fluid. Starting from x_0, the probability that the (projection of the) position is in I at time $t > 0$ is given by

$$\int_I p(x_0, t, x)dx$$

where

$$\frac{\partial}{\partial t}p = D\,\frac{\partial^2}{\partial x^2}p$$

$p(t, x_0, x) = \frac{1}{2\sqrt{\pi Dt}}e^{-\frac{-|x-x_0|^2}{4Dt}}$, where D is a constant written, in particular, in terms of Avogrado's number;

$\longrightarrow$ Jean Perrin (Nobel prize, 1926) determines this number/ confirms the atomic nature of matter

“C’est un cas où il est vraiment naturel de penser à ces fonctions continues sans dérivées que les mathématiciens ont imaginées, et que l’on regardait à tort comme de simples curiosités mathématiques, puisque l’expérience peut les suggérer."

Jean Perrin

Definition A Brownian motion (or Wiener process) starting at x_0 at time 0 is a stochastic process $(t, \omega) \to B_t(\omega),\ t \geq 0$, such that

1. a.s. $B_0 = x_0$
2. Increments $B_{t+s} - B_t, s \geq 0$ are, for every t, independent of $B_u, u < t$
3. $B_{t+s} - B_t$ is normally distributed with mean 0 and variance s
4. With probability one $t \to B_t$ is continuous.

In particular, $P(B_t(\omega) \in A) = \int_A p_t(x_0, y)dy$ with $\frac{\partial}{\partial t} p = \frac{1}{2}\Delta p$ (heat equation)

In semigroup terminology,
$T_t f(x_0) = e^{\frac{t}{2}\Delta} f(x_0) = E^P f(B_t(\omega)) = \int f(y)p_t(x_0, y)dy.$

Mathematical construction:

- Norbert Wiener (1923)
- Andrei Kolmogorov (1930's)

Brownian motion as limit of symmetric random walk (1d):

Consider the random variables

$$X_k = \sqrt{\delta} \text{ with probability} \frac{1}{2}, \quad X_k = -\sqrt{\delta} \text{ with probability} \frac{1}{2}$$

in the time intervals $](k-1)\delta, k\delta]$.

At time $t = n\delta$ define $W(t) = \sum_{k=1}^{n} X_k$, with $W(0) = 0$

Make δ go to zero (...).

By the central limit theorem $W(t) \simeq \mathcal{N}(0, t)$. Since the coin tosses are independent, $W(t)$ has independent increments.

Examples of other relations with (linear) partial differential equations:

$\partial_t v = \frac{1}{2}\Delta v + V, v(0) = f$: v can be represented by **Feynman-Kac's formula**

$$v(t,x) = E_x\Big(f(B_t)\exp\big(\int_0^t V(B_s)ds\big)\Big)$$

Another example: the (stationary) **Dirichlet problem** $\frac{1}{2}\Delta v + Vv = 0$ in a domain D with boundary value $v = g$ is written as

$$v(x) = E_x\Big(g(B_\tau)\exp\big(\int_0^\tau V(B_s)ds\big)\Big),$$

$\tau = \inf\{t : B_t \notin D\}$, a random time, $B_0 = x$.

Kiyosi Itô (end 40's, 50's): birth of **Stochastic Analysis**

Consider the (elliptic) second-order linear operator

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^T)_{i,j}(x)\partial^2_{i,j} f + b.\nabla f$$

One can associate a stochastic differential equation

$$dX_t = \sigma(X_t).dB_t + b(X_t)dt$$

dB_t: Itô stochastic differentiation

(remember: $t \to B_t$ non differentiable !) $\to$ **Itô calculus**

In particular similar formulae as above relate pde's and stochastic processes, when replacing Δ by $\sum_{i,j=1}^{n}(\sigma\sigma^T)_{i,j}(x)\partial^2_{i,j}$.

Remark: relation with (Riemannian) geometry

In Itô calculus, if $dX_t = dB_t + b(X_t)dt$ and $f(t,x)$ is a smooth function,

$$df(t,X_t) = \nabla f(t,X_t).dB_t + (\partial_t f + b.\nabla f)(t,X_t)dt + \frac{1}{2}\Delta f(t,X_t)dt$$

(Itô's formula)

II. The Euler equation

One of the first pde to be written:
Leonhard Euler, *“Principes généraux du mouvement des fluides,"*
Mémoires de l’Académie des Sciences de Berlin (1757)

In Eulerian coordinates,

$$\frac{\partial u}{\partial t} + u.\nabla u = -\nabla p, \quad u(t_0) = u_0, \quad (\mathrm{div}\ u = 0)$$

Lagrangian coordinates: correspond to the velocity of a flow $g(t, x)$

$$\frac{\partial g}{\partial t} = u(t, g(t, x)), \;\; g(0, x) = x$$

$$\frac{d^2 g}{d^2 t} = (\frac{\partial u}{\partial t} + u.\nabla u)(t, g) = -\nabla p(t, g) \quad \text{(Newton's law)}$$

Variational approach: the flow minimizes the action functional

$$S[g] = \frac{1}{2}\int_0^T \int |\dot{g}(t,x)|^2 dx\, dt$$

where g are (volume preserving) diffeomorphisms on the underlying manifold **(form a group).**

Remember: geodesics are curves that minimise the lenght.

Euler-Lagrange equations give $\frac{d}{dt}[\dot{g}] = 0$ i.e., precisely,

$$\frac{\partial^2 g}{\partial^2 t} = (\frac{\partial u}{\partial t} + u.\nabla u)(t,g) = 0$$

($g(t)$ is a geodesic then the vector field $u(t) = \dot{g}(t)\ o\ g^{-1}(t)$ solves Euler eq.)

Euler equation $\Leftrightarrow$ geodesic equation for the L^2 metric

(Vladimir Arnold 1966)

• From this description Arnold (and co-authors) derive the instability of the *Lagrangian* motion from strict geometric arguments, namely the fact that the curvature of the underlying spaces is negative.

• Motion is chaotic even though the dynamics is deterministic (as in Lorenz simplified model for atmospheric convection with a small number of degrees of freedom).

• Unpredictability of the weather ("butterfly effect").

III. The Navier-Stokes equation

It models the motion of a (incompressible) **viscous** fluid. It was established one century after Euler equation.

$$\partial_t u + u.\nabla u = \nu \Delta u - \nabla p, \quad u(t_0) = u_0, \quad (\mathrm{div}\ u = 0)$$

($\nu > 0$ = viscosity)

The situation concerning variational principles is not clear in Physics literature.

We consider stochastic Lagrangian flows and replace $\frac{\partial}{\partial t}$ by a mean derivative D_t.

Stochastic Lagrangian flows:

$$dg(t,x) = \sqrt{2\nu}dB_t + Y_t(x)dt$$

$$g(0,x) = x$$

Define

$$D_t g(t) = Y_t$$

or

$$D_t g(t,x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} E_t\Big(g(t+\epsilon,x) - g(t,x)\Big)$$

where E_t denotes conditional expectation given the past of t.

Stochastic action functional

$$S[g] = \frac{1}{2} E[\int_0^T \int |D_t g(t)(x)|^2 \, dx \, dt]$$

Variations:

for v smooth, with div $v = 0$, $v(0) = v(T) = 0$,

$$e_t(\epsilon v) = id + \epsilon \int_0^t \dot{v}(s, e_s(\epsilon v) ds \quad \simeq id + \epsilon v(t, .)$$

$$g^\epsilon(t, x) = e_t(\epsilon v)(g(t, x))$$

Derivatives:

$$(D_L) S[g] = \frac{d}{d\epsilon}\Big|_{\epsilon=0} S[e_\cdot(\epsilon v) \circ g(\cdot)]$$

Consider a solution of a stochastic differential equation of the form

$$dg_u(t,x) = \sqrt{2\nu}\,dB_t + u(t,g(t,x))\,dt, \qquad g(0,x) = x$$

By Itô calculus

$$dg^\epsilon(t) = \sqrt{2\nu}\nabla e_t(\epsilon v)(g(t))\,dB_t + [\partial_t + (u.\nabla) + \nu\Delta](e_t(\epsilon v))\,dt$$

so

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} Dg^\epsilon(t) = [\partial_t v + (u.\nabla)v + \nu\Delta v](g(t))$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} S[g^\epsilon] = E \int_0^T \int (Dg(t)(x).[\partial_t v + (u.\nabla)v + \nu\Delta v])(g(t,x))dxdt$$

$$= \int_0^T \int (u.[\partial_t v + (u.\nabla)v + \nu\Delta v])(t,x)dxdt$$

since $Dg(t)(x) = u(t, g(t,x))$. Therefore, using integration by parts,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} S[g^\epsilon] = 0$$

iff

$$\partial_t u + (u.\nabla)u - \nu\Delta u = -\nabla p$$

Theorem. Let $(t, x) \mapsto u(t, x)$ be a smooth time-dependent divergence-free vector field on $\mathbb{R}^3$, defined on $[0, T] \times \mathbb{R}^3$. Let $g_u(t)$ be a stochastic Brownian flow with diffusion constant $\nu > 0$ and drift u. The stochastic process $g_u(t)$ is critical for the energy functional S if and only if the vector field $u(t)$ verifies the Navier-Stokes equation

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \Delta u - \nabla p$$

First version for the torus: with F. Cipriano; on a Riemannian manifold: with M. Arnaudon; generalisations to Lie groups: with M. Arnaudon, X. Chen, T. Ratiu.
Other variational principles: T. Funaki, D. Gomes.

IV. Relations with entropy

For Euler equation, Arnold's geodesic approach does not always provide solutions. In 1989 Yann Brenier relaxed the problem, considering the so-called generalised solutions.

One minimises a kinetic energy averaged by probability measures Q on the path space $\Omega = C([0,1]; M)$

$$\min E_Q \int_0^1 |\dot{X}_t|^2 dt, \quad Q_{01} = \pi,$$

$Q_{01} := (X_0, X_1)_* Q.$

Here $dQ_t = dx \quad \forall t \quad (Q_t = (X_t)_* Q)$ and π is a probability measure on $M \times M$ s.t. its marginals satisfy $d\pi_0 = d\pi_1 = dx$.

The solutions P only charge absolutely continuous paths, since the kinetic energy is understood to be ∞ otherwise.

Then $\begin{cases} dP_t = dx \;\; \forall t \quad \text{and} \;\; P_{01} = \pi \\ \ddot{X}_t + \nabla p(t, X_t) = 0, \; \forall t, \; P - a.e. \end{cases}$

We consider Brownian-type paths (not abs. continuous). For Q the corresponding law on the path space, kinetic energy is replaced by the"mean" velocity:

$$v_t^Q := \lim_{\epsilon \to 0} \frac{1}{\epsilon} E_Q(X_{t+\epsilon} - X_t \mid X_{[0,t]})$$

Consider the reference measure R

$$R = \int R^x dx,$$

R^x the law of the Brownian motion starting from x with diffusion constant $\nu > 0$.

On the other hand recall the notion of relative entropy of a measure Q with respect to a measure R

This is the Kullback-Leibler (1951) notion in Information Theory

$$H(Q|R) := \int \log(\frac{dQ}{dR})\, dQ \in (-\infty, \infty]$$

By Girsanov theorem, to any measure Q on Ω with a finite relative entropy w.r.t. R corresponds a (time dependent) vector field v s.t. Q is the law of the process $dX_t = \sqrt{2\nu}dB_t + v(t, X_t)dt$

$$H(Q|R) = H(Q_0|R_0) + \frac{1}{2\nu}E_Q \int_0^1 |v(t, X_t)|^2 \, dt$$

(in our case $dR_0 = dx$).
So we naturally consider the problem

$$\min \frac{1}{2\nu}E_Q \int_0^1 |v(t, X_t)|^2 \, dt$$

with $Q_{01} = \pi$ and $Q_t = \mu_t$ prescribed measures on M (Lebesgue measure for incompressibility constraint), which is an entropy minimisation problem, relaxing our Lagrangian approach and extending to the viscous case Brenier's generalised solutions.

(recent ongoing work)