SCHUBERT POLYNOMIALS AND SYMMETRIC FUNCTIONS
NOTES FOR THE LISBON COMBINATORICS SUMMER SCHOOL 2012

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1. SYMMETRIC POLYNOMIALS EXERCISES

Polynomial rings such as R have a useful property: every polynomial p can be uniquely written as a sum $\sum_{\mathbb{N}} p_d$ of **homogeneous** polynomials p_d, in which every monomial in p_d has the same degree d. More generally, define a **graded** ring Q as one that contains a list $(Q_d)_{d\in\mathbb{N}}$ of subspaces, such that $Q_d Q_e \leq Q_{d+e}$ and every $q \in Q$ is uniquely the sum $q = \sum_{\mathbb{N}} q_d$, $q_d \in Q_d$.

Exercise 1.1. *R_k has a $\mathbb{Z}$-basis of size $\binom{n+k-1}{n}$. (Hint: correspond a monomial like $x_1^5 x_2^3 x_4^2 x_5^2$, for $n = 6$, to a word like $*****|***||**|**||$.)*

Exercise 1.2. *For each polynomial in $(R^{S_2})_2$, show that it can be written uniquely as a polynomial in $x_1 + x_2$ and $x_1^2 + x_2^2$ using rational coefficients. Find one that despite having integer coefficients itself, cannot be written as $p(x_1 + x_2, x_1^2 + x_2^2)$ where $p(a, b) \in \mathbb{Z}[a, b]$ has integer coefficients.*

We define the *lexicographically first* monomial m in a nonzero polynomial p, and denote it $\operatorname{init} p$. It is the m with the highest power of x_1 available, then among ties it has the highest power of x_2 available, and so on. (Writing monomials $x_1^2 x_2 x_3^3$ like $x_1x_1x_2x_3x_3x_3$, this is almost dictionary order, except that in dictionaries x_1x_1 comes after x_1. It is indeed dictionary order when restricted to monomials of a fixed degree.)

Exercise 1.3.
(1) $\operatorname{init} e_k = \prod_{i=1}^k x_i$.
(2) $\operatorname{init} (pq) = \operatorname{init} p \cdot \operatorname{init} q$.
(3) *If p is symmetric, and $\operatorname{init} p = c \prod_i x_i^{m_i}$ $(c \in \mathbb{Z})$, then $m_1 \geq m_2 \geq \ldots \geq m_n$.*

Exercise 1.4. *Write $\sum_i x_i^n$ as a polynomial in the elementary symmetric polynomials, for $n \leq 4$.*

Exercise 1.5. *Let $n = 2$, and $Z_2 = \{1, \tau\}$ act on R by $\tau \cdot x = -x$, $\tau \cdot y = -y$. Show that the τ-invariant subring R^{Z_2} is generated by x^2, xy, y^2, and is isomorphic to $\mathbb{Z}[a, b, c]/\langle b^2 - ac\rangle$. Show that this graded ring is not graded-isomorphic to a polynomial ring.*

In fact there is a complete classification, due to C. Chevalley, of which finite groups G acting on R and preserving degree have R^G isomorphic to a polynomial ring.