$\mathbb{Z}_p = \{(a_n) \in \prod \mathbb{Z}/p^n : a_{n+1} \equiv a_n \ (p^n)\} = \varprojlim_n \mathbb{Z}/p^n$ (projective limit)

$R_1 = \{0, 1, 2, \dots, p-1\}$ complete set of representatives of the residue classes mod p

- each integer m is congruent mod p to a unique elt of R.

$R_n = \{\sum_{i=0}^{n-1} r_i p^i \ r_i \in R_i\}$ complete set of representatives of residue classes mod p^n

$a \in \mathbb{Z}_p \quad a = (r_n)\ ;\ r_n \in R_n$

- if $r_{n+1} = \sum_{i=0}^{n} r_i p^i$ then $r_n = \sum_{i=0}^{n-1} r_i p^i$; $a \mapsto \sum_{i=0}^{\infty} r_i p^i \quad r_i \in R_i$

$\mathbb{Z}_p \Leftrightarrow \{\sum_{i=0}^{\infty} r_i p^n : r_i \in R_i\}$

(a_n) when $a_n = \sum_{i=0}^{n-1} r_i p^i$

obvious multiplication / addition or RHS is the same as that in $\mathbb{Z}_p$ "base p".

$-1 \leftrightarrow 1\cdot 2^0 + 1\cdot 2 + 1\cdot 2^2 + \dots + 1\cdot 2^n + \dots$

$\left(\overset{?}{=} \frac{1}{1-2} = -1\right)$

$(r_0 + r_1 p + \dots)(s_0 + s_1 p + s_2 p^2) = (r_0 s_0 + (r_0 s_1 + r_1 s_0) p + \dots)$

p-adic valuation

$\mathrm{ord}_p(\dots) \quad \mathbb{Q}_p \longrightarrow \mathbb{Z} \cup \{+\infty\}$

$0 \neq x \in \mathbb{Q}_p \quad x = p^k u\ ;\ u \in \mathbb{Z}_p^\times ;\ k \in \mathbb{Z}$ - unique.

$\mathrm{ord}_p(x) := k \quad - \mathrm{ord}_p(0) = +\infty$

$\mathrm{ord}_p(xy) = \mathrm{ord}_p(x) + \mathrm{ord}_p(y)$: $x = p^k u \quad y = p^l v\ ;\ xy = p^{k+l} uv$

$- \mathrm{ord}_p(x) = 0 \Leftrightarrow x \in \mathbb{Z}_p^\times$

$- \mathrm{ord}_p(1/x) = -\mathrm{ord}_p(x)$

ord_p = homomorphism : $\mathbb{Q}_p^\times \mapsto \mathbb{Z}$

$- \mathrm{ord}_p(x) \geq 0 \Leftrightarrow x \in \mathbb{Z}_p$; $- \mathrm{ord}_p(x) \geq n \Leftrightarrow x \in p^n \mathbb{Z}_p$

$- \mathrm{ord}_p(x+y) \geq \min\{\mathrm{ord}_p(x), \mathrm{ord}_p(y)\} =$ if $\mathrm{ord}_p(x) \neq \mathrm{ord}_p(y)$

$x = p^k u\ ;\ y = p^l v\ ;\ l \leq k\ ;\ x + y = p^l \underbrace{(v + p^{k-l} u)}_{\in \mathbb{Z}_p}$

Examples: $\mathrm{ord}_2(3/5) = 0$

$\mathrm{ord}_3(3/5) = 1$

$\mathrm{ord}_5(3/5) = -1$

$\mathrm{ord}_2(3/5 + 1) = \mathrm{ord}_2(8/5) = 3$

$\mathrm{ord}_3(3/5 + 1) = \mathrm{ord}_3(8/5) = 0$

$\mathrm{ord}_5(3/5 + 1) = \mathrm{ord}_5(8/5) = -1$

p-adic absolute value:

$|\cdot|_p : \mathbb{Q}_p \longmapsto \mathbb{R}_{\geq 0}$

$|x|_p = p^{-\mathrm{ord}_p(x)} \quad (|0|_p = 0)$

$|x|_p = p^{-k}$

$- |xy|_p = |x|_p \cdot |y|_p$

$- |x|_p = 0 \Leftrightarrow x = 0 \quad - |x+y|_p$

$- |x+y|_p \leq p^{-\min\{\mathrm{ord}_p(x), \mathrm{ord}_p(y)\}}$

$= $ if $|x|_p \neq |y|_p$

Ostrowski's Thm

$f: \mathbb{Q} \to \mathbb{R}_{\geq 0}$ s.t.

(i) $f(x)=0 \iff x=0$

(ii) $f(xy)=f(x)\cdot f(y)$

(iii) $f(x+y) \leq f(x)+f(y)$ then either

(a) $f(x)=1 \;\forall x \neq 0$

(b) $f(x)=|x|_p^\alpha$, $0<\alpha \leq 1$

(c) $f(x)=|x|_p^\beta$, $\beta \geq 1$

$1+2^{-1}+2^{-2}+\ldots+2^{-n}+\ldots=2$

$S_n = 1+2^{-1}+\ldots+2^{-n} = \frac{2^{-n-1}-1}{1/2-1} = 2-2^{-n}$

$2-2^{-n}$ is getting "close" to 2 in the sense measured by conditional absolute value

given $\varepsilon>0$, $|2-(2-2^{-n})|<\varepsilon$ if $n \gg 0$

We'll replace the usual absolute value $|\,|$ with the p-adic absolute value $|\cdot|_p$

$x, y \in \mathbb{Z}_p$

"x & y close" p-adically $|x-y|_p$ small

this should be a small real number

$|x-y|_p \leq p^{-m}$

$\Updownarrow$

$ord_p(x-y) \geq m \iff x-y \in p^m\mathbb{Z}_p$ (i.e. $p^m | x-y$)

-1 $\quad 1+2+2^2+\ldots$

2-adic expansion

$b_n = 1+2+2^2+\ldots+2^n = \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1$

$|-1-b_n|_2 = |-1-(-1+2^{n+1})|_2 = |2^{n+1}|_2 = \underbrace{2^{-1-n}}_{\text{small if } n \text{ large}}$

Sequences series (p-adically)

$\{a_n\}$ $a_n \in \mathbb{Q}_p$ sequence, $n \geq 0$

converges to a! means

fn $\varepsilon>0$ s.t. $|a-a_n|_p<\varepsilon$ if $n>N_\varepsilon$: $\exists N_\varepsilon>0$

natural candidates for convergent sequences:

Cauchy sequences: $\{a_n\}$ s.t. for $\varepsilon>0$ $\exists N_\varepsilon>0$ s.t. $|a_n-a_m|<\varepsilon$ $\forall n,m>N_\varepsilon$

In $\mathbb{Q}_p$ every Cauchy sequence converges. $a_n = p^{k_n}u_n$ $\quad k_n \in \mathbb{Z}$, $u_n \in \mathbb{Z}_p^\times$

k_n unbounded: $\{a_n\} \to 0$ p-adically

k_n bounded: fn $\varepsilon<<1$ $\quad |p^{k_n}u_n - p^{k_m}u_m|_p < p^{-j}$ $\quad j \gg 0$

$=|p|^{k_n}|u_n - p^{k_n-k_m}|$

$|p^k u_n - p^k u_m|_p \le p^{-j}$ $j >> 0$ C. Skinner

$|u_n - u_m|_p \le p^{k-j}$ (5)

$u_n \equiv u_m \mod p^{k-j}$ $\forall\, n, m >> 0$

Every element of $\mathbb{Q}_p$ is the limit of a Cauchy sequence of rational numbers.

$x \in \mathbb{Q}_p$, $\exists \{r_n\}$ C.S. s.t. $\{r_n\} \mapsto x$; $r_n \in \mathbb{Q}$

$x = p^k u$; $u = \{u_n\}$ $u_n \in \mathbb{Z}/p^n$; $u_{n+1} \equiv u_n \mod p^n$

[can assume] $u_n \equiv r_n \mod p^n$; $r_n \in R_n$

Th. $\{r_n\}$ is a sequence of rational numbers

$r_n = \sum_{i=0}^{n-1} c_i p^i$, $c_i \in R_1 = \{0, 1, \ldots, p-1\}$.

$|r_n - r_m| = |\sum_{i=n}^{m-1} c_i p^i|_p = |p^n|_p \cdot |c_n + c_{n+1} p + \ldots + c_{m-1} p^{m-n-1}|_p \le p^{-n}$

$u = \{u_n\} = \{r_n \mod p^n\}$; $u - r_m = \{\underbrace{r_n - r_m}_{0 \text{ if } m \ge n} \mod p^n\}$

$p^m \mid u - r_m$

$|u - r_m|_p \le p^{-m}$ so $\{r_m\} \mapsto u$

$\sum_{i=0}^{\infty} c_i p^i = 0$ in $\mathbb{Q}_p$

$\mathcal{C}_p$ = set of (p-adic) Cauchy sequences $\{r_n\}$, $r_n \in \mathbb{Q}$ w.r.t. $|\cdot|_p$

- ring under mult. + addition of sequences.

$\mathcal{C}_p \twoheadrightarrow \mathbb{Q}_p$ kernel of (*) = set of Cauchy sequences that converge to 0

$\{r_n\} \longmapsto$ its limit (p-adic)

$\{\{r_n\}$ C.S. $\&$ $|r_n|_p \mapsto 0\} = \mathcal{N}_p$

Thm. $\mathcal{C}_p / \mathcal{N}_p \xrightarrow{\sim} \mathbb{Q}_p$