Lisbon Summer School (2011) Problem Set 4

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1. Show $\mathbf{Q}[\sqrt{-2}]$, $\mathbf{Q}[\sqrt{-3}]$, and $\mathbf{Q}[\sqrt{5}]$ have Minkowski bound less than 2, so all of them have class number 1. What is $\mathcal{O}_K$ in each case?

2. a) Show $\mathbf{Q}[\sqrt{-5}]$ has Kronecker bound 10.4 and Minkowski bound 2.8. Since $\mathbf{Z}[\sqrt{-5}]$ has nonprincipal ideals from yesterday's problem set, the Minkowski bound implies that $\mathbf{Q}[\sqrt{-5}]$ has class number 2.

   b) For the nonprincipal ideals $\mathfrak{p} = (2, 1+\sqrt{-5})$ and $\mathfrak{q} = (3, 1+\sqrt{-5})$ in $\mathbf{Z}[\sqrt{-5}]$, find explicit nonzero x and y in $\mathbf{Z}[\sqrt{-5}]$ such that $x\mathfrak{p} = y\mathfrak{q}$. (Hint: Use some of the prime ideal factorizations in problem 4 of yesterday's problem set.)

3. In $\mathbf{Z}[\sqrt{-14}]$, verify the prime ideal factorizations

$$(2) = (2, \sqrt{-14})^2, \quad (3) = (3, 1+\sqrt{-14})(3, 1-\sqrt{-14}).$$

   b) Verify the following formulas for powers of $\mathfrak{q} = (3, 1+\sqrt{-14})$:

$$\mathfrak{q}^2 = (9, 2-\sqrt{-14}), \quad \mathfrak{q}^3 = (27, 16+\sqrt{-14}), \quad \mathfrak{q}^4 = (5+2\sqrt{-14}).$$

   c) Show $\mathfrak{q}$, $\mathfrak{q}^2$, and $\mathfrak{q}^3$ are all nonprincipal ideals. (Hint: Suppose there were a principal ideal with the same norm and see what a generator of that ideal would have to be.)

   d) Show $(2+\sqrt{-14}) = (2, \sqrt{-14})(3, 1-\sqrt{-14})^2$ and conclude that $[(2, \sqrt{-14})] = [\mathfrak{q}]^2$ in the ideal class group of $\mathbf{Q}[\sqrt{-14}]$.

   e) Show $\mathbf{Q}[\sqrt{-14}]$ has Minkowski bound 4.7 and use the previous calculations to show the ideal class group of $\mathbf{Q}[\sqrt{-14}]$ is cyclic of order 4 with generator $[\mathfrak{q}]$.

4. a) Show $\mathbf{Q}[\sqrt{-11}]$ has class number 1.

   b) Use part a to show that the integral solutions of $y^2 = x^3 - 11$ are $(3, \pm 4)$ and $(15, \pm 58)$.

5. a) Show $\mathbf{Q}[\sqrt{-13}]$ has class number 2.

   b) Use part a to show that the integral solutions of $y^2 = x^3 - 13$ are $(17, \pm 70)$.