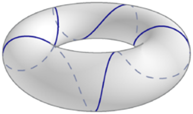
Abundance of minimal surfaces

André Neves

THE UNIVERSITY OF CHICAGO

Introduction

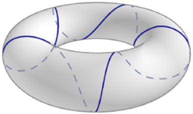
A geodesic on a manifold is a curve with the property that it minimizes the distance between any two nearby points on the curve.



- Closed geodesics can be seen from two different perspectives.

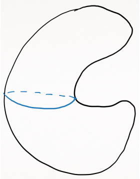
Introduction

A geodesic on a manifold is a curve with the property that it minimizes the distance between any two nearby points on the curve.

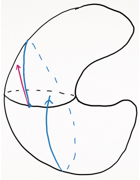


- Closed geodesics can be seen from two different perspectives.

Variational: Closed geodesics are critical points for length functional.



Dynamical: Closed geodesics are periodic orbits to geodesic flow (Hamiltonian flow on cotangent bundle)



Introduction

Theorem (Birkhoff, 1912)

Every 2-sphere has a closed geodesic.

Introduction

Theorem (Birkhoff, 1912)

Every 2-sphere has a closed geodesic.

Theorem (Lusternick-Schnirelman, 30's)

Every 2-sphere has 3 **simple** *closed geodesics.*

Introduction

Theorem (Birkhoff, 1912)

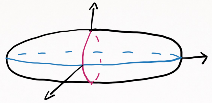
Every 2-sphere has a closed geodesic.

Theorem (Lusternick-Schnirelman, 30's)

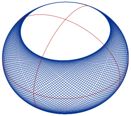
Every 2-sphere has 3 **simple** *closed geodesics.*

- Both proofs are variational.

  - Optimal on some ellipsoids.



  - There are a lot more geodesics that are not simple.



Introduction

Theorem (Bowen, '73)

Assuming the ambient curvature of the closed manifold is negative, almost all closed geodesics of large length become equidistributed.

Introduction

Theorem (Bowen, '73)

Assuming the ambient curvature of the closed manifold is negative, almost all closed geodesics of large length become equidistributed.

Theorem (Pugh-Robinson, '83)

For a C^2-generic set of metrics on surfaces, closed geodesics are dense.

Introduction

Theorem (Bowen, ’73)

Assuming the ambient curvature of the closed manifold is negative, almost all closed geodesics of large length become equidistributed.

Theorem (Pugh-Robinson, ’83)

For a C^2-generic set of metrics on surfaces, closed geodesics are dense.

- Both results use the dynamical point of view for closed geodesics.

Introduction

Theorem (Bowen, '73)

Assuming the ambient curvature of the closed manifold is negative, almost all closed geodesics of large length become equidistributed.

Theorem (Pugh-Robinson, '83)

For a C^2-generic set of metrics on surfaces, closed geodesics are dense.

- Both results use the dynamical point of view for closed geodesics.

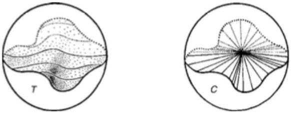
Theorem (Franks, '92, Bangert, '93)

Every 2-sphere has infinitely many closed geodesics.

Introduction

In the same way that geodesics are critical points of the length/energy functional, minimal hypersurfaces T are critical points of the area functional.

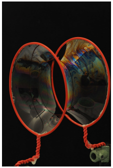
They have the property that for any intersection with a small ball B, $T \cap B$ has less area than any other hypersurface C with the same boundary.



There is no “dynamical” point of view for minimal hypersurfaces.

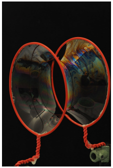
Introduction

They can be found, for instance, as solutions to the Plateau problem, proposed by Lagrange in 1760.

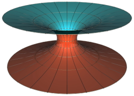


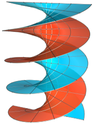
Introduction

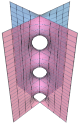
They can be found, for instance, as solutions to the Plateau problem, proposed by Lagrange in 1760.

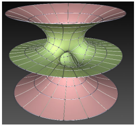


Besides the plane, the first examples in $\mathbb{R}^3$ were the catenoid and the helicoid (Euler and Mesnieur in 18th century).









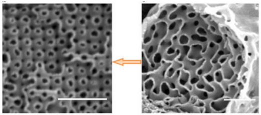
The first rigorous mathematical solution was only found by Douglas and Rado in 30's.

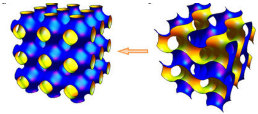
Introduction

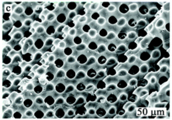
Minimal surfaces are ubiquitous in Science, from General Relativity (called apparent horizons) to Biomedicine or Material Science.

Yu *et al.*, Nature, ’14

Lai *et al.*, Chem. Comm, ’07







Minimal surfaces in Geometry have been used to prove the Positive Mass Theorem (Schoen–Yau), Frankel conjecture (Siu–Yau), or Willmore conjecture (Marques-N.), among many other results.

Introduction

Yau Conjecture, '82

Every closed 3*-manifold admits an infinite number of distinct minimal surfaces.*

Introduction

Yau Conjecture, '82

Every closed 3*-manifold admits an infinite number of distinct minimal surfaces.*

Theorem (Pitts, '81, Schoen-Simon, '82)

Every (M^{n+1}, g) *has one closed minimal embedded hypersurfaces, smooth outside a set of codimension* 7*.*

Introduction

Yau Conjecture, '82

Every closed 3-manifold admits an infinite number of distinct minimal surfaces.

Theorem (Pitts, '81, Schoen-Simon, '82)

Every (M^{n+1}, g) has one closed minimal embedded hypersurfaces, smooth outside a set of codimension 7.

Theorem (Marques-N., '14)

Every (M^{n+1}, g) has $n+1$ closed minimal embedded hypersurfaces, smooth outside a set of codimension 7.

If $Ric(g) > 0$, there are infinitely many closed minimal embedded hypersurfaces, smooth outside a set of codimension 7.

New Progress

Theorem (Irie–Marques–N., '17)

Let $3 \leq n+1 \leq 7$. *For a* C^∞*-generic set of metrics on* (M^{n+1}, g), *closed minimal embedded hypesurfaces are dense.*

In particular, Yau's conjecture holds for generic metrics.

New Progress

Theorem (Irie–Marques–N., '17)

Let $3 \leq n+1 \leq 7$. *For a* C^∞*-generic set of metrics on* (M^{n+1}, g), *closed minimal embedded hypesurfaces are dense.*

In particular, Yau's conjecture holds for generic metrics.

Theorem (Marques–N.–Song, '17)

Let $3 \leq n+1 \leq 7$. *For a* C^∞*-generic set of metrics on* (M^{n+1}, g), *there is a sequence* $\{\Sigma_j\}_{j\in\mathbb{N}}$ *of closed minimal embedded hypesurfaces that becomes equidistributed. More precisely, for all* $f \in C^\infty(M)$

$$\lim_{q\to\infty} \frac{1}{\sum_{j=1}^{q} vol(\Sigma_j)} \sum_{j=1}^{q} \int_{\Sigma_j} f \, d\Sigma_j = \frac{1}{vol(M)} \int_M f \, dM.$$

New Progress

- The analogous results for geodesics (denseness and equidistribution) use the dynamical point of view, which has no analogue for minimal hypersurfaces.

New Progress

- The analogous results for geodesics (denseness and equidistribution) use the dynamical point of view, which has no analogue for minimal hypersurfaces.

Theorem (Song, '18)

Let $3 \leq n+1 \leq 7$. *Every* (M^{n+1}, g) *has an infinite number of closed minimal hypersurfaces.*

In particular, Yau's conjecture holds.

New Progress

• The analogous results for geodesics (denseness and equidistribution) use the dynamical point of view, which has no analogue for minimal hypersurfaces.

Theorem (Song, '18)

Let $3 \leq n+1 \leq 7$. *Every* (M^{n+1}, g) *has an infinite number of closed minimal hypersurfaces.*

In particular, Yau's conjecture holds.

• The crucial new ingredient comes from combining min-max theory with the notion of volume spectrum (introduced by Gromov) and the Weyl Law for the volume spectrum (proven by Liokumovich-Marques-N.).

Weyl Spectrum

(M^{n+1}, g) closed Riemannian manifold.

- $W^{1,2}(M) = \left\{\text{all functions } f \text{ with } \int_M (f^2 + |\nabla f|^2) < \infty\right\}.$

Weyl Spectrum

(M^{n+1}, g) closed Riemannian manifold.

- $W^{1,2}(M) = \left\{\text{all functions } f \text{ with } \int_M (f^2 + |\nabla f|^2) < \infty\right\}.$
- The p-th eigenvalue is given by

$$\lambda_p(M) = \inf_{\{(p+1)\text{-plane } P \subset W^{1,2}\}} \sup_{f \in P - \{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Weyl Spectrum

(M^{n+1}, g) closed Riemannian manifold.

- $W^{1,2}(M) = \left\{\text{all functions } f \text{ with } \int_M (f^2 + |\nabla f|^2) < \infty\right\}.$
- The p-th eigenvalue is given by

$$\lambda_p(M) = \inf_{\{(p+1)\text{-plane } P \subset W^{1,2}\}} \sup_{f \in P - \{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Weyl Law (Weyl, 1911, Minakshisundaram-Pleijel, 1949)

The asymptotic behavior of $\{\lambda_p\}_{p\in\mathbb{N}}$ *depends only on the volume of* M*:*

$$\lim_{p\to\infty} \lambda_p(M) p^{-\frac{2}{n+1}} = a(n) vol(M)^{-\frac{2}{n+1}},$$

where $a(n) = 4\pi^2 vol(B)^{-\frac{2}{n+1}}$ *and* B *is the unit ball in* $\mathbb{R}^{n+1}$*.*

Weyl Spectrum

• On $W^{1,2}(M) - \{0\}$ identify a function with its constant multiples and denote the resulting space by $\mathcal{P}$, which is weakly homotopic to $\mathbb{RP}^{\infty}$.

Weyl Spectrum

- On $W^{1,2}(M) - \{0\}$ identify a function with its constant multiples and denote the resulting space by $\mathcal{P}$, which is weakly homotopic to $\mathbb{RP}^\infty$.
- $(p+1)$-planes in $W^{1,2}(M)$ become p-dim projective spaces $\mathbb{RP}^p$ in $\mathcal{P}$.

Weyl Spectrum

- On $W^{1,2}(M) - \{0\}$ identify a function with its constant multiples and denote the resulting space by $\mathcal{P}$, which is weakly homotopic to $\mathbb{RP}^{\infty}$.
- $(p+1)$-planes in $W^{1,2}(M)$ become p-dim projective spaces $\mathbb{RP}^{p}$ in $\mathcal{P}$.
- The Raleigh quotient $R([f]) = \frac{\int_M |\nabla f|^2}{\int_M f^2}$ is well defined on $\mathcal{P}$ because it is scale invariant.

Weyl Spectrum

- On $W^{1,2}(M) - \{0\}$ identify a function with its constant multiples and denote the resulting space by $\mathcal{P}$, which is weakly homotopic to $\mathbb{RP}^\infty$.
- $(p+1)$-planes in $W^{1,2}(M)$ become p-dim projective spaces $\mathbb{RP}^p$ in $\mathcal{P}$.
- The Raleigh quotient $R([f]) = \frac{\int_M |\nabla f|^2}{\int_M f^2}$ is well defined on $\mathcal{P}$ because it is scale invariant.
- The p-th eigenvalue is now given by

$$\lambda_p(M) = \inf_{\mathbb{RP}^p \subset \mathcal{P}} \sup_{[f] \in \mathbb{RP}^p} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Weyl Spectrum

- On $W^{1,2}(M) - \{0\}$ identify a function with its constant multiples and denote the resulting space by $\mathcal{P}$, which is weakly homotopic to $\mathbb{RP}^\infty$.
- $(p+1)$-planes in $W^{1,2}(M)$ become p-dim projective spaces $\mathbb{RP}^p$ in $\mathcal{P}$.
- The Raleigh quotient $R([f]) = \frac{\int_M |\nabla f|^2}{\int_M f^2}$ is well defined on $\mathcal{P}$ because it is scale invariant.
- The p-th eigenvalue is now given by

$$\lambda_p(M) = \inf_{\mathbb{RP}^p \subset \mathcal{P}} \sup_{[f] \in \mathbb{RP}^p} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Advantage: Linear structure is gone! To define p-th eigenvalue one only needs

- A space $\mathcal{Z}$ that is (weakly) homeomorphic to $\mathbb{RP}^\infty$.
- A functional $F : \mathcal{Z} \to [0, \infty]$;

Volume Spectrum

- $\mathcal{Z}_n(M;\mathbb{Z}_2)=\{\partial\Omega \ : \Omega$ is a region in $M\}$.

Theorem (Almgren, '61)

$\mathcal{Z}_n(M;\mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$. Thus its cohomology ring has a single generator λ.

Volume Spectrum

- $\mathcal{Z}_n(M;\mathbb{Z}_2)$=$\{\partial\Omega\ \ :\Omega$ is a region in $M\}$.

Theorem (Almgren, '61)

$\mathcal{Z}_n(M;\mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$. Thus its cohomology ring has a single generator λ.

- A continuous map $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is called a p-sweepout if $\Phi^*\lambda^p \neq 0$ in $H^p(X;\mathbb{Z}_2)$.

Volume Spectrum

- $\mathcal{Z}_n(M;\mathbb{Z}_2)=\{\partial\Omega \ : \Omega$ is a region in $M\}$.

Theorem (Almgren, '61)

$\mathcal{Z}_n(M;\mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$. Thus its cohomology ring has a single generator λ.

- A continuous map $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is called a p-sweepout if $\Phi^*\lambda^p \neq 0$ in $H^p(X;\mathbb{Z}_2)$.
- For every $p \in \mathbb{N}$, Gromov introduced the volume spectrum $\{\omega_p(M)\}_{p\in\mathbb{N}}$

$$\omega_p(M) := \inf_{\{\Phi \ p\text{-sweepout}\}} \sup_{x\in dmn(\Phi)} vol(\Phi(x)).$$

Volume Spectrum

- $\mathcal{Z}_n(M;\mathbb{Z}_2)$=$\{\partial\Omega \ : \Omega$ is a region in $M\}$.

Theorem (Almgren, '61)

$\mathcal{Z}_n(M;\mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$. Thus its cohomology ring has a single generator λ.

- A continuous map $\Phi : X \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is called a p-sweepout if $\Phi^*\lambda^p \neq 0$ in $H^p(X;\mathbb{Z}_2)$.
- For every $p \in \mathbb{N}$, Gromov introduced the volume spectrum $\{\omega_p(M)\}_{p\in\mathbb{N}}$

$$\omega_p(M) := \inf_{\{\Phi \ p\text{-sweepout}\}} \sup_{x\in dmn(\Phi)} vol(\Phi(x)).$$

Conjecture (Gromov's Weyl Law, '03)

There is $\alpha(n)$ such that $\lim_{p\to\infty} \omega_p(M)p^{-\frac{1}{n+1}} = \alpha(n)vol(M)^{\frac{n}{n+1}}$.

- Gromov and Guth showed that for some $C = C(M,g)$,

$$C^{-1}p^{\frac{1}{n+1}} \leq \omega_p(M) \leq Cp^{\frac{1}{n+1}}.$$

New ingredient

Theorem (Liokumovich–Marques–N, '16)

Gromov's Weyl Law holds for every (M^{n+1}, g) *(with possibly* $\partial M \neq 0$*) i.e., there is* $\alpha(n)$ *such that* $\lim_{p\to\infty} \omega_p(M) p^{-\frac{1}{n+1}} = \alpha(n) vol(M)^{\frac{n}{n+1}}$.

New ingredient

Theorem (Liokumovich–Marques–N, ’16)

Gromov’s Weyl Law holds for every (M^{n+1}, g) *(with possibly* $\partial M \neq 0$*) i.e., there is* $\alpha(n)$ *such that* $\lim_{p\to\infty} \omega_p(M)p^{-\frac{1}{n+1}} = \alpha(n)vol(M)^{\frac{n}{n+1}}$.

- We are able to prove a Weyl Law without knowing the value of $\alpha(n)$.

New ingredient

Theorem (Liokumovich–Marques–N, '16)

Gromov's Weyl Law holds for every (M^{n+1}, g) *(with possibly* $\partial M \neq 0$*) i.e., there is* $\alpha(n)$ *such that* $\lim_{p\to\infty} \omega_p(M) p^{-\frac{1}{n+1}} = \alpha(n) vol(M)^{\frac{n}{n+1}}$.

• We are able to prove a Weyl Law without knowing the value of $\alpha(n)$.

• In the special case where M is a region $U \subset \mathbb{R}^{n+1}$ we follow a proof in the spirit of Weyl's original proof.

• When M is a general closed manifold, the methods used in the Laplacian spectrum do not apply and we came up with a new idea based on cut-and-paste arguments.

Min-Max Theorem

Min-Max Theorem (Pitts, Schoen–Simon, Marques–N.)

Let $3 \leq n+1 \leq 7$. For every $p \in \mathbb{N}$, there is Σ_p a closed embedded minimal hypersurface with possible multiplicities such that $\omega_p(M) = vol(\Sigma_p)$.

Min-Max Theorem

Min-Max Theorem (Pitts, Schoen–Simon, Marques–N.)

Let $3 \leq n+1 \leq 7$. For every $p \in \mathbb{N}$, there is Σ_p a closed embedded minimal hypersurface with possible multiplicities such that $\omega_p(M) = vol(\Sigma_p)$.

- The combined work of Pitts and Schoen–Simon proved the theorem above for homotopies. The Morse index estimates proven by Marques and myself allow to extend the result to cohomology classes.

- It is possible that $\Sigma_p = 2\Sigma' + 3\Sigma''$ where $\Sigma'' \cap \Sigma' = \emptyset$. Thus the min-max method may give critical points that are **not** geometrically distinct.

Min-Max Theorem

Min-Max Theorem (Pitts, Schoen–Simon, Marques–N.)

Let $3 \leq n+1 \leq 7$. For every $p \in \mathbb{N}$, there is Σ_p a closed embedded minimal hypersurface with possible multiplicities such that $\omega_p(M) = vol(\Sigma_p)$.

• The combined work of Pitts and Schoen–Simon proved the theorem above for homotopies. The Morse index estimates proven by Marques and myself allow to extend the result to cohomology classes.

• It is possible that $\Sigma_p = 2\Sigma' + 3\Sigma''$ where $\Sigma'' \cap \Sigma' = \emptyset$. Thus the min-max method may give critical points that are **not** geometrically distinct.

This feature is what makes the problem challenging and the reason it was open for so long.

Denseness Theorem

Theorem (Irie–Marques–Neves, '17)

Let $3 \le n+1 \le 7$. *For a* C^∞*-generic set of metrics on* (M^{n+1}, g)*, closed minimal embedded hypesurfaces are dense.*

Basic strategy:

Denseness Theorem

Theorem (Irie–Marques–Neves, '17)

Let $3 \leq n+1 \leq 7$. For a C^∞-generic set of metrics on (M^{n+1}, g), closed minimal embedded hypesurfaces are dense.

Basic strategy:

- Choose $U \subset M$ an open set and pick a metric g_0. It is possible that no minimal hypesurface intersects U.
- Why are there arbitrarily small perturbations of g_0 which have minimal hypesurfaces intersecting U?

Denseness Theorem

Theorem (Irie–Marques–Neves, '17)

Let $3 \leq n+1 \leq 7$*. For a* C^∞*-generic set of metrics on* (M^{n+1}, g)*, closed minimal embedded hypesurfaces are dense.*

Basic strategy:

- Choose $U \subset M$ an open set and pick a metric g_0. It is possible that no minimal hypesurface intersects U.

- Why are there arbitrarily small perturbations of g_0 which have minimal hypesurfaces intersecting U?

- Do a tiny deformation $(g_t)_{0 \leq t \leq 1}$ inside U so that $vol(g_1) > vol(g_0)$. The Weyl Law implies that for some p large, $\omega_p(M, g_1) > \omega_p(M, g_0)$.

Denseness Theorem

Theorem (Irie–Marques–Neves, '17)

Let $3 \leq n+1 \leq 7$. For a C^∞-generic set of metrics on (M^{n+1}, g), closed minimal embedded hypesurfaces are dense.

Basic strategy:

- Choose $U \subset M$ an open set and pick a metric g_0. It is possible that no minimal hypesurface intersects U.
- Why are there arbitrarily small perturbations of g_0 which have minimal hypesurfaces intersecting U?
- Do a tiny deformation $(g_t)_{0 \leq t \leq 1}$ inside U so that $vol(g_1) > vol(g_0)$. The Weyl Law implies that for some p large, $\omega_p(M, g_1) > \omega_p(M, g_0)$.
- If no g_t has a minimal surface intersecting U, the minimal hypersurfaces with respect to g_t or g_0 are the same because $g_t = g_0$ outside U.

Denseness Theorem

Theorem (Irie–Marques–Neves, '17)

Let $3 \leq n+1 \leq 7$. For a C^∞-generic set of metrics on (M^{n+1}, g), closed minimal embedded hypesurfaces are dense.

Basic strategy:

- Choose $U \subset M$ an open set and pick a metric g_0. It is possible that no minimal hypesurface intersects U.
- Why are there arbitrarily small perturbations of g_0 which have minimal hypesurfaces intersecting U?
- Do a tiny deformation $(g_t)_{0 \leq t \leq 1}$ inside U so that $vol(g_1) > vol(g_0)$. The Weyl Law implies that for some p large, $\omega_p(M, g_1) > \omega_p(M, g_0)$.
- If no g_t has a minimal surface intersecting U, the minimal hypersurfaces with respect to g_t or g_0 are the same because $g_t = g_0$ outside U.
- We then argue that $t \mapsto \omega_p(M, g_t)$ is constant, which contradicts the 3rd step.

Thank you!