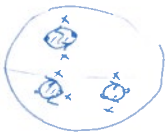
Dan Dugger - "Surfaces with involutions"

Compact 2-manifolds: $S^2, T, T_g =$ 

$\mathbb{RP}^2, N_r = (\mathbb{RP}^2)^{\# r}$

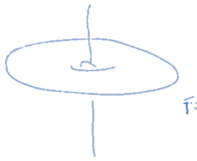


sphere with
r cross-caps

A surface with involution or C_2-manifold is a pair (X, f) $f: X \to X$ with $f^2 = id$.

When are 2 such isomorphic? $(X,f) \cong (Y,g)$ if $\exists$ $h: X \to Y$ homeomorphism such that $h \circ f = g \circ h$

Example: Involutions on the torus

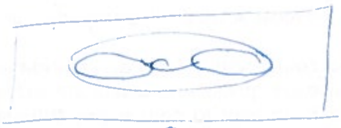


180° rotation about central axis

$F=0, C=0, Q=+$

$F=4, C=0, Q=+$

How do I know they are different?
1st has no fixed points but 2nd has 4 fixed points



reflect on the plane

$F=0$ $C=2$ $Q=-$

2 circles of fixed points
so clearly a different action

There's always the trivial action ~~$f = id$.~~

There are two more:

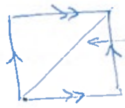
origin

$F=0, C=0, Q=-$

antipodal action
with respect to origin
at the center of the torus
(no fixed points)

Why is it different from the first one? the 1st action is orientation preserving and the last one is not.

One more: $F=0$, $C=1$, $Q=-$



reflect on diagonal line

This has a fixed circle and so it's different from all the others.

It's a theorem that these are all.

Goal: Classify involutions on all the surfaces

(This seems a reasonable goal. There aren't that many surfaces and there does
First worked on in 90s (1890s) by Felix Klein and then his students who worked on this (they were interested in Real Alg. Geometry). But they didn't finish. I want to tell you what they did and what they missed.)

What should this mean? Given a surface X

1. Count how many involutions on X
2. Algorithm for naming them and listing them
3. Have a complete set of invariants (a way of telling whether two such surfaces are the same)

Klein and others did some of this but not all

I believe I have done all these things.

Solution to 1 (to give an idea of the level of complexity)

Theorem: (a) On T_g there are $4+2g$ involutions (Klein 1890s) [doesn't seem so bad]

(b) On N_r there are

$$\begin{cases} 1+\frac{(r+3)^3}{64} & \text{if } r \equiv 1 \mod 4 \\ 1+\frac{(r+1)(r+3)(r+5)}{64} & \text{if } r \equiv 1 \mod 4 \\ \frac{1}{64}(r^3+18r^2+152r) & \text{if } r \equiv 0 \mod 4 \\ \frac{1}{64}(r^3+18r^2+156r-8) & \text{if } r \equiv 2 \mod 4 \end{cases}$$

Completely crazy!
A lot is still not understood about this

Example: On K (r=2) $\frac{1}{64}(2^3 + 18\cdot 4 + 156\cdot 2 - 8) = 6$

on ~~K#~~ K#K get 15 and this grows very fast

In the rest of talk want to give you a taste of the answers to (2) and (3).

Invariants: $F = \#$ fixed points
$C = \#$ fixed circles ← ovals
(one can show that fixed points are either isolated or parts of surfaces)

Given a loop on the surface if we draw a normal vector along a loop. Get either

a cylinder

two sided ovals

or a Möbius band

one sided ovals

Can decompose C into C_+ and C_-
↑ number of fixed circles which are 2 sided.

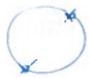
$X^{\circlearrowleft f}$ the quotient is $X/_{y\sim f(y)}$ $y \in X$ $= X/C_2$. This is a surface. Define $Q = \pm$ according to whether X/C_2 is orientable or not

These invariants are enough to distinguish the (a) cases and this is what Klein did 100 years ago.

Involutions on circles:

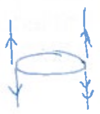
$S^{1,0}$ trivial

$S^{1,1}$ reflect

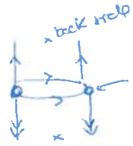


S^1_a S^1 antipodal.

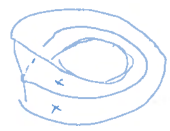
Surgery techniques



$S^{1,0}$ antitube — tube with action turning upside down



$S^{1,1}$ anti-tube



reflect in the core circle.

$S^{1,0}(M)$ Möbius version of $S^{1,0}$

The boundary is S^1_a

Keeping those picture in mind here are 5 surgery techniques

[DCC]

disjoint conjugate neighborhoods

cut them out and insert cross caps which are switched by involution

add "dual cross caps"

[$S^{1,0}$-AT]

add $S^{1,0}$ anti tube and similarly $S^{1,1}$ antitube [$S^{1,1}$-AT]

[DT]

add dual tori

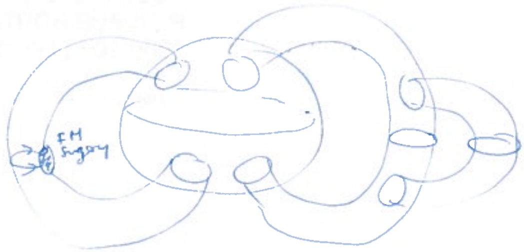
[FM]

• isolated fixed point when we take out a small disk. This will have boundary S^1_a. Sew in an $S^{1,0}M$

(not stated precisely)

Theorem: You can make any surface with involution by starting with a std one and then applying these operations

E.g. $S^2_a + [DCC] + [S^{1,1}\text{-}AT] + 2[S^{1,0}\text{-}AT] + [FM]$



Have a nice nomenclature to talk about these

Theorem (Scherer, 1920s) $\overset{X}{\underset{f}{\circlearrowleft}}$ surface $\beta = \dim H_1(X; \mathbb{F}_2)$

$F + 2C \leq \beta + 2$, $F \equiv C_- \equiv \beta \mod 2$ (this gives nice restrictions on what these can look like. For instance on a torus we can't have 20 fixed points)

Example: torus $\beta = 2$. So $F + 2C \leq 4 \Rightarrow 2C \leq 4 \Rightarrow C = 2$ (so can't get more than 2 fixed circles)
Similarly $F \leq 4$

$F \equiv 2$ so F is even hence $F = 0, 2$ or 4 are the only possibilities (it turns out 2 is not possible but this doesn't come from this theorem)

Proof of this theorem: think about surgery:

① Check in basic cases S^2_a, $S^{2,2}$ =  rotate $180°$ $S^{2,1}$ 

② Check invariance under surgery. For instance $\underline{\underline{DCC}}$ doesn't change F, C, etc. Changes β into $\beta + 2$ ✓
each cross cap increases β by one

$\underline{\underline{FM}}$ $F \to F - 1$, $C \to C + 1$, $C_- \to C_- + 1$, $\beta \to \beta + 1$, so the conditions in the theorem still holds.

Have i) given some way of producing surfaces with involutions
ii) given some invariants.

Given X, list all F, $C = C_+ + C_-$ satisfying conditions in Scherer. This is a finite list.
($Q = +$ or $Q = -$)

A little extra work $\Rightarrow$ there is only one possible action and we can write it down. One can use a computer to ~~[illegible]~~ list all of them

This is the best answer you can have because it's telling you the invariants determine the action.

It works almost all of the time. It doesn't work when $X = N_r$, $F = C = 0$. Then get multiple actions

This is what was completely missed previously.

Example: $S^2_a + 2[DCC] + [S^{1,0}\text{-}AT]$ vs $T_1^{\text{anti}} + [DCC] + [S^{1,0}\text{-}AT]$ ($F = 0$, $C = C_+ = 1$, $C_- = 0$, $Q = -$)

But they are not homeomorphic. Why? (there's an example like this in each genus)

$F: X \to X$

$$H_1(X;\mathbb{F}_2) \xrightarrow{F_*} H_1(X,\mathbb{F}_2)$$

Have an intersection product.

Let's write it as a bilinear form. It is a general property of homology that $\langle f_* x, f_* y\rangle = \langle x, y\rangle$

So $f_* \in \mathrm{Aut}(H_1(X;\mathbb{F}_2), \langle -,-\rangle)$

isometry group of this linear form.

$X = N_r$ the group is called $O_r = \left\{ \begin{matrix} r\times r \\ \text{matrices} \end{matrix} \right.$ with entries 0 or 1 and $AA^T = I \Big\}$

$f^2 = id \Rightarrow (f_*)^2 = id.$

Algebraic version of this problem: Classify conjugacy classes of involutions in $O_r(\mathbb{F}_2)$?

Surely group theorists have answered this! Apparently not. This is not a simple group mostly and group theorists mostly care about those.

1) $D(A) = \mathrm{rank}(A - I)$ Dickson invariant of a matrix gives a complete classification of involutions in GL

2) $\mathbb{F}_2^r \longrightarrow \mathbb{F}_2$ this doesn't look linear but it is because we are in characteristic 2.

$v \longrightarrow \langle v, \sigma v\rangle$

involution

$$\langle v+w, \sigma(v+w)\rangle = \langle v,\sigma v\rangle + \cancel{\langle v,\sigma w\rangle + \langle w, \sigma v\rangle} + \langle w, \sigma w\rangle$$

because σ is an isometry.

Can take rank of this map

Call it $\alpha(A)$ (or $\alpha(\sigma)$)

When r is odd this answers the question (this was already known) In this case O_r is simple.

3) $\underline{r \text{ even}}$ $O_r(\mathbb{F}_2) \longrightarrow O_r(\mathbb{F}_2)$

$A \longrightarrow mA$ change 1s into 0s

$\underline{\text{Fact}}$: m preserves involutions (this only became clear after exhaustive computer search)

Look at $[D(A), \alpha(A), D(mA), \alpha(mA)] = DD(A)$

$\underline{\text{Theorem}}$ (a) DD classifies involutions

(b) F, C_+, C_-, Q, DD completely classifies topological involutions. (distinguishes the examples mentioned above)